Research Article

Dongkwang Kim* and Jaewook Ahn

Chemotaxis-consumption system with Robin boundary conditions coupled to the (Navier–)Stokes equations

https://doi.org/10.1515/anona-2025-0152
Received September 25, 2025; accepted February 28, 2026; published online April 7, 2026

Abstract: In this paper, we consider the chemotaxis-consumption system on a bounded smooth domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with fluid coupling

$$\begin{cases} \rho_t + u \cdot \nabla \rho - \nabla \cdot (D(\rho) \nabla \rho) = \nabla \cdot (\rho S(x, \rho, c) \cdot \nabla c), \\ u \cdot \nabla c - \Delta c = -\rho c, \\ u_t + \kappa (u \cdot \nabla) u - \Delta u + \nabla \pi = \rho \nabla \Phi, \quad \nabla \cdot u = 0, \end{cases}$$

subject to the boundary conditions $\nu \cdot (D(\rho)\nabla\rho + \rho S(x, \rho, c)\nabla c)|_{\partial\Omega} = 0$, $(\nu \cdot \nabla c + c)|_{\partial\Omega} = \gamma$, and $u|_{\partial\Omega} = 0$. When $(n, \kappa) = (2, 1)$, we establish the global existence and uniform boundedness of classical solutions for all suitably regular initial data, under general structural conditions on the tensor-valued sensitivity S and a strictly positive lower bound on the diffusivity D. In case $(n, \kappa) = (3, 0)$, we show that the same result holds provided that D meets a certain diffusion enhancement condition depending on γ. Moreover, we construct finite-time blow-up solutions for the radially symmetric, fluid-free system when $n = 2, 3$, $D(\xi) \lesssim (1 + \xi)^{m-1}$ with $0 < m < \frac{2}{n}$ and $S \equiv \mathbb{I}_{n\times n}$. We prove that, for any prescribed initial mass, blow-up occurs when γ is sufficiently large.

Keywords: chemotaxis-fluid system; blow-up; global boundedness

MSC 2020: 35B44; 35K51; 92C17

1 Introduction

The movement of bacteria toward or away from chemical signals, a mechanism known as chemotaxis, plays a crucial role in their survival. For instance, some aquatic bacteria migrate toward the water–air interface for oxygen uptake [1], and other bacteria employ a more sophisticated strategy to find an optimal oxygen concentration, exhibiting a repellent response to high concentrations and an attractant response to low concentrations [2]. These chemotactic motions of bacteria are often observed in the presence of a moving fluid. From a modeling perspective, such situations can be described by chemotaxis-consumption systems coupled to fluid equations:

$$\rho_t + u \cdot \nabla \rho - \nabla \cdot (D(\rho) \nabla \rho) = \nabla \cdot (\rho S(x, \rho, c) \cdot \nabla c),$$

***Corresponding author: Dongkwang Kim**, Department of Mathematical Sciences, Ulsan National Institute of Science and Technology (UNIST), Ulsan, Republic of Korea, E-mail: dkkim@unist.ac.kr
Jaewook Ahn, Department of Mathematics, Dongguk University, Seoul, Republic of Korea, E-mail: jaewookahn@dgu.ac.kr

Open Access. © 2026 the author(s), published by De Gruyter. This work is licensed under the Creative Commons Attribution 4.0 International License.

$$\tau c_t + u \cdot \nabla c - \Delta c = -\rho c,$$

$$u_t + \kappa(u \cdot \nabla)u - \Delta u + \nabla \pi = \rho \nabla \Phi, \qquad \nabla \cdot u = 0.$$

Here, ρ denotes the density of the motile species, c represents the concentration of a chemical substance, u is the incompressible fluid velocity, and π stands for the associated pressure. The function D models the diffusivity, and S is the tensor-valued chemotactic sensitivity. The scalar function Φ represents a prescribed gravitational potential. The model was originally proposed by Tuval et al. [1] to describe the dynamics of aerobic bacteria *Bacillus subtilis* living in water. Over the past years, a number of works have investigated various forms of this model under different assumptions on its structures, such as D and S, or on the presence or absence of the fluid. For an overview of this system and its variants, see [3]. We briefly review these results below.

Most previous studies have focused on the setting with no-flux boundary conditions for both ρ and c (combined with no-slip boundary conditions for u), which is well-suited for the construction of Lyapunov-type functionals. Indeed, for linear diffusivity $D \equiv 1$ and scalar sensitivity $S = -\chi(c)$ with suitably chosen positive χ (e.g., the prototype $\chi(c) \equiv 1$), such functionals have been used to prove global solvability as well as stability and asymptotic behavior; see, e.g., [4,5] and references therein. When χ is singular of the form $\chi(c) = c^{-\alpha}$ with $\alpha > 0$, global solutions at various regularity levels have been constructed (e.g., [6,7]).

Such global solvability, and even boundedness, can remain valid when diffusion is strengthened (e.g., the porous-medium type $D(\rho) \approx \rho^{m-1}$ with $m > 1$). For $n = 2$, $\kappa = 0$ (Stokes flow), and sufficiently smooth $\chi \geq 0$, it has been shown that global bounded weak solutions exist whenever $m > 1$ [8]. For $n = 3$ and $\kappa = 0$, global weak solvability has been proved for $m \in \left(\frac{3}{2}, 2\right]$ under suitable conditions on scalar sensitivity χ [9]. In the special case $\chi \equiv 1$ and $\kappa = 0$, the existence of locally bounded global solutions has been established when $m > \frac{8}{7}$ [10]. This result was later extended to the case $m > \frac{9}{8}$ by [11], where the authors also proved the boundedness and stabilization of solutions to spatially homogeneous equilibria. For the existence of global bounded weak solutions when $m \geq \frac{65}{63}$, see [12]. We also remark that in chemotaxis systems with signal production mechanisms, nonlinear diffusion has been shown to play a similarly important role in determining global boundedness (e.g., [13,14]).

For tensor-valued sensitivity S, there are relatively few results. When S decays as $|S(x, \rho, c)| \leq (1 + \rho)^{-\alpha}$ with $\alpha > \frac{1}{6}$, global classical solvability has been proved for $n = 3$ and $\kappa = 0$ under linear diffusion [15], later extended to $\alpha > 0$ [16]. Under the more general condition $|S(x, \rho, c)| \leq \tilde{S}(c)$ with a nondecreasing $\tilde{S}$, global existence and boundedness of weak solutions have been established for $n = 2$, $\kappa = 1$, $m > 1$ [17] and for $n = 3$, $\kappa = 0$, $m > \frac{7}{6}$ [18] (see also [19] for small-data result). In the same setting for S, global mass-preserving generalized solutions have been constructed for $\kappa = 0$ [20] and $\kappa = 1$ [21] in two dimensions, with eventual smoothness and stabilization in the former [22]. For $n = 3$ and $\kappa = 0$, global boundedness has been obtained even when S exhibits singularity $|S(x, \rho, c)| \leq \tilde{S}(c)/c^{1/2}$, provided diffusion is slightly enhanced, including the case $m > 1$ [23]. Very recently, this result has been extended to the case $n = 2$, $\kappa = 1$, and $|S(x, \rho, c)| \leq \tilde{S}(c)/c^{5/6}$ in ref. [24].

We also note that there have been results on global solvability and large-time behavior subject to either Robin or Dirichlet boundary conditions for c (see, e.g., [25,26]). However, it remains open whether linear diffusion can ensure classical, or even weak, global solvability for general sensitivity S. A partial result is given in ref. [27], where global bounded classical solutions have been constructed in the fluid-free system under general S and linear diffusion for the Robin case; see also ref. [28] for the Dirichlet case.

In the radially symmetric, fluid-free system with the Dirichlet boundary condition for c, recent works have demonstrated that finite-time blow-up may occur when the chemotaxis term is repulsive (e.g., $S(x, \rho, c) = \chi(c) > 0$). If $\tau = 0$ and $n \geq 2$, then $D(\rho) \approx \rho^{m-1}$ with $0 < m < 1$ and $\chi(c) = c^{-1}$ can lead to blow-up [29]. When $\tau = 0$, $n = 2$, and $\chi(c) \equiv 1$, blow-up has been observed for $0 < m < 1$, while global boundedness has been shown for $m \geq 1$ [30]. This result has recently been partially extended to $n \geq 3$ [31]. We also refer to ref. [32] for a finite-time blow-up result when $\tau = 0$, $n = 2$, $m = 1$, and $S(x, \rho, c) = (\rho + 1)^{l-1}$ with $l > 1$.

The purpose of this paper is to establish results on global boundedness and finite-time blow-up in the chemotaxis-consumption system coupled to the (Navier–)Stokes equations with no-flux/Robin/no-slip boundary conditions. Our analysis covers tensor-valued sensitivities S under general structural assumptions for the boundedness results, while the blow-up scenario is demonstrated in the repulsive case $S \equiv \mathbb{I}_{n\times n}$. Specifically, we obtain three types of results: (i) global boundedness in two dimensions under a strictly positive lower bound on the diffusivity D, which in particular includes the linear diffusion case; (ii) global boundedness in three dimensions provided D is sufficiently strong in a sense depending on γ; and (iii) finite-time blow-up in the radially symmetric and fluid-free setting when D is sufficiently weak. These results complement and extend previous works in the fluid-free framework and for models with Dirichlet boundary conditions for c [27,30,31].

To state our main results, we specify our problem. Consider the situation where the diffusion of c occurs on a much faster time scale than that of ρ, namely $\tau \equiv 0$. The problem is then given by the following parabolic–elliptic chemotaxis-consumption fluid system with tensor-valued sensitivity:

$$\begin{cases} \rho_t + u\cdot\nabla\rho - \nabla\cdot(D(\rho)\nabla\rho) = \nabla\cdot(\rho S(x,\rho,c)\cdot\nabla c), & x\in\Omega,\ t>0,\\ u\cdot\nabla c - \Delta c = -\rho c, & x\in\Omega,\ t>0,\\ u_t + \kappa(u\cdot\nabla)u - \Delta u + \nabla\pi = \rho\nabla\Phi, \qquad \nabla\cdot u = 0, & x\in\Omega,\ t>0,\\ \rho(x,0) = \rho_0(x), \quad u(x,0) = u_0(x), & x\in\Omega,\\ \nu\cdot(D(\rho)\nabla\rho + \rho S(x,\rho,c)\cdot\nabla c) = 0, \quad \nu\cdot\nabla c = \gamma - c, \quad u = 0, & x\in\partial\Omega,\ t>0. \end{cases} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^n$, $n = 2,3$, is a bounded domain with smooth boundary, $\kappa \in \mathbb{R}$, and $\gamma \in \mathbb{R}_+ := (0,\infty)$. We suppose that D and S fulfill

$$D \in C^2([0,\infty)), \quad D(\xi) > 0 \quad \text{for all } \xi \in \overline{\mathbb{R}_+}, \tag{1.2}$$

$$\begin{gathered} S \in C^2\left(\overline{\Omega}\times\overline{\mathbb{R}_+}\times\overline{\mathbb{R}_+}\right), \text{ and there exists } S_0 \in C\left(\overline{\mathbb{R}_+}\right) \text{ such that} \\ |S(x,r,s)| \le S_0(s) \quad \text{for all } (x,r,s)\in\overline{\Omega}\times\overline{\mathbb{R}_+}\times\overline{\mathbb{R}_+}. \end{gathered} \tag{1.3}$$

The initial data (ρ_0, u_0) and the potential function Φ are assumed to satisfy

$$\rho_0 \in W^{1,\infty}(\Omega), \quad (0 \not\equiv)\rho_0 \ge 0 \quad \text{on } \overline{\Omega}, \quad u_0 \in D(A^\alpha) \quad \text{for some } \alpha \in \left(\frac{n}{4}, 1\right), \quad \text{and } \Phi \in C^2(\overline{\Omega}), \tag{1.4}$$

where $A := -\mathcal{P}\Delta$ denotes the realization of the Stokes operator in the solenoidal subspaces $L^2_\sigma(\Omega)(\Omega)^n := \{\varphi \in L^2(\Omega)^n \mid \nabla\cdot\varphi = 0\}$ of $L^2(\Omega)^n$.

We begin by stating the local well-posedness result for the system (1.1).

Proposition 1.1. *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded smooth domain. Suppose that D and S satisfy* (1.2) *and* (1.3)*. Then, for any initial data (ρ_0, u_0) satisfying* (1.4)*, there exist a maximal time $T_{\max} = T_{\max}(\rho_0,u_0) \in (0,\infty]$ and a classical solution (ρ, c, u, π) of* (1.1) *in $\Omega\times(0,T_{\max})$ such that*

$$\begin{aligned} &\rho \in C\left(\overline{\Omega}\times[0,T_{\max})\right)\cap C^{2,1}\left(\overline{\Omega}\times(0,T_{\max})\right), \qquad c \in C^{2,0}\left(\overline{\Omega}\times(0,T_{\max})\right),\\ &u \in C\left(\overline{\Omega}\times[0,T_{\max})\right)\cap C^{2,1}\left(\overline{\Omega}\times(0,T_{\max})\right), \qquad \pi \in C^{1,0}\left(\overline{\Omega}\times(0,T_{\max})\right). \end{aligned}$$

Moreover, if $T_{\max} < \infty$, then ρ blows up in the sense that

$$\|\rho(\cdot,t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as } t \to T_{\max}.$$

In addition, it holds that $\rho \ge 0$ and $c > 0$ in $\overline{\Omega}\times(0,T_{\max})$, and that

$$\int_\Omega \rho(x,t)\mathrm{d}x = \int_\Omega \rho_0(x)\mathrm{d}x \quad \text{for all } t\in(0,T_{\max}), \tag{1.5}$$

$$\|c(\cdot,t)\|_{L^\infty(\Omega)} \le \gamma \quad \text{for all } t\in(0,T_{\max}), \tag{1.6}$$

$$\int_{\Omega} |\nabla c(x,t)|^2 \mathrm{d}x + \int_{\Omega} \rho c^2(x,t)\mathrm{d}x \le \gamma^2 |\partial\Omega| \quad \text{for all } t \in \left(0, T_{\max}\right), \tag{1.7}$$

as well as

$$\|S(\cdot, \rho(\cdot,t), c(\cdot,t))\|_{L^\infty(\Omega)} \le \|S_0\|_{C([0,\gamma])} \quad \text{for all } t \in \left(0, T_{\max}\right). \tag{1.8}$$

In the fluid-free system ($u \equiv 0 \equiv \nabla\Phi$), if we further assume that ρ_0 in (1.4) *is radially symmetric, then (ρ, c) are radially symmetric for all $t \in \left(0, T_{\max}\right)$.*

Proof. Local existence and blow-up criteria can be found, e.g., in ref. [31]. The nonnegativity of the solutions and (1.6) follow from the maximum principle. The mass conservation (1.5) is obtained by integrating the equation for ρ in (1.1) over Ω. Estimate (1.7) results from testing the equation for c in (1.1) with c. Finally, (1.8) is an immediate consequence of the assumption on S in (1.3) together with (1.6). □

We next state our main results concerning global existence and blow-up phenomena.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain, and suppose that D and S satisfy* (1.2) *and* (1.3). *Suppose in addition that*

$$D(\xi) \ge k_D \quad \text{for all } \xi \in \overline{\mathbb{R}_+}, \quad \text{for some } k_D > 0. \tag{1.9}$$

Then for any initial data (ρ_0, u_0) fulfilling (1.4), *the classical solution (ρ, c, u, π) of* (1.1) *exists globally in time and satisfies*

$$\sup_{t>0}\left(\|\rho(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|u(\cdot,t)\|_{L^\infty(\Omega)}\right) < \infty.$$

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain, $\kappa = 0$, and suppose that D and S satisfy* (1.2) *and* (1.3). *Then, for any $\gamma > 0$, there exists $M = M(\gamma) > 0$ such that if D additionally fulfills*

$$\liminf_{\xi\to\infty} D(\xi) \ge M, \tag{1.10}$$

then for any initial data (ρ_0, u_0) satisfying (1.4), *the classical solution (ρ, c, u, π) is global in time and satisfies*

$$\sup_{t>0}\left(\|\rho(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|u(\cdot,t)\|_{L^\infty(\Omega)}\right) < \infty.$$

Remark 1.1. We can extend Theorems 1.1 and 1.2 to more general Robin boundary conditions $\nu \cdot \nabla c = (\gamma - c)g$ with $0 < g \in C^{1+\theta}(\partial\Omega)$ for some $\theta \in (0,1)$; however, for simplicity, we assume $g \equiv 1$.

Remark 1.2. In Theorem 1.2, the lower bound M depends not only on γ but also on the structure of S; see (3.8). In particular, if $|S| \le 1$ (e.g., the prototype $S = \pm\mathbb{I}_{n\times n}$), the condition can be simplified to

$$\liminf_{\xi\to\infty} D(\xi) \ge 2\gamma.$$

Theorem 1.3. *Let $\Omega = B_R \subset \mathbb{R}^n$, $n \in \{2,3\}$, and $m \in \left(0, \frac{2}{n}\right)$. Suppose that $S \equiv \mathbb{I}_{n\times n}$, and D satisfies* (1.2) *and*

$$D(\xi) \le K_D(1+\xi)^{m-1} \quad \text{for all } \xi > 0, \quad \text{for some } K_D > 0. \tag{1.11}$$

Then, for any $L > 0$, there exists $\gamma^* = \gamma^*(n, m, R, L) > 0$ such that for all $\gamma \in [\gamma^*, \infty)$ and any radially symmetric initial data ρ_0 satisfying (1.4) *and*

$$\|\rho_0\|_{L^1(\Omega)} = L,$$

the corresponding solution (ρ, c) to (1.1) *in the fluid-free setting ($u \equiv 0 \equiv \nabla\Phi$) blows up in finite time, i.e., $T_{\max} < \infty$.*

Remark 1.3. A similar argument can establish the blow-up of solutions, in either finite or infinite time, for higher dimensions $n \geq 4$. However, we focus on $n = 2, 3$ cases as they are the most physically relevant.

Strategy. To prove Theorem 1.1, a crucial step is the derivation of spatially localized, time-uniform L^2-estimate for ∇c (Lemma 2.7), extending earlier result available only in the fluid-free case [27, Prop. 3.1]. This estimate further leads to additional localized estimates, including in particular $\rho \ln \rho$ in $L_t^\infty L^1_{\mathrm{loc}}(\Omega)$ (Lemmas 2.9 and 2.10). These, in turn, yield L^∞-bound of ρ uniform-in-time.

For Theorem 1.2, the absence of a convection term in the fluid equations, together with the assumption (1.10) on D, makes it possible to close the energy estimate for ρ (Lemmas 3.1–3.4). This step requires the control of $\|u\|_{L^6(\Omega)}$, which, as observed in ref. [23], can be bounded in terms of $\|\nabla \rho\|_{L^2(\Omega)}$ (Lemma 3.3). This energy estimate is sufficient to conclude uniform boundedness for ρ.

Theorem 1.3 states a finite-time blow-up result when $\gamma \geq \gamma^*(n, m, \Omega, \|\rho_0\|_{L^1(\Omega)})$. We remark that the dependence of γ^* on ρ_0 is considerably simpler than that in previous works [31, proof of Thm. 3], where the required largeness of the boundary condition for blow-up depended on ρ_0 in a quite nonlinear way. To this end, after obtaining a uniform-in-time lower bound for c (Lemma 4.1), we separately consider two cases $c(0, t) \leq h\gamma$ and $c(0, t) > h\gamma$ for some $h \in (0, 1)$, to derive an ODI $\phi'(t) \gtrsim -\|\rho_0\|^m_{L^1(\Omega)} + \gamma \|\rho_0\|^2_{L^1(\Omega)}/(\|\rho_0\|_{L^1(\Omega)} + 1)^3$ for a moment-type bounded functional ϕ of the mass accumulation function (Lemma 4.2). This in turn leads to blow-up since ϕ linearly grows in time whenever $\gamma \geq \gamma^*$.

Throughout this paper, C denotes a generic positive constant, which may vary from line to line, and the notation $C(\cdot, \cdot, \ldots)$ indicates its dependence on the specified quantities. For $n \geq 2$, we denote by ω_n the surface measure of the unit sphere in $\mathbb{R}^n$ and by B_R the ball of radius R centered at the origin. Whenever no confusion arises, the norm $\|\cdot\|_{L^p(\Omega)}$ will be abbreviated by $\|\cdot\|_{L^p}$.

Unless stated otherwise, we assume throughout that (ρ_0, u_0) and Φ satisfy (1.4), and that D and S fulfill (1.2) and (1.3), respectively. Let (ρ, c, u) in $\Omega \times (0, T_{\max})$ denote the corresponding solution given in Proposition 1.1.

2 Proof of Theorem 1.1

In this section, $n = 2$ and consider (1.1) with $\kappa = 1$, corresponding to the full Navier–Stokes coupling.

The following lemma is derived from the basic properties of the solutions stated in Proposition 1.1.

Lemma 2.1. *Suppose that* (1.9) *holds. There exists* $C = C(\gamma) > 0$ *such that*

$$\int_t^{t+\tau} \int_\Omega |\nabla \ln(\rho + 1)|^2 \leq C \quad \text{for all } t \in (0, T_{\max} - \tau),$$

where $\tau := \min\left\{1, \frac{1}{2} T_{\max}\right\}$.

Proof. From the equation for ρ in (1.1), Young's inequality, (1.8), and (1.9), we obtain

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \ln(\rho + 1) &\leq -k_D \int_\Omega |\nabla \ln(\rho + 1)|^2 + \|S_0\|_{C([0,\gamma])} \int_\Omega |\nabla \ln(\rho + 1)||\nabla c| \\ &\leq -\frac{k_D}{2} \int_\Omega |\nabla \ln(\rho + 1)|^2 + \frac{\|S_0\|^2_{C([0,\gamma])}}{2k_D} \int_\Omega |\nabla c|^2. \end{aligned}$$

Integrating over $[t, t + \tau]$ and using (1.7) and a trivial inequality $\rho \geq \ln(\rho + 1)$ along with (1.5) leads to the claim. □

The following functional inequality is a variant of the Trudinger–Moser inequality (cf. [33, Lem. 2.2]).

Lemma 2.2. *For each $\eta > 0$, there exists $C = C(\eta) > 0$ such that for all nonnegative $f \in C(\overline{\Omega})$, $f \not\equiv 0$, and $g \in W^{1,2}(\Omega)$, and each $a > 0$, one has*

$$\int_\Omega f|g| \le \frac{1}{a}\int_\Omega f\ln\left(\frac{f}{\overline{f}}\right) + \frac{(1+\eta)a}{8\pi}\int_\Omega f \int_\Omega |\nabla g|^2 + Ca\int_\Omega f\left(\int_\Omega |g|\right)^2 + \frac{C}{a}\int_\Omega f,$$

where $\overline{f} := \frac{1}{|\Omega|}\int_\Omega f$.

Thanks to Lemmas 2.1 and 2.2, we can deduce the following logarithmic integrability property for ρ.

Lemma 2.3. *Suppose that* (1.9) *holds. There exists $C = C(\gamma) > 0$ such that*

$$\int_t^{t+\tau}\int_\Omega \rho\ln(\rho+1) \le C \quad \text{for all } t \in (0, T_{\max} - \tau),$$

where $\tau := \min\{1, \frac{1}{2}T_{\max}\}$.

Proof. Using Lemma 2.2 with $f = \rho$, $g = \ln(\rho+1)$ and $a = \eta = 2$, and (1.5), we arrive at

$$\int_\Omega \rho\ln(\rho+1)$$

$$\le \frac{1}{2}\int_\Omega \rho\ln\rho - \frac{1}{2}(\ln\overline{\rho_0})\int_\Omega \rho_0 + \frac{3}{4\pi}\int_\Omega \rho_0\int_\Omega |\nabla\ln(\rho+1)|^2 + 2C\int_\Omega \rho_0\left(\int_\Omega \ln(\rho+1)\right)^2 + \frac{C}{2}\int_\Omega \rho_0.$$

The claim follows from the trivial inequalities $a\ln a \le a\ln(a+1)$ and $a \ge \ln(a+1)$ for $a \ge 0$, (1.5), and Lemma 2.1. □

We next recall a standard ODE-type lemma for absorbing inequalities (see [34, Lem. 3.4]).

Lemma 2.4. *Let $h \in L^1_{\mathrm{loc}}(\mathbb{R})$ be nonnegative and suppose there exist $\tau > 0$ and $b > 0$ such that*

$$\frac{1}{\tau}\int_t^{t+\tau} h(s)ds \le b \quad \text{for all } t \in (0, T-\tau).$$

Assume further that $y \in C([0,T)) \cap C^1((0,T))$ satisfies

$$y'(t) + ay(t) \le h(t) \quad \text{with some } a > 0.$$

Then,

$$y(t) \le y(0) + \frac{b\tau}{1-e^{-a\tau}} \quad \text{for all } t \in (0,T).$$

We now derive a uniform-in-time energy estimate for the velocity field.

Lemma 2.5. *Suppose that* (1.9) *holds. There exists $C = C(\gamma) > 0$ such that*

$$\sup_{t\in(0,T_{\max})} \|u(\cdot,t)\|_{L^2(\Omega)} \le C \tag{2.1}$$

and

$$\int_t^{t+\tau}\int_\Omega |\nabla u|^2 \le C \quad \text{for all } t \in (0, T_{\max} - \tau), \tag{2.2}$$

where $\tau := \min\{1, \frac{1}{2}T_{\max}\}$.

Proof. Testing the equation for u in (1.1) with u yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 \le \|\nabla\Phi\|_{L^\infty(\Omega)}\int_\Omega \rho|u|.$$

Using Lemma 2.2 with $f = \rho$, $g = u$, $\eta = 1$, together with (1.5), we obtain

$$\int_\Omega \rho|u| \le \frac{1}{a}\int_\Omega \rho \ln\left(\frac{\rho}{\overline{\rho_0}}\right) + \frac{a}{4\pi}\int_\Omega \rho_0 \int_\Omega |\nabla u|^2 + Ca\int_\Omega \rho_0\left(\int_\Omega |u|\right)^2 + \frac{C}{a}\int_\Omega \rho_0.$$

By Poincaré and Hölder's inequalities, we have

$$\left(\int_\Omega |u|\right)^2 \le C\int_\Omega |\nabla u|^2.$$

By choosing $a = a(\|\rho_0\|_{L^1(\Omega)}, \|\nabla\Phi\|_{L^\infty(\Omega)}) > 0$ sufficiently small and applying Poincaré inequality once more, we, therefore, obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2 + \int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 \le C\left(1 + \int_\Omega \rho \ln(\rho + 1)\right)$$

with some $C > 0$. The claim follows by applying Lemmas 2.3 and 2.4. □

To derive spatially localized estimates, we prepare the following test function (cf., [27]).

Lemma 2.6. *Let $\eta \in (0, e^{-1})$. Then, the nonnegative radial function*

$$\psi_\eta(x) := \begin{cases} \ln(-\ln|x|) - \ln(-\ln\eta), & \text{if } x \in B_\eta(0)\backslash\{0\}, \\ 0, & \text{otherwise,} \end{cases}$$

satisfies $\psi_\eta(x) \nearrow \infty$ as $|x| \to 0$, and moreover,

$$\left\|\psi_\eta\right\|_{H^1(\mathbb{R}^2)} + \left\|\nabla\left(\psi_\eta^2\right)\right\|_{L^2(\mathbb{R}^2)} \to 0 \quad \text{as } \eta \to 0.$$

Proof. It was shown in ref. [27, Prop. 3.1] that $\psi_\eta \nearrow \infty$ as $|x| \to 0$ while $\left\|\psi_\eta\right\|_{H^1(\mathbb{R}^2)} \to 0$ as $\eta \to 0$. It remains to show that $\left\|\nabla\left(\psi_\eta^2\right)\right\|_{L^2(\mathbb{R}^2)} \to 0$ as $\eta \to 0$. Passing to radial coordinates, we compute

$$\begin{aligned}\left\|\nabla\left(\psi_\eta^2\right)\right\|_{L^2(\mathbb{R}^2)}^2 &= 4\omega_2\int_0^\eta \psi_\eta^2(r)(\partial_r\psi_\eta(r))^2 r\mathrm{d}r \\ &\le 4\omega_2\int_0^\eta \frac{|\ln(-\ln r)|^2}{|r\ln r|^2} r\mathrm{d}r.\end{aligned}$$

Substituting $\zeta = -\ln(-\ln r)$, we obtain

$$\left\|\nabla\left(\psi_\eta^2\right)\right\|_{L^2(\mathbb{R}^2)}^2 \le 4\omega_2 \int\limits_{\ln(-\ln\,\eta)}^{\infty} \frac{\zeta^2}{e^\zeta}\,\mathrm{d}\zeta,$$

which vanishes as $\eta \to 0$. □

We show that the spatially localized L^2-norm of ∇c can be made arbitrarily small uniformly in time.

Lemma 2.7. *Suppose that* (1.9) *holds. Then, for each* $\varepsilon > 0$, *there exists* $\delta_\star = \delta_\star(\varepsilon,\gamma) > 0$ *such that for all* $q \in \overline{\Omega}$ *and all* $\delta \in (0,\delta_\star)$,

$$\sup_{t\in(0,T_{\max})} \|\nabla c(\cdot,t)\|_{L^2(\Omega\cap B_\delta(q))} \le \varepsilon.$$

Proof. Fix $q \in \overline{\Omega}$, $\eta \in (0,e^{-1})$, and define $\psi_\eta(x)$ as in Lemma 2.6. Denote $\psi(x) := \psi_\eta(x-q)$. Multiplying the second equation in (1.1) by $c\psi^2$ and integrating over Ω yields

$$\int\limits_\Omega |\nabla c|^2\psi^2 + \int\limits_\Omega \rho c^2\psi^2 = \frac{1}{2}\int\limits_\Omega c^2 u\cdot\nabla\left(\psi^2\right) + \int\limits_{\partial\Omega} c(\gamma-c)\psi^2 - 2\int\limits_\Omega c\psi\nabla c\cdot\nabla\psi.$$

To estimate the last term, we apply Young's inequality and use (1.6):

$$\left|-2\int\limits_\Omega c\psi\nabla c\cdot\nabla\psi\right| \le \frac{1}{2}\int\limits_\Omega |\nabla c|^2\psi^2 + 2\gamma^2\|\nabla\psi\|_{L^2(\Omega)}^2.$$

For the boundary term, (1.6) and the trace embedding $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ implies for some $C > 0$

$$\int\limits_{\partial\Omega} c(\gamma-c)\psi^2 \le \gamma^2 C\|\psi\|_{H^1(\mathbb{R}^2)}^2.$$

To estimate the remaining term, we employ (1.6), (2.1), and Hölder's inequality:

$$\left|\frac{1}{2}\int\limits_\Omega c^2u\cdot\nabla\left(\psi^2\right)\right| \le \frac{\gamma^2}{2}\|u\|_{L^2(\Omega)}\left\|\nabla\left(\psi^2\right)\right\|_{L^2(\Omega)} \le C\left\|\nabla\left(\psi^2\right)\right\|_{L^2(\Omega)}$$

with some $C > 0$. Putting the estimates together, we find $C > 0$ such that

$$\int\limits_\Omega |\nabla c|^2\psi^2 \le C\left(\|\psi\|_{H^1(\mathbb{R}^2)}^2 + \left\|\nabla\left(\psi^2\right)\right\|_{L^2(\Omega)}\right).$$

Since $\psi(x) \nearrow \infty$ radially as $x \to q$, we can conclude the desired result. □

In preparation for localization arguments, we introduce a family of smooth cutoff functions as follows; see, e.g., [35].

Lemma 2.8. *Let* $\delta > 0$. *Then there exists a radially symmetric, nonnegative function* $\varphi_\delta \in C_0^\infty\left(\mathbb{R}^2\right)$ *such that*

$$\varphi_\delta(x) = \begin{cases} 1, & x \in B_{\frac{\delta}{2}}(0), \\ 0, & x \in \mathbb{R}^2\backslash B_\delta(0), \end{cases}$$

with $0 \le \varphi_\delta \le 1$ *in* $\mathbb{R}^2$, *and*

$$\left|\nabla\varphi_\delta(x)\right| \le K\varphi_\delta(x)^{1/2} \quad \text{for all } x \in \mathbb{R}^2,$$

where $K = \mathcal{O}(\delta^{-1})$.

We now establish localized estimates for ρ and ∇c. Its fluid-free version can be found in ref. [27, Lem. 3.2].

Lemma 2.9. *Suppose that* (1.9) *holds. Fix any* $\delta > 0$. *Let* φ_δ *be the cutoff function from Lemma* 2.8, *and for any* $q \in \overline{\Omega}$, *define* $\varphi(x) := \varphi_\delta(x - q)$. *Then there exist* $K_1 > 0$ *and* $K_2 = K_2(\gamma) > 0$, *independent of* δ *and* q, *such that*

$$\int_\Omega \rho^2(\cdot,t)\varphi^3(\cdot) \le K_1\left(\int_\Omega \frac{|\nabla\rho|^2}{\rho}(\cdot,t)\varphi^3(\cdot) + \left\|\varphi^{\frac{3}{2}}\right\|^2_{W^{1,\infty}(\mathbb{R}^2)}\right) \tag{2.3}$$

and

$$\begin{aligned}\int_\Omega |\nabla c|^4(\cdot,t)\varphi^3(\cdot) \le{}& K_2\|\nabla c(\cdot,t)\|_{L^2(\Omega\cap B_\delta)}\left(\int_\Omega \frac{|\nabla\rho|^2}{\rho}(\cdot,t)\varphi^3(\cdot) + \left\|\varphi^{\frac{3}{2}}\right\|^2_{W^{1,\infty}(\mathbb{R}^2)} + \|\varphi\|^3_{W^{2,\frac{3}{2}}(\mathbb{R}^2)}\right)\\ &+ K_2\left(\int_\Omega |u(\cdot,t)|^{\frac{12}{5}}\right)^5\end{aligned} \tag{2.4}$$

for all $t \in (0, T_{\max})$.

Proof. To verify (2.3), we invoke the Sobolev embedding $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$, which gives for some $C > 0$

$$\int_\Omega \rho^2\varphi^3 \le C\left(\left\|\nabla\left(\rho\varphi^{3/2}\right)\right\|^2_{L^1(\Omega)} + \left\|\rho\varphi^{3/2}\right\|^2_{L^1(\Omega)}\right).$$

Using Hölder's inequality and (1.5), we can find $C > 0$, independent of δ and q, satisfying

$$\left\|\nabla\left(\rho\varphi^{\frac{3}{2}}\right)\right\|^2_{L^1(\Omega)} + \left\|\rho\varphi^{\frac{3}{2}}\right\|^2_{L^1(\Omega)} \le C\left(\|\rho_0\|_{L^1(\Omega)}\int_\Omega \frac{|\nabla\rho|^2}{\rho}\varphi^3 + \|\rho_0\|^2_{L^1(\Omega)}\left\|\varphi^{\frac{3}{2}}\right\|^2_{W^{1,\infty}(\Omega)}\right),$$

which yields the desired estimate (2.3).

To establish (2.4), we employ Hölder's inequality and (1.6) to see

$$\begin{aligned}\int_\Omega |\nabla c|^4\varphi^3 &\le \|\nabla c\|_{L^2(\Omega\cap B_\delta)}\left(\int_\Omega |\nabla c|^6\varphi^6\right)^{\frac{1}{2}}\\ &= \|\nabla c\|_{L^2(\Omega\cap B_\delta)}\|\nabla(c\varphi) - c\nabla\varphi\|^3_{L^6(\Omega)}\\ &\le 4\|\nabla c\|_{L^2(\Omega\cap B_\delta)}(\|c\varphi\|^3_{W^{1,6}(\Omega)} + \gamma^3\|\nabla\varphi\|^3_{L^6(\Omega)}).\end{aligned} \tag{2.5}$$

Using the elliptic regularity result (see, e.g., [36, Thm. 2.3.3.6]) and the Sobolev embedding $W^{2,3/2}(\Omega) \hookrightarrow W^{1,6}(\Omega)$, we find $C > 0$, independent of δ and q, such that

$$\begin{aligned}&\|c\varphi\|^3_{W^{1,6}(\Omega)} + \gamma^3\|\nabla\varphi\|^3_{L^6(\Omega)}\\ &\le C\left(\|\Delta(c\varphi)\|^3_{L^{\frac{3}{2}}(\Omega)} + \|c\varphi\|^3_{L^{\frac{3}{2}}(\Omega)} + \|(\gamma - c)\varphi\|^3_{W^{\frac{1}{3},\frac{3}{2}}(\partial\Omega)} + \|c\nabla\varphi\cdot\nu\|^3_{W^{\frac{1}{3},\frac{3}{2}}(\partial\Omega)}\right) + \gamma^3\|\varphi\|^3_{W^{2,\frac{3}{2}}(\mathbb{R}^2)},\end{aligned}$$

where we used

$$\nabla(c\varphi)\cdot\nu = \nabla c\cdot\nu\varphi + c\nabla\varphi\cdot\nu = (\gamma - c)\varphi + c\nabla\varphi\cdot\nu \quad \text{on} \quad \partial\Omega.$$

By (1.6), we see

$$\|c\varphi\|^3_{L^{\frac{3}{2}}} \le \gamma^3\|\varphi\|^3_{W^{2,\frac{3}{2}}(\mathbb{R}^2)}.$$

Since the trace inequality yields with some $C > 0$

$$\|(\gamma-c)\varphi\|^3_{W^{\frac{1}{3},\frac{3}{2}}(\partial\Omega)}+\|c\nabla\varphi\cdot\nu\|^3_{W^{\frac{1}{3},\frac{3}{2}}(\partial\Omega)}\le C\Big(\|(\gamma-c)\varphi\|^3_{W^{1,\frac{3}{2}}(\Omega)}+\|c\nabla\varphi\|^3_{W^{1,\frac{3}{2}}(\Omega)}\Big),$$

if we further use Hölder's inequality, (1.6) and (1.7), we can obtain that

$$\|(\gamma-c)\varphi\|^3_{W^{\frac{1}{3},\frac{3}{2}}(\partial\Omega)}+\|c\nabla\varphi\cdot\nu\|^3_{W^{\frac{1}{3},\frac{3}{2}}(\partial\Omega)}\le C\|\varphi\|^3_{W^{2,\frac{3}{2}}(\mathbb{R}^2)}$$

with some $C>0$. By Hölder's inequality again, we see that

$$\|\rho\varphi\|^3_{L^{\frac{3}{2}}(\Omega)}\le\|\rho\|_{L^1(\Omega)}\int_\Omega\rho^2\varphi^3. \tag{2.6}$$

With this, we use a trivial inequality $(a+b+c)^3\le 9(a^3+b^3+c^3)$ for $a,b,c\ge 0$, Hölder's inequality, (1.5)–(1.7), and the Sobolev embedding $W^{2,\frac{3}{2}}(\Omega)\hookrightarrow W^{1,6}(\Omega)$ to deduce

$$\begin{aligned}\frac{1}{9}\|\Delta(c\varphi)\|^3_{L^{\frac{3}{2}}(\Omega)}&\le\|\Delta c\varphi\|^3_{L^{\frac{3}{2}}(\Omega)}+\|\nabla c\cdot\nabla\varphi\|^3_{L^{\frac{3}{2}}(\Omega)}+\|c\Delta\varphi\|^3_{L^{\frac{3}{2}}(\Omega)}\\&\le\Big(\|u\cdot\nabla c\varphi\|_{L^{\frac{3}{2}}(\Omega)}+\gamma\|\rho\varphi\|_{L^{\frac{3}{2}}(\Omega)}\Big)^3+\|\nabla c\|^3_{L^2(\Omega)}\|\nabla\varphi\|^3_{L^6(\Omega)}+\gamma^3\|\Delta\varphi\|^3_{L^{\frac{3}{2}}(\Omega)}\\&\le\Big(\|\varphi\|^{\frac{1}{4}}_{L^\infty(\Omega)}\Big\|\nabla c\varphi^{\frac{3}{4}}\Big\|_{L^4(\Omega)}\|u\|_{L^{\frac{12}{5}}(\Omega)}+\gamma\|\rho_0\|^{\frac{1}{3}}_{L^1(\Omega)}\Big\|\rho^{\frac{2}{3}}\varphi\Big\|_{L^3(\Omega)}\Big)^3+C\|\varphi\|^3_{W^{2,\frac{3}{2}}(\mathbb{R}^2)}\end{aligned}$$

with some $C>0$. Combining the above estimates, it follows from (2.5) that there exists $C>0$ independent of δ and q satisfying

$$\int_\Omega|\nabla c|^4\varphi^3\le C\|\nabla c\|_{L^2(\Omega\cap B_\delta)}\left[\Big(\int_\Omega|\nabla c|^4\varphi^3\Big)^{\frac{3}{4}}\Big(\int_\Omega|u|^{\frac{12}{5}}\Big)^{\frac{5}{4}}+\int_\Omega\rho^2\varphi^3+\|\varphi\|^3_{W^{2,\frac{3}{2}}(\mathbb{R}^2)}\right].$$

Applying Young's inequality, along with (1.7) and (2.3), yields the desired bound (2.4). □

We next provide a localized $L\log L$ estimate for ρ.

Lemma 2.10. *Suppose that* (1.9) *holds. For $\delta>0$ and $q\in\overline{\Omega}$, let $\varphi(x)=\varphi_\delta(x-q)$ be as in* Lemma 2.8. *Then there exists $\delta_*>0$, independent of q, such that for all $\delta<\delta_*$, one can find $C=C(\delta,\gamma)>0$, independent of q, satisfying*

$$\sup_{t\in(0,T_{\max})}\int_\Omega(\rho\ln\rho(\cdot,t)+e^{-1})\varphi^3(\cdot)\le C$$

and

$$\int_t^{t+\tau}\int_\Omega\frac{|\nabla\rho|^2}{\rho}\varphi^3\le C\quad\text{for all }t\in(0,T_{\max}-\tau),$$

where $\tau:=\min\{1,\frac{1}{2}T_{\max}\}$.

Proof. We begin by testing the first equation in (1.1) against $(\ln\rho-1)\varphi^3$. Through standard calculations, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\rho\ln\rho\varphi^3-\frac{d}{dt}\int_\Omega\rho\varphi^3+k_D\int_\Omega\frac{|\nabla\rho|^2}{\rho}\varphi^3\\&\le-\int_\Omega\nabla\rho\cdot(S(x,\rho,c)\cdot\nabla c)\varphi^3-3\int_\Omega\ln\rho\nabla\rho\cdot\nabla\varphi\varphi^2-3\int_\Omega\rho\ln\rho(S(x,\rho,c)\cdot\nabla c)\cdot\nabla\varphi\varphi^2.\end{aligned}\tag{2.7}$$

Each of the three terms on the right-hand side can be controlled as follows.

Let K_1 and K_2 be as in Lemma 2.9. Using (1.8), Hölder's and Young's inequalities, along with Lemma 2.9 and (1.5), we compute the first term as

$$\begin{aligned}\left|\int_\Omega \nabla\rho\cdot(S(x,\rho,c)\cdot\nabla c)\varphi^3\right| &\le \|S_0\|_{C([0,\gamma])}\left(\int_\Omega \frac{|\nabla\rho|^2}{\rho}\varphi^3\right)^{\frac12}\left(\int_\Omega \rho^2\varphi^3\right)^{\frac14}\left(\int_\Omega |\nabla c|^4\varphi^3\right)^{\frac14}\\
&\le \frac{k_D}{8}\int_\Omega \frac{|\nabla\rho|^2}{\rho}\varphi^3 + \frac{1}{8K_1}\int_\Omega \rho^2\varphi^3 + K_3\int_\Omega |\nabla c|^4\varphi^3\\
&\le \frac{k_D}{4}\int_\Omega \frac{|\nabla\rho|^2}{\rho}\varphi^3 + K_2K_3\|\nabla c\|_{L^2(\Omega\cap B_\delta)}\int_\Omega \frac{|\nabla\rho|^2}{\rho}\varphi^3 + K_2K_3\left(\int_\Omega |u|^{\frac{12}{5}}\right)^5 + C\end{aligned}$$

with some $K_3 > 0$, independent of δ and q, and $C = C(\delta) > 0$. The second term in (2.7) can be estimated via Young's inequality, a trivial inequality $a|\ln a|^2 \le 16e^{-2}a^{3/2} + 4e^{-2}$ for $a \ge 0$, along with Lemma 2.8, (2.3), and (2.6), as

$$\begin{aligned}&\left|-3\int_\Omega \ln\rho\nabla\rho\cdot\nabla\varphi\varphi^2\right|\\
&\quad\le \frac{k_D}{8}\int_\Omega \frac{|\nabla\rho|^2}{\rho}\varphi^3 + \frac{18}{k_D}\int_\Omega \left(16e^{-2}\rho^{\frac32} + 4e^{-2}\right)\varphi|\nabla\varphi|^2\\
&\quad\le \frac{k_D}{8}\int_\Omega \frac{|\nabla\rho|^2}{\rho}\varphi^3 + \frac{18}{k_D}\cdot 16e^{-2}K^2\|\rho_0\|_{L^1(\Omega)}^{\frac12}\left(\int_\Omega \rho^2\varphi^3\right)^{\frac12} + \frac{72}{k_D}e^{-2}K^2\int_{\mathbb{R}^2}\varphi^2\\
&\quad\le \frac{k_D}{4}\int_\Omega \frac{|\nabla\rho|^2}{\rho}\varphi^3 + C\end{aligned}$$

with some $C = C(\delta) > 0$. A similar argument applies to the last term in (2.7). By (1.8) and Young's inequality,

$$3\int_\Omega \rho\ln\rho(S(x,\rho,c)\cdot\nabla c)\cdot\nabla\varphi\varphi^2 \le \frac14\int_\Omega |\nabla c|^4\varphi^3 + \frac34\left(3\|S_0\|_{C([0,\gamma])}\right)^{\frac43}\int_\Omega \rho^{\frac43}|\ln\rho|^{\frac43}\varphi^{\frac53}|\nabla\varphi|^{\frac43}.$$

Using a trivial inequality $a^{\frac43}|\ln a|^{\frac43} \le 16e^{-\frac43}a^{\frac32} + e^{-\frac43}$ for $a \ge 0$, Lemma 2.8 and (2.6), we note that

$$\begin{aligned}\int_\Omega \rho^{\frac43}|\ln\rho|^{\frac43}\varphi^{\frac53}|\nabla\varphi|^{\frac43} &\le K^{\frac43}\int_\Omega \left(16e^{-\frac43}\rho^{\frac32} + e^{-\frac43}\right)\varphi^{\frac32}\\
&\le 16K^{\frac43}\|\rho_0\|_{L^1(\Omega)}^{\frac12}\left(\int_\Omega \rho^2\varphi^3\right)^{\frac12} + C\end{aligned}$$

with some $C = C(\delta) > 0$. If we further use Lemma 2.9 and Young's inequality, then we arrive at

$$\left|3\int_\Omega \rho\ln\rho(S(x,\rho,c)\cdot\nabla c)\cdot\nabla\varphi\varphi^2\right| \le \left(\frac{K_2}{4}\|\nabla c\|_{L^2(\Omega\cap B_\delta)} + \frac{k_D}{8}\right)\int_\Omega \frac{|\nabla\rho|^2}{\rho}\varphi^3 + \frac{K_2}{4}\left(\int_\Omega |u|^{\frac{12}{5}}\right)^5 + C,$$

where $C = C(\delta) > 0$.

Combining above estimates gives that with some $C > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho\ln\rho\varphi^3 - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho\varphi^3 + \frac{3k_D}{8}\int_\Omega \frac{|\nabla\rho|^2}{\rho}\varphi^3$$
$$\le \left(K_2K_3 + \frac{K_2}{4}\right)\|\nabla c\|_{L^2(\Omega\cap B_\delta)}\int_\Omega \frac{|\nabla\rho|^2}{\rho}\varphi^3 + C\left(\int_\Omega |u|^{\frac{12}{5}}\right)^5 + C.$$

We now apply Lemma 2.7 to select $\delta_* > 0$ such that

$$\left(K_2K_3 + \frac{K_2}{4}\right)\sup_{t\in(0,T_{\max})}\|\nabla c(\cdot,t)\|_{L^2(\Omega\cap B_\delta)} \le \frac{k_D}{8} \quad \text{for } \delta < \delta_*,$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho\ln\rho\varphi^3 - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho\varphi^3 + \frac{k_D}{4}\int_\Omega \frac{|\nabla\rho|^2}{\rho}\varphi^3 \le C\left(\int_\Omega |u|^{\frac{12}{5}}\right)^5 + C. \tag{2.8}$$

We observe from Young's inequality, an inequality $a\ln a \le 2e^{-1}a^{\frac{3}{2}}$ for $a \ge 0$, (2.3), (2.6), and Young's inequality that there exists $C = C(\delta) > 0$ such that

$$\int_\Omega (\rho\ln\rho + e^{-1})\varphi^3 - \int_\Omega \rho\varphi^3 \le \int_\Omega 2e^{-1}\rho^{\frac{3}{2}}\varphi^{\frac{3}{2}} + e^{-1}\int_\Omega \varphi^3$$
$$\le 2e^{-1}\|\rho_0\|_{L^1(\Omega)}^{\frac{1}{2}}\left(\int_\Omega \rho^2\varphi^3\right)^{\frac{1}{2}} + e^{-1}\int_\Omega \varphi^3 \tag{2.9}$$
$$\le \frac{k_D}{8}\int_\Omega \frac{|\nabla\rho|^2}{\rho}\varphi^3 + C.$$

Combining (2.8) and (2.9), we find $C = C(\delta) > 0$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t) + \mathcal{F}(t) + \frac{k_D}{8}\int_\Omega \frac{|\nabla\rho|^2}{\rho}\varphi^3 \le C\left(\int_\Omega |u|^{\frac{12}{5}}\right)^5 + C,$$

where

$$\mathcal{F}(t) := \int_\Omega (\rho\ln\rho(\cdot,t) + e^{-1})\varphi^3(\cdot) - \int_\Omega \rho(\cdot,t)\varphi^3(\cdot).$$

Recalling the Gagliardo–Nirenberg inequality

$$\left(\int_\Omega |u|^{\frac{12}{5}}\right)^5 \le C\|u\|_{L^2(\Omega)}^{10}\|\nabla u\|_{L^2(\Omega)}^2 + C\|u\|_{L^2(\Omega)}^{12}, \tag{2.10}$$

and using Lemmas 2.4, 2.5, and (1.5), we can deduce the desired result. □

Corollary 2.11. *Suppose that* (1.9) *holds. There exists* $C = C(\gamma) > 0$ *such that*

$$\sup_{t\in(0,T_{\max})}\int_\Omega (\rho\ln\rho(\cdot,t) + e^{-1}) \le C,$$

as well as

$$\int_t^{t+\tau}\int_\Omega \frac{|\nabla\rho|^2}{\rho} \le C, \quad \int_t^{t+\tau}\int_\Omega |\nabla c|^4 \le C, \quad \text{and} \quad \int_t^{t+\tau}\int_\Omega \rho^2 \le C,$$

for all $t \in (0, T_{\max} - \tau)$*, where* $\tau = \min\{1, \frac{1}{2}T_{\max}\}$.

Proof. Note that every open covering $\bigcup_{q\in\overline{\Omega}} B_{\frac{\delta}{2}}(q)$, $\delta > 0$, of compact set $\overline{\Omega}$ has a finite subcovering. Since $a \ln a + e^{-1} \ge 0$ for $a \ge 0$ and $\varphi = 1$ in $B_{\frac{\delta}{2}}(q)$, the first and second assertions are a direct consequence of Lemma 2.10. By a similar reason, we can obtain the third assertion from (1.7), (2.4), (2.10), and Lemma 2.5. The last assertion results from the second assertion, (1.5), and the following interpolation inequality

$$\|\rho\|^2_{L^2(\Omega)} \le C\|\rho\|_{L^1(\Omega)} \int_\Omega \frac{|\nabla\rho|^2}{\rho} + C\|\rho\|^2_{L^1(\Omega)}.$$

□

We next collect additional regularity properties for the solution constructed in Proposition 1.1. As a preliminary, we recall the following interpolation-type inequality (see, e.g., [27, Lem. 3.4]).

Lemma 2.12. *There exists* $C > 0$ *such that for any* $p \ge 1$, $s > 1$, $\varepsilon > 0$, *and any nonnegative function* $f \in C^1(\overline{\Omega})$, *one has*

$$\int_\Omega f^{p+1} \le C\frac{(p+1)^2}{\ln s}\int_\Omega (f\ln f + e^{-1})\int_\Omega f^{p-2}|\nabla f|^2 + (4C)^{1+\varepsilon/2}\left(\int_\Omega f^{\frac{\varepsilon}{2}\frac{p+1}{1+\varepsilon}}\right)^{\frac{2(1+\varepsilon)}{\varepsilon}} + 6s^{p+1}|\Omega|.$$

We are now in a position to derive further bounds for the solution.

Lemma 2.13. *Suppose that* (1.9) *holds. There exists* $C = C(\gamma) > 0$ *such that*

$$\sup_{t\in(0,T_{\max})} \|\rho(\cdot,t)\|_{L^2(\Omega)} \le C.$$

Proof. Testing the first equation in (1.1) with ρ, integrating by parts, and using (1.8) and Hölder's inequality, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^2 + k_D\int_\Omega |\nabla\rho|^2 \le -\int_\Omega \rho\nabla\rho\cdot(S(x,\rho,c)\cdot\nabla c) \le \|S_0\|_{C([0,\gamma])}\|\rho\|_{L^4(\Omega)}\|\nabla\rho\|_{L^2(\Omega)}\|\nabla c\|_{L^4(\Omega)}.$$

By the Gagliardo–Nirenberg inequality $\|\rho\|_{L^4(\Omega)} \le C(\|\nabla\rho\|^{\frac{1}{2}}_{L^2(\Omega)}\|\rho\|^{\frac{1}{2}}_{L^2(\Omega)} + \|\rho\|_{L^1(\Omega)})$ together with (1.5), we find $C > 0$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^2 + k_D\int_\Omega |\nabla\rho|^2 \le C\Big(\|\rho\|^{1/2}_{L^2(\Omega)}\|\nabla\rho\|^{3/2}_{L^2(\Omega)} + \|\nabla\rho\|_{L^2(\Omega)}\Big)\|\nabla c\|_{L^4(\Omega)}.$$

Young's inequality then implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) \le Cg(t)y(t) + C$$

with some $C > 0$, where $y(t) := \|\rho(\cdot,t)\|^2_{L^2(\Omega)} + 1$ and $g(t) := \|\nabla c(\cdot,t)\|^4_{L^4(\Omega)}$. Note from Corollary 2.11 that there exists $C > 0$ such that

$$\int_t^{t+\tau} y(s)\mathrm{d}s \le C \quad \text{and} \quad \int_t^{t+\tau} g(s)\mathrm{d}s \le C \quad \text{for all } t \in (0, T_{\max} - \tau),$$

where $\tau := \min\{1, \frac{1}{2}T_{\max}\}$. In case $T_{\max} \le 2$, we apply an ODE-comparison argument to show

$$y(t) \le y(0)e^{C\int_0^t g(s)\mathrm{d}s} + C\int_0^t e^{C\int_\sigma^t g(s)\mathrm{d}s}\,\mathrm{d}\sigma \quad \text{for all } t \in (0, T_{\max}).$$

Then, along with the fact that

$$\int_0^t g(s)\mathrm{d}s \le \int_0^\tau g(s)\mathrm{d}s + \int_{t-\tau}^t g(s)\mathrm{d}s \le 2C \quad \text{for all } t \in (0, T_{\max}),$$

we can see that

$$y(t) \le (y(0) + C)e^{2C^2} \quad \text{for all } t \in \big(0, T_{\max}\big). \tag{2.11}$$

In case $T_{\max} > 2$, where $\tau = 1$, note that for any unit interval $[t, t+1] \subset \big(0, T_{\max}\big)$, there exists $t_0 \in [t, t+1]$ satisfying

$$y(t_0) \le C.$$

Thus, again by an ODE-comparison argument, we have

$$y(t) \le 2Ce^{C^2} \quad \text{for all } t \in \big(0, T_{\max}\big).$$

Combining it with (2.11), we can deduce the desired result. □

Lemma 2.14. *Suppose that* (1.9) *holds. Let* $\alpha \in \left(\frac{1}{2}, 1\right)$ *be as in* (1.4). *Then, there exists* $C = C(\gamma) > 0$ *such that*

$$\sup_{t\in(0,T_{\max})} \|A^\alpha u(\cdot, t)\|_{L^2(\Omega)} \le C, \tag{2.12}$$

and

$$\sup_{t\in(0,T_{\max})} \|u(\cdot, t)\|_{L^\infty(\Omega)} \le C. \tag{2.13}$$

Proof. We first project the equation for u via the Helmholtz operator $\mathcal{P}$ and then test it against $Au = \mathcal{P}\Delta u$, yielding

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^2 + \int_\Omega |Au|^2 \le \int_\Omega |(\mathcal{P}(u\cdot\nabla)u)||Au| + \int_\Omega |\mathcal{P}(\rho\nabla\Phi)||Au|. \tag{2.14}$$

Here, we have used that $\|\nabla u\|^2_{L^2(\Omega)} = (u, Au)_{L^2(\Omega)}$ in $L^2_\sigma(\Omega)$. It holds from the Gagliardo–Nirenberg inequality and the elliptic regularity for the Stokes operator that

$$\begin{aligned}\|u\|^2_{L^\infty(\Omega)} &\le \|u\|_{W^{2,2}}\|u\|_{L^2(\Omega)}\\ &\le \|Au\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}.\end{aligned}$$

Using this estimate, and applying Young's inequality and $L^2(\Omega)$-boundedness of $\mathcal{P}$, we estimate the first term on the right-hand side of (2.14) as

$$\begin{aligned}\int_\Omega |\mathcal{P}((u\cdot\nabla)u)||Au| &\le \int_\Omega |(u\cdot\nabla)u|^2 + \frac{1}{4}\int_\Omega |Au|^2\\ &\le \|u\|^2_{L^\infty(\Omega)}\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{4}\int_\Omega |Au|^2\\ &\le C\|u\|^2_{L^2(\Omega)}\|\nabla u\|^4_{L^2(\Omega)} + \frac{1}{2}\int_\Omega |Au|^2\end{aligned}$$

with some $C > 0$. Using young's inequality and $L^2(\Omega)$-boundedness again, the last term in (2.14) is estimated as

$$\int_\Omega |\mathcal{P}(\rho\nabla\Phi)||Au| \le \|\nabla\Phi\|^2_{L^\infty(\Omega)}\|\rho\|^2_{L^2(\Omega)} + \frac{1}{4}\int_\Omega |Au|^2.$$

According to (2.1) and Lemma 2.13, we infer that $h(t) := \|\nabla u(\cdot,t)\|^2_{L^2(\Omega)}$ satisfies

$$h'(t) \le C(h^2(t) + 1)$$

with some $C > 0$. In view of (2.2), Grönwall-type reasoning, as in the proof of Lemma 2.13, again guarantees the existence of $C > 0$ such that

$$\sup_{t\in(0,T_{\max})} \|\nabla u(\cdot,t)\|_{L^2(\Omega)} \le C. \tag{2.15}$$

Next, we use the smoothing estimates for the Stokes semigroup (see, e.g., [19, Lem. 2.3]) and Hölder's inequality to obtain $\mu > 0$ and $C > 0$ such that

$$\begin{aligned}\left\|A^\alpha u(\cdot,t)\right\|_{L^2(\Omega)} \le \left\|A^\alpha u_0\right\|_{L^2(\Omega)} &+ C\int_0^t (t-s)^\alpha e^{-\mu t}\|u\|_{L^\infty(\Omega)}\|\nabla u\|_{L^2(\Omega)}\mathrm{d}s \\ &+ C\int_0^t (t-s)^\alpha e^{-\mu t}\|\nabla\Phi\|_{L^\infty(\Omega)}\|\rho\|_{L^2(\Omega)}\mathrm{d}s.\end{aligned}$$

Pick $q \in (2, \frac{1}{1-\alpha}]$. From the Gagliardo–Nirenberg inequality and the well-known embedding $D(A^\alpha) \hookrightarrow W^{1,q}(\Omega)$, we find $C > 0$ satisfying

$$\|u\|_{L^\infty(\Omega)} \le C\|u\|^{1-\theta}_{L^2(\Omega)}\left\|A^\alpha u\right\|^\theta_{L^2(\Omega)}, \qquad \theta = \frac{1}{2\left(1-\frac{1}{q}\right)} \in (0,1). \tag{2.16}$$

It then follows by (2.1), Lemma 2.13, and (2.15) that with some $C > 0$

$$\sup_{t\in(0,T_{\max})} \left\|A^\alpha u(\cdot,t)\right\|_{L^2(\Omega)} \le C + C\left(\sup_{t\in(0,T_{\max})} \left\|A^\alpha u(\cdot,t)\right\|_{L^2(\Omega)}\right)^\theta.$$

Therefore, due to Young's inequality, we obtain (2.12). The bound (2.13) is immediately obtained by combining (2.1) and (2.12) with (2.16). □

Lemma 2.15. *Suppose that* (1.9) *holds. There exists* $C = C(\gamma) > 0$ *such that*

$$\sup_{t\in(0,T_{\max})} \|\rho(\cdot,t)\|_{L^3(\Omega)} \le C.$$

Proof. Testing (1.1) with ρ^2 and employing Hölder's inequality and (1.8), we have

$$\begin{aligned}\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^3 + \frac{8k_D}{9}\int_\Omega \left|\nabla\rho^{\frac{3}{2}}\right|^2 &\le -\frac{4}{3}\int_\Omega \rho^{\frac{3}{2}}\nabla\rho^{\frac{3}{2}}\cdot S(x,\rho,c)\cdot\nabla c \\ &\le \frac{4}{3}\left\|S_0\right\|_{C([0,\gamma])}\|\rho\|^{\frac{3}{2}}_{L^4(\Omega)}\left\|\nabla\rho^{\frac{3}{2}}\right\|_{L^2(\Omega)}\|c\|_{W^{1,8}(\Omega)}.\end{aligned}$$

By the Gagliardo–Nirenberg interpolation $\|c\|_{W^{1,8}(\Omega)} \le C\|c\|^{\frac{1}{2}}_{W^{2,4}}\|c\|^{\frac{1}{2}}_{L^\infty(\Omega)}$ with (1.6) and the standard elliptic regularity theory, we can find $C > 0$ satisfying

$$\|c\|_{W^{1,8}(\Omega)} \le C\left(\|\rho\|^{\frac{1}{2}}_{L^4(\Omega)} + \|u\cdot\nabla c\|^{\frac{1}{2}}_{L^4(\Omega)} + \|\gamma - c\|^{\frac{1}{2}}_{W^{\frac{3}{4},4}(\partial\Omega)} + 1\right),$$

which infers via the trace embedding $W^{1,4}(\Omega) \hookrightarrow W^{\frac{3}{4},4}(\partial\Omega)$, (2.13), and Young's inequality that with some $C > 0$

$$\|c\|_{W^{1,8}(\Omega)} \le C\Big(\|\rho\|_{L^4(\Omega)}^{\frac{1}{2}} + 1\Big).$$

From Lemma 2.12 with $(f, p, \varepsilon) = (\rho, 3, 1)$ and Corollary 2.11, we obtain $C > 0$, independent of $s > 1$, such that

$$\int_\Omega \rho^4 \le \frac{C}{\log s}\int_\Omega \left|\nabla \rho^{\frac{3}{2}}\right|^2 + (4C)^{\frac{3}{2}}\left(\int_\Omega \rho_0\right)^4 + 6s^4|\Omega|.$$

Combining the above estimates, taking sufficiently large s, and using Young's inequality, we can find $C > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^3 + k_D\int_\Omega \left|\nabla\rho^{\frac{3}{2}}\right|^2 \le C.$$

This implies, by the Gagliardo–Nirenberg type inequality

$$\|f\|_{L^3(\Omega)}^3 \le \left\|\nabla f^{\frac{3}{2}}\right\|_{L^2(\Omega)}^2 + C\|f\|_{L^1(\Omega)}^3 \quad \text{for all} \quad f \in C^1(\overline{\Omega}),$$

and (1.5), that with some $C > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^3 + k_D\int_\Omega \rho^3 \le C.$$

Hence, the desired result follows. □

Proof of Theorem 1.1. We first establish the uniform-in-time bound

$$\sup_{t\in(0,T_{\max})} \|\nabla c(\cdot,t)\|_{L^\infty(\Omega)} \le C. \tag{2.17}$$

To this end, we apply the elliptic regularity to the equation for c in (1.1) and use (1.6). This yields with some $C > 0$,

$$\|c\|_{W^{2,3}(\Omega)} \le C\Big(\|\rho\|_{L^3(\Omega)} + \|u\cdot\nabla c\|_{L^3(\Omega)} + \|\gamma - c\|_{W^{\frac{2}{3},3}(\partial\Omega)} + 1\Big).$$

By the trace inequality

$$\|\gamma - c\|_{W^{\frac{2}{3},3}(\partial\Omega)} \le C\|\gamma - c\|_{W^{1,3}(\Omega)},$$

together with (1.6), (2.13), and Lemma 2.15, we infer that

$$\|c\|_{W^{2,3}(\Omega)} \le C\big(\|\nabla c\|_{L^3(\Omega)} + 1\big).$$

Moreover, the Gaglairdo–Nirenberg inequality

$$\|\nabla c\|_{L^3(\Omega)} \le C\|\nabla c\|_{L^2(\Omega)}^{\frac{3}{4}}\|c\|_{W^{2,3}(\Omega)}^{\frac{1}{4}}$$

combined with Young's inequality and (1.7) ensures the uniform-in-time boundedness of $\|c\|_{W^{2,3}(\Omega)}$. Since $W^{2,3}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, (2.17) follows.

Finally, for $p > 1$, we compute L^p-estimate for ρ. Applying Hölder's inequality, we observe

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^p + \frac{4k_D(p-1)}{p^2}\int_\Omega \left|\nabla\rho^{\frac{p}{2}}\right|^2 &\le -\frac{2(p-1)}{p}\int_\Omega \rho^{\frac{p}{2}}\nabla\rho^{\frac{p}{2}}\cdot\big(S(x,\rho,c)\cdot\nabla c\big)\\ &\le \frac{2k_D(p-1)}{p^2}\int_\Omega \left|\nabla\rho^{\frac{p}{2}}\right|^2 + \frac{(p-1)^2}{k_D p}\|S(x,\rho,c)\cdot\nabla c\|_{L^\infty(\Omega)}^2\int_\Omega \rho^p.\end{aligned}$$

Therefore, in view of (1.8) and (2.17), by applying a Moser-type iteration, we can deduce

$$\sup_{t\in(0,T_{\max})} \|\rho(\cdot,t)\|_{L^\infty(\Omega)} \le C$$

with some $C > 0$. Hence, $T_{\max} = \infty$ by Proposition 1.1, and we complete the proof. □

3 Proof of Theorem 1.2

Throughout this section, we consider the system (1.1) with $\kappa = 0$ in a smooth bounded domain $\Omega \subset \mathbb{R}^3$.

Lemma 3.1. *Suppose that* (1.10) *holds with some* $M > 0$. *Then, there exists* $K_4 = K_4(\gamma, M) > 0$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^2(\cdot,t) + \int_\Omega D(\rho)|\nabla\rho|^2(\cdot,t) \le K_4 + \frac{\|S_0\|^2_{C([0,\gamma])}}{M}\int_\Omega \rho^2|\nabla c|^2(\cdot,t) \quad \text{for all } t\in\big(0,T_{\max}\big). \tag{3.1}$$

Proof. Multiplying the first equation in (1.1) by ρ and integrating over Ω yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^2 + 2\int_\Omega D(\rho)|\nabla\rho|^2 = -2\int_\Omega \rho S(x,\rho,c)\nabla c\cdot\nabla\rho.$$

By Young's inequality and (1.8), this implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^2 + \int_\Omega D(\rho)|\nabla\rho|^2 \le \|S_0\|^2_{C([0,\gamma])}\int_\Omega \frac{\rho^2}{D(\rho)}|\nabla c|^2.$$

Owing to (1.10), we can choose $L = L(M) > 0$ such that $D(\xi) \ge M$ whenever $\xi > L$. Set

$$k_D := \min_{\xi\ge 0} D(\xi) > 0. \tag{3.2}$$

Splitting the last integral over $\{\rho \le L\}$ and $\{\rho > L\}$ and using (1.7) for the first part, we obtain

$$\begin{aligned}\int_\Omega \frac{\rho^2}{D(\rho)}|\nabla c|^2 &\le \frac{L^2}{k_D}\int_{\{\rho\le L\}} |\nabla c|^2 + \frac{1}{M}\int_{\{\rho>L\}} \rho^2|\nabla c|^2\\ &\le \frac{L^2}{k_D}\gamma^2|\partial\Omega| + \frac{1}{M}\int_\Omega \rho^2|\nabla c|^2.\end{aligned}$$

Combining these estimates proves (3.1). □

The last term in (3.1) can be estimated by means of the diffusion contribution $\int_\Omega D(\rho)|\nabla\rho|^2$ and the fluid term $\int_\Omega |u|^6$, as stated below.

Lemma 3.2. *Suppose that* (1.10) *holds with some* $M > 0$. *Given any* $\eta > 0$, *there exist* $K_5 = K_5(\gamma,\eta) > 0$ *and* $K_6 = K_6(\gamma,\eta,M) > 0$ *such that*

$$\int_\Omega \rho^2|\nabla c|^2(\cdot,t) + \int_\Omega \rho^3 c^2(\cdot,t) \le \left(\eta + \frac{4\gamma^2}{M}\right)\int_\Omega D(\rho)|\nabla\rho|^2(\cdot,t) + K_5\int_\Omega |u(\cdot,t)|^6 + K_6 \quad \text{for all } t\in\big(0,T_{\max}\big).$$

Proof. Testing the equation for c in (1.1) with $\rho^2 c$ gives

$$\int_\Omega \rho^2|\nabla c|^2 + \int_\Omega \rho^3c^2 = \int_{\partial\Omega} \rho^2 c\frac{\partial c}{\partial\nu} - 2\int_\Omega \rho c\nabla\rho\cdot\nabla c - \int_\Omega \rho^2 c(u\cdot\nabla c). \tag{3.3}$$

For the boundary term, the trace embedding $W^{r+\frac{1}{2},2}(\Omega) \hookrightarrow L^2(\partial\Omega)$ for $r > 0$ (see [37]) along with $c \le \gamma$ ensures the existence of $C > 0$ such that

$$\begin{aligned}\int_{\partial\Omega} \rho^2 c \frac{\partial c}{\partial \nu} &= \int_{\partial\Omega} \rho^2 c(\gamma - c) \\ &\le \gamma^2 \|\rho\|^2_{L^2(\partial\Omega)} \\ &\le C\gamma^2 \|\rho\|^2_{W^{r+\frac{1}{2},2}(\Omega)}.\end{aligned}$$

Fix $r \in (0, \frac{1}{2})$. By the fractional version of Gagliardo–Nirenberg inequality [37, Lem. 2.5], there is $C > 0$ such that

$$\|\rho\|^2_{W^{r+\frac{1}{2},2}(\Omega)} \le C\|\nabla\rho\|^{2\theta}_{L^2(\Omega)} \|\rho\|^{2(1-\theta)}_{L^1(\Omega)} + C\|\rho\|^2_{L^1(\Omega)},$$

where $\theta = \frac{2r+4}{5} \in \left(r + \frac{1}{2}, 1\right)$. Let $L = L(M) > 0$ be as in the proof of Lemma 3.1. Given $\eta > 0$, combining the above two estimates and using Young's inequality and (1.5), we have

$$\begin{aligned}\int_{\partial\Omega} \rho^2 c \frac{\partial c}{\partial \nu} &\le \frac{\eta k_D}{6} \int_\Omega |\nabla\rho|^2 + C \\ &\le \frac{\eta}{6} \int_\Omega D(\rho)|\nabla\rho|^2 + C\end{aligned} \tag{3.4}$$

for some $C = C(\gamma, \eta) > 0$. We now split the domain Ω into $\{\rho > L\}$ and $\{\rho \le L\}$ and apply Young's inequality along with (1.6) and (1.7) to obtain

$$\begin{aligned}&-2\int_\Omega \rho c \nabla\rho \cdot \nabla c \\ &\quad = -2\int_{\{\rho > L\}} \rho c \nabla\rho \cdot \nabla c - 2\int_{\{\rho \le L\}} \rho c \nabla\rho \cdot \nabla c \\ &\quad \le \frac{2\gamma}{\sqrt{M}} \int_{\{\rho > L\}} \sqrt{D(\rho)}\rho|\nabla\rho||\nabla c| + \frac{2\gamma L}{\sqrt{k_D}} \int_{\{\rho \le L\}} \sqrt{D(\rho)}|\nabla\rho||\nabla c| \\ &\quad \le \frac{1}{2}\int_\Omega \rho^2|\nabla c|^2 + \frac{2\gamma^2}{M}\int_\Omega D(\rho)|\nabla\rho|^2 + \frac{\eta}{6}\int_\Omega D(\rho)|\nabla\rho|^2 + \frac{6\gamma^4 L^2|\partial\Omega|}{\eta k_D}.\end{aligned} \tag{3.5}$$

Finally, for the last term in (3.3), an integration by parts together with Young's inequality gives

$$-\int_\Omega \rho^2 c(u \cdot \nabla c) = 2\int_\Omega \rho c^2(u \cdot \nabla\rho) \le \frac{1}{2}\int_\Omega \rho^3 c^2 + \frac{\eta}{6}\int_\Omega D(\rho)|\nabla\rho|^2 + C\int_\Omega |u|^6 \tag{3.6}$$

with some $C = C(\gamma, \eta) > 0$. Combining (3.4)–(3.6) with (3.3) yields the desired result. □

The following bounds for u can be found in ref. [23, Lem. 4.1].

Lemma 3.3. *Suppose that* (1.10) *holds with some* $M > 0$. *For any* $\delta > 0$ *and* $\lambda > 0$, *there exists* $C > 0$ *such that*

$$\int_0^T \int_\Omega e^{\lambda t}\left(|\nabla u|^3 + |u|^6\right) \le \delta \int_0^T \int_\Omega e^{\lambda t}|\nabla\rho|^2 + Ce^{\lambda T} \quad \text{for all } T \in (0, T_{\max}).$$

Now, we obtain a uniform $L^2(\Omega)$-bound for ρ in the following lemma.

Lemma 3.4. *For each $\gamma > 0$, one can choose $M = M(\gamma) > 0$ such that whenever D satisfies* (1.10) *with this M, there exists $C > 0$ fulfilling*

$$\sup_{t\in(0,T_{\max})} \|\rho(\cdot,t)\|_{L^2(\Omega)} \le C \tag{3.7}$$

Proof. We fix $\gamma > 0$ and choose $M = M(\gamma) > 0$ sufficiently large to ensure

$$M > 2\gamma \|S_0\|_{C([0,\gamma])}. \tag{3.8}$$

Here, we may assume that $\|S_0\|_{C([0,\gamma])} > 0$. Let K_4 be as in Lemma 3.1. Since $M^2 > 4\gamma^2\|S_0\|^2_{C([0,\gamma])}$, we can pick $\eta \in (0,1)$ small to satisfy

$$\eta < \frac{M}{3\|S_0\|^2_{C([0,\gamma])}} \left(1 - \frac{4\gamma^2\|S_0\|^2_{C([0,\gamma])}}{M^2}\right). \tag{3.9}$$

For this choice of η, let K_5, K_6 be from Lemma 3.2, and apply Lemma 3.3 with $\lambda = 1$ and $\delta = \frac{\eta}{K_5}$. This yields $K_7 = K_7(\gamma,\eta) > 0$ such that

$$K_5 \int_0^T \int_\Omega e^t |u|^6 \le \eta \int_0^T \int_\Omega e^t D(\rho)|\nabla\rho|^2 + K_7 e^T \quad \text{for all } T \in (0, T_{\max}), \tag{3.10}$$

Next, Ehrling's inequality combined with the compact and continuous embeddings $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega) \hookrightarrow L^1(\Omega)$ and (1.5) gives $K_8 = K_8(\gamma,\eta) > 0$ such that

$$\int_\Omega \rho^2 \le \frac{\eta\|S_0\|^2_{C([0,\gamma])}}{M} \int_\Omega D(\rho)|\nabla\rho|^2 + K_8 \tag{3.11}$$

Combining (3.3) and (3.11) with (3.1) gives

$$\begin{aligned} \frac{d}{dt}\int_\Omega \rho^2 + \int_\Omega \rho^2 + \int_\Omega D(\rho)|\nabla\rho|^2 \le{}& \frac{\|S_0\|^2_{C([0,\gamma])}}{M}\left(2\eta + \frac{4\gamma^2}{M}\right)\int_\Omega D(\rho)|\nabla\rho|^2 \\ &+ \frac{K_5\|S_0\|^2_{C([0,\gamma])}}{M}\int_\Omega |u|^6 + K_9, \end{aligned} \tag{3.12}$$

where $K_9 = K_4 + K_8 + \frac{K_6\|S_0\|^2_{C([0,\gamma])}}{M} > 0$. Multiplying (3.12) by e^t, integrating over $(0,T)$, and using (3.10), we deduce

$$e^T \int_\Omega \rho^2(\cdot,T) + \left(1 - \frac{3\eta\|S_0\|^2_{C([0,\gamma])}}{M} + \frac{4\gamma^2\|S_0\|^2_{C([0,\gamma])}}{M^2}\right)\int_0^T\int_\Omega e^t D(\rho)|\nabla\rho|^2 \le \left(\frac{K_7\|S_0\|^2_{C([0,\gamma])}}{M} + K_9\right)e^T + \int_\Omega \rho_0^2$$

for all $T \in (0,T_{\max})$. Finally, thanks to (3.9), this in particular implies that with some $C > 0$

$$\int_\Omega \rho^2(\cdot,T) \le C \quad \text{for all } T \in (0,T_{\max}).$$

□

As a preparation for deriving an L^p-estimate for ρ, we establish boundedness properties for u and ∇c.

Lemma 3.5. *For each $\gamma > 0$, there exists $M = M(\gamma) > 0$ such that whenever D satisfies* (1.10) *with this M, there exists $C > 0$ such that*

$$\sup_{t\in(0,T_{\max})} \|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \tag{3.13}$$

and

$$\int_\Omega |\nabla c(\cdot,t)|^6 \le C \quad \text{for all } t \in \big(0, T_{\max}\big). \tag{3.14}$$

Proof. The variation-of-constants formula for u reads

$$u(\cdot,t) = e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A}\mathcal{P}(\rho(\tau)\nabla\Phi)\mathrm{d}\tau.$$

By the continuous embedding $D(A^\alpha) \hookrightarrow L^\infty(\Omega)^3$ for $\alpha \in \left(\frac{3}{4}, 1\right)$, it follows that with some $\mu > 0$ and $C > 0$

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C\big\|A^\alpha u_0\big\|_{L^2(\Omega)} + C\int_0^t (t-\tau)^{-\alpha}e^{-\mu\tau}\|\rho(\cdot,\tau)\|_{L^2(\Omega)}\mathrm{d}\tau.$$

This entails (3.13) by (3.7). Next, the Sobolev embedding $W^{2,2}(\Omega) \hookrightarrow W^{1,6}(\Omega)$ and the elliptic regularity for c together with (1.8) yield

$$\|\nabla c\|_{L^6(\Omega)} \le C\|c\|_{W^{2,2}(\Omega)} \le C\Big(\|\rho\|_{L^2(\Omega)} + \|u\nabla c\|_{L^2(\Omega)} + \|\gamma - c\|_{W^{\frac{1}{2},2}(\partial\Omega)} + 1\Big)$$

with some $C > 0$. The trace embedding $W^{1,2}(\Omega) \hookrightarrow W^{\frac{1}{2},2}(\partial\Omega)$, (3.7), (3.13), and (1.7) imply (3.14). □

It turns out that the regularity properties of u and c established in Lemma 3.5 are sufficient to derive $L^p(\Omega)$-bounds for ρ for all $p \in (2, \infty)$.

Lemma 3.6. *For each $\gamma > 0$ and $p > 2$, there exists $M = M(\gamma) > 0$ such that whenever D satisfies* (1.10) *with this M, there exists $C > 0$ such that*

$$\sup_{t\in(0,T_{\max})} \|\rho(\cdot,t)\|_{L^p(\Omega)} \le C. \tag{3.15}$$

Proof. Let k_D be the positive constant in (3.2). Testing the equation for ρ in (1.1) by $p\rho^{p-1}$, adding $\int_\Omega \rho^p$ to both sides, and using (1.8) and Young's inequality, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^p + \int_\Omega \rho^p + \frac{4(p-1)}{p}\int_\Omega D(\rho)\Big|\nabla\rho^{\frac{p}{2}}\Big|^2
&\le 2(p-1)\big\|S_0\big\|_{C([0,\gamma])}\int_\Omega \rho^{\frac{p}{2}}\Big|\nabla\rho^{\frac{p}{2}}\Big||\nabla c| + \int_\Omega \rho^p \\
&\le \frac{(p-1)k_D}{p}\int_\Omega \Big|\nabla\rho^{\frac{p}{2}}\Big|^2 + C\int_\Omega \rho^p\Big(|\nabla c|^2 + 1\Big) \\
&\le \frac{p-1}{p}\int_\Omega D(\rho)\Big|\nabla\rho^{\frac{p}{2}}\Big|^2 + C\int_\Omega \rho^p\Big(|\nabla c|^2 + 1\Big)
\end{aligned} \tag{3.16}$$

with some $C = C(p, \gamma) > 0$. By Hölder's inequality and (3.14), we find $C = C(\gamma, M) > 0$ such that

$$\int_\Omega \rho^p\left(|\nabla c|^2 + 1\right) \le \left\||\nabla c|^2 + 1\right\|_{L^3(\Omega)} \|\rho^p\|_{L^{\frac{3}{2}}(\Omega)} \le C\left\|\rho^{\frac{p}{2}}\right\|^2_{L^3(\Omega)}. \tag{3.17}$$

Applying the Gagliardo–Nirenberg inequality, we have

$$\left\|\rho^{\frac{p}{2}}\right\|^2_{L^3(\Omega)} \le C\left\|\nabla\rho^{\frac{p}{2}}\right\|^{2\theta}_{L^2(\Omega)}\left\|\rho^{\frac{p}{2}}\right\|^{2(1-\theta)}_{L^{\frac{2}{p}}(\Omega)} + C\left\|\rho^{\frac{p}{2}}\right\|^2_{L^{\frac{2}{p}}(\Omega)},$$

where $\theta = \frac{3p-2}{3p-1} \in (0, 1)$, and thus by Young's inequality and (1.5), we infer

$$\left\|\rho^{\frac{p}{2}}\right\|^2_{L^3(\Omega)} \le \frac{p-1}{p}\int_\Omega D(\rho)\left|\nabla\rho^{\frac{p}{2}}\right|^2 + C \tag{3.18}$$

with some $C = C(p) > 0$. Substituting (3.17) and (3.18) into (3.16), we obtain the following inequality for $y(t) = \int_\Omega \rho^p(\cdot, t)$:

$$y'(t) + y(t) \le C,$$

which entails (3.15). □

Proof of Theorem 1.2. For any $\gamma > 0$, let $M = M(\gamma) > 0$ be chosen according to (3.8), and suppose that (1.10) holds with this M. To begin, we establish that $\nabla c \in L^\infty(0, T_{\max};\ L^\infty(\Omega))$. Fix $t \in (0, T_{\max})$ and $p > 3$. As in the proof of (3.14), elliptic regularity for c, along with (1.6), ensures the existence of $C > 0$ such that

$$\|c\|_{W^{2,p}(\Omega)} \le C\left(\|\rho\|_{L^p(\Omega)} + \|u\nabla c\|_{L^p(\Omega)} + \|\gamma - c\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} + 1\right).$$

Thus, thanks to the trace embedding $W^{1,p}(\Omega) \hookrightarrow W^{1-\frac{1}{p},p}(\partial\Omega)$ as well as the bounds (3.13) and (3.15), we deduce that

$$\|c\|_{W^{2,p}(\Omega)} \le C\left(1 + \|\nabla c\|_{L^p(\Omega)}\right)$$

for some $C > 0$. From Ehrling's inequality along with the compact and continuous embeddings $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega) \hookrightarrow L^2(\Omega)$, we note that for each $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\|\nabla c\|_{L^p(\Omega)} \le \varepsilon\|c\|_{W^{2,p}(\Omega)} + C_\varepsilon\|\nabla c\|_{L^2(\Omega)}.$$

Therefore, choosing ε sufficiently small, and using (1.7) and the Sobolev embedding $W^{2,p}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, we can find $C > 0$ such that

$$\sup_{t\in(0,T_{\max})} \|\nabla c(\cdot, t)\|_{L^\infty(\Omega)} \le C.$$

Finally, with the uniform $L^\infty(\Omega)$-bound on ∇c established, a standard Moser iteration argument applied to the equation for ρ completes the proof of Theorem 1.2. □

4 Proof of Theorem 1.3

Throughout this section, we consider $\Omega = B_R \subset \mathbb{R}^n$, $R > 0$, with $n = 2, 3$, and let $S \equiv \mathbb{I}_{n\times n}$. The initial data ρ_0 in (1.4) are taken to be radially symmetric. We consider the following fluid-free system for $(\rho, c)(r, t) = (\rho, c)(|x|, t)$:

$$\begin{cases} \partial_t\rho = r^{1-n}(r^{n-1}D(\rho)\rho_r)_r + r^{1-n}\left(r^{n-1}\rho c_r\right)_r, & r \in (0, R),\ t \in (0, T_{\max}), \\ 0 = r^{1-n}\left(r^{n-1}c_r\right)_r - \rho c, & r \in (0, R),\ t \in (0, T_{\max}), \\ (D(\rho)\rho_r + \rho c_r)|_{r=R} = 0, \quad (c_r + c)|_{r=R} = \gamma, & t \in (0, T_{\max}), \\ \rho|_{t=0} = \rho_0, & r \in (0, R). \end{cases} \tag{4.1}$$

Note that $\rho_r|_{r=0} = c_r|_{r=0} = 0$ for $t < T_{\max}$. To detect blow-up of ρ, we analyze the time evolution of a suitable functional $\phi(t)$ (see Lemma 4.2), which is defined in terms of the mass accumulation function

$$w(s,t) := \frac{1}{\omega_n} \int_{B_{s^{1/n}}} \rho(x,t)\mathrm{d}x = \int_0^{s^{\frac{1}{n}}} r^{n-1}\rho(r,t)\mathrm{d}r.$$

The equation for ρ in (4.1) with its no-flux boundary condition then transforms to the following Dirichlet problem for w:

$$\begin{cases} w_t = n^2 s^{2-\frac{2}{n}} D(nw_s)w_{ss} + ns^{1-\frac{1}{n}} w_s c_s\left(s^{\frac{1}{n}}, t\right), & s \in (0, R^n),\ t \in \left(0, T_{\max}\right), \\ w|_{s=0} = 0, \quad w|_{s=R^n} = \omega_n^{-1}\|\rho_0\|_{L^1(\Omega)}, & t \in \left(0, T_{\max}\right), \\ w|_{t=0} = \int_0^{s^{\frac{1}{n}}} s^{n-1}\rho_0(s)\mathrm{d}s, & s \in (0, R^n). \end{cases} \tag{4.2}$$

The next lemma ensures that c maintains a positive lower bound at the boundary $r = R$.

Lemma 4.1. *If the initial data ρ_0 is such that* (1.4) *holds, then*

$$c(R,t) \geq \frac{\gamma}{1 + R^{-n+1}\omega_n^{-1}\|\rho_0\|_{L^1(\Omega)}} \quad \text{for all } t \in \left(0, T_{\max}\right). \tag{4.3}$$

Proof. Since $\rho \geq 0$ and $c \geq 0$, we have

$$r^{n-1}c_r(r,t) = \int_0^r s^{n-1}\rho(s,t)c(s,t)\mathrm{d}s \geq 0,$$

which shows that $c(r,t)$ is nondecreasing in r. Along with the boundary condition for c in (4.1), we have

$$\begin{aligned} \gamma - c(R,t) &= c_r(R,t) \\ &= R^{-n+1} \int_0^R r^{n-1}\rho(r,t)c(r,t)\mathrm{d}r \\ &\leq R^{-n+1}\omega_n^{-1}\|\rho_0\|_{L^1(\Omega)} c(R,t), \end{aligned}$$

which directly implies (4.3). □

We now introduce a moment-type functional whose time evolution yields a differential inequality sufficient for proving blow-up.

Lemma 4.2. *Let $m \in \left(0, \frac{2}{n}\right)$ and $\alpha \in \left(2 - \frac{4}{n}, 2 - \frac{2}{n} - m\right)$. Then, there exists $\Lambda = \Lambda(n,m,\alpha,R) > 0$ with the property that whenever D satisfies* (1.11), *the initial data ρ_0 is such that* (1.4) *holds, and $\gamma > 0$, then the functional*

$$\phi(t) := \int_0^{R^n} s^{-\alpha} w(s,t)\mathrm{d}s$$

belongs to $C^0([0, T_{\max})) \cap C^1(\left(0, T_{\max}\right))$, and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \geq -\Lambda\|\rho_0\|_{L^1(\Omega)}^m + \frac{1}{\Lambda}\frac{\|\rho_0\|_{L^1(\Omega)}^2}{\left(\|\rho_0\|_{L^1(\Omega)} + 1\right)^3}\gamma \quad \text{for all } t \in \left(0, T_{\max}\right). \tag{4.4}$$

Proof. Let $m \in \left(0, \frac{2}{n}\right)$ and $\alpha \in \left(2 - \frac{4}{n}, 2 - \frac{2}{n} - m\right)$. Then, $\phi(t) \in C^0([0, T_{\max})) \cap C^1((0, T_{\max}))$ by the dominated convergence theorem.

We define

$$\tilde{D}(\xi) := \int_0^\xi D(\sigma)\mathrm{d}\sigma \quad \text{for } \xi \geq 0,$$

and differentiate $\phi(t)$ to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) &= n\int_0^{R^n} s^{2-\frac{2}{n}-\alpha}\partial_s(\tilde{D}(nw_s(s,t)))\mathrm{d}s + n\int_0^{R^n} s^{1-\frac{1}{n}-\alpha}w_s(s,t)c_s\left(s^{\frac{1}{n}}, t\right)\mathrm{d}s \\ &=: I(t) + J(t).\end{aligned} \tag{4.5}$$

Estimate of $I(t)$. From (1.11), we have

$$\tilde{D}(\xi) \leq K_D\int_0^\xi (1+\sigma)^{m-1}\mathrm{d}\sigma \leq \frac{K_D}{m}\xi^m.$$

Integration by parts then yields

$$\begin{aligned} I(t) &= -n\left(2 - \frac{2}{n} - \alpha\right)\int_0^{R^n} s^{1-\frac{2}{n}-\alpha}\tilde{D}(nw_s(s,t))\mathrm{d}s \\ &\quad + \left\{ns^{2-\frac{2}{n}-\alpha}\tilde{D}(nw_s(s,t))\right\}\Big|_{s=0}^{R^n} \\ &\geq -\frac{K_D(2n-2-n\alpha)n^m}{m}\int_0^{R^n} s^{1-\frac{2}{n}-\alpha}(w_s(s,t))^m\mathrm{d}s,\end{aligned} \tag{4.6}$$

where the boundary term at $s = 0$ vanishes due to $\alpha < 2 - \frac{2}{n} - m$. Applying Hölder's inequality and using again $\alpha < 2 - \frac{2}{n} - m$, we deduce

$$\begin{aligned} I(t) &\geq -C\left(\int_0^{R^n} s^{\left(1-\frac{2}{n}-\alpha\right)\frac{1}{1-m}}\right)^{1-m}\left(\int_0^{R^n} w_s(s,t)\mathrm{d}s\right)^m \\ &= -C\|\rho_0\|_{L^1(\Omega)}^m \end{aligned} \tag{4.7}$$

with some $C = C(n, m, \alpha, R) > 0$.

Estimate of $J(t)$. We consider two cases depending on the value of $c(0,t)$, separately. Let

$$\beta := \frac{1}{1 + R^{-n+1}\omega_n^{-1}\|\rho_0\|_{L^1(\Omega)}}$$

and

$$S_k := \left\{t \in (0, T_{\max}); c(0,t) \leq \frac{\beta\gamma}{k}\right\},$$

where $k > 1$ will be specified later.

We first deal with the case $t \in S_k$. From the equation for c in (4.1), it holds that

$$nw_s(s^n, t) = \rho(s,t) \ge s^{1-n}(s^{n-1}(\ln\, c)_s)_s(s,t) \tag{4.8}$$

for all $s \in (0, R)$. Using (4.8) and integrating by parts, we estimate

$$\begin{aligned}
J(t) &= n\int_0^{R^n} s^{1-\frac{1}{n}-\alpha} w_s(s,t)c_s\left(s^{\frac{1}{n}}, t\right)\mathrm{d}s \\
&= n\int_0^{R} s^{2n-2-n\alpha}\rho(s,t)c_s(s,t)\mathrm{d}s \\
&\ge n\int_0^{R} s^{n-1-n\alpha}\partial_s(s^{n-1}(\ln\, c)_s)(s,t)c_s(s,t)\mathrm{d}s \\
&= -n(n-1-n\alpha)\int_0^{R} s^{2n-3-n\alpha}\frac{(c_s(s,t))^2}{c(s,t)}\mathrm{d}s \\
&\quad - n\int_0^{R} s^{2n-2-n\alpha}(\ln\, c)_s(s,t)c_{ss}(s,t)\mathrm{d}s \\
&\quad + n\left\{s^{2n-2-n\alpha}(\ln\, c)_s(s,t)c_s(s,t)\right\}\Big|_{s=0}^{R}.
\end{aligned} \tag{4.9}$$

By the equation for c, we obtain

$$\begin{aligned}
&- n\int_0^{R} s^{2n-2-n\alpha}(\ln\, c)_s(s,t)c_{ss}(s,t)\mathrm{d}s \\
&\quad = -n\int_0^{R} s^{2n-2-n\alpha}(\ln\, c)_s(s,t)\left(\rho(s,t)c(s,t) - \frac{n-1}{s}c_s(s,t)\right)\mathrm{d}s \\
&\quad = -J(t) + n(n-1)\int_0^{R} s^{2n-3-n\alpha}(\ln\, c)_s(s,t)c_s(s,t)\mathrm{d}s.
\end{aligned}$$

The last term on the right hand side of (4.9) is nonnegative because the positiveness of c gives

$$\lim_{s\to 0} s^{2n-2-n\alpha}\frac{c_s^2(s,t)}{c(s,t)} = 0.$$

Hence, we deduce

$$J(t) \ge \frac{n^2\alpha}{2}\int_0^{R} s^{2n-3-n\alpha}\frac{c_s^2(s,t)}{c(s,t)}\mathrm{d}s = 2n^2\alpha\int_0^{R} s^{2n-3-n\alpha}\left(\left(c^{\frac{1}{2}}\right)_s\right)^2(s,t)\mathrm{d}s. \tag{4.10}$$

We observe from Hölder's inequality and the condition $2 - \frac{4}{n} < \alpha$ that

$$
\begin{aligned}
\left(c^{\frac{1}{2}}(R,t)-c^{\frac{1}{2}}(0,t)\right)^2 &= \left(\int_0^R \left(c^{\frac{1}{2}}\right)_s(s,t)\mathrm{d}s\right)^2 \\
&\le \left(\int_0^R s^{2n-3-n\alpha}\left(\left(c^{\frac{1}{2}}\right)_s\right)^2(s,t)\mathrm{d}s\right)\cdot\left(\int_0^R s^{-2n+3+n\alpha}\mathrm{d}s\right) \\
&\le C\int_0^R s^{2n-3-n\alpha}\left(\left(c^{\frac{1}{2}}\right)_s\right)^2(s,t)\mathrm{d}s
\end{aligned}
\tag{4.11}
$$

with some $C = C(n,m,\alpha,R) > 0$. Combining (4.10) and (4.11), it follows from Lemma 4.1 that, for all $t \in S_k$,

$$
J(t) \ge C\left(c^{\frac{1}{2}}(R,t)-c^{\frac{1}{2}}(0,t)\right)^2 \ge C_a\left(1-\frac{1}{\sqrt{k}}\right)^2\beta\gamma \tag{4.12}
$$

with some $C_a = C_a(n,m,\alpha,R) > 0$.

On the other hand, if $t \notin S_k$, i.e., $c(0,t) > \frac{\beta\gamma}{k}$, then it follows that

$$
c_s(s,t) = s^{-n+1}\int_0^s r^{n-1}\rho(r,t)c(r,t)\mathrm{d}r \ge \frac{\beta\gamma}{k}s^{-n+1}w(s^n,t),
$$

which yields via integration by parts that

$$
\begin{aligned}
J(t) &= n\int_0^{R^n} s^{1-\frac{1}{n}-\alpha}w_s(s,t)c_s\left(s^{\frac{1}{n}},t\right)\mathrm{d}s \\
&\ge \frac{n\beta\gamma}{k}\int_0^{R^n} s^{-\alpha}w_s(s,t)w(s,t)\mathrm{d}s \\
&= \frac{n\beta\gamma}{2k}\int_0^{R^n} s^{-\alpha}\partial_s(w^2(s,t))\mathrm{d}s \\
&= \frac{n\alpha\beta\gamma}{2k}\int_0^{R^n} s^{-\alpha-1}w^2(s,t)\mathrm{d}s + \frac{n\beta\gamma}{2k}\left\{s^{-\alpha}w^2(s,t)\right\}\Big|_{s=0}^{R^n} \\
&\ge C_b\frac{\|\rho_0\|_{L^1(\Omega)}^2\beta\gamma}{k}
\end{aligned}
\tag{4.13}
$$

with some $C_b = C_b(n,m,\alpha,R) > 0$. We choose

$$
k = \left(\sqrt{\frac{C_a}{C_b}}\|\rho_0\|_{L^1(\Omega)}+1\right)^2
$$

so that the right-hand sides of (4.12) and (4.13) coincide. This choice guarantees the existence of $C = C(n,m,\alpha,R) > 0$ such that

$$
J(t) \ge C\frac{\|\rho_0\|_{L^1(\Omega)}^2}{\left(\|\rho_0\|_{L^1(\Omega)}+1\right)^3}\gamma. \tag{4.14}
$$

Combining (4.7) and (4.14) with (4.5) yields the desired result. □

We are now in a position to complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Let $m \in \left(0, \frac{2}{n}\right)$. Set $\alpha \in \left(2 - \frac{4}{n}, \min\left\{2 - \frac{2}{n} - m, 1\right\}\right)$ as

$$\alpha = \frac{2 - \frac{4}{n} + \min\left\{2 - \frac{2}{n} - m, 1\right\}}{2}.$$

Let $\Lambda = \Lambda(n, m, R) > 0$ be as provided by Lemma 4.2. Given any $L > 0$, define

$$\gamma^* := 2\Lambda^2 \frac{(L+1)^3}{L^{1-m}}. \tag{4.15}$$

Now take $\gamma \geq \gamma^*$ and consider any radially symmetric initial data ρ_0 satisfying (1.4) and $\|\rho_0\|_{L^1(\Omega)} = L$. Let (ρ, c) be the corresponding solution to (4.1) guaranteed by Proposition 1.1. From Lemma 4.2 and (4.15), the functional $\phi(t) = \int_0^{R^n} s^{-\alpha} w(s,t) \mathrm{d}s$ fulfills

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \geq -\Lambda L^m + \frac{1}{\Lambda}\frac{L}{(L+1)^3}\gamma^* = \Lambda L^m \quad \text{for all } t \in (0, T_{\max}). \tag{4.16}$$

Therefore, using the fact $\alpha < 1$, integration of (4.16) over time yields

$$\Lambda L^m t \leq \phi(t) = \int_0^{R^n} s^{-\alpha} w(s,t) \mathrm{d}s \leq \frac{L}{\omega_n (1-\alpha)} R^{n(1-\alpha)}$$

for all $t \in (0, T_{\max})$, which induces that $T_{\max}$ must be finite, as desired. □

Acknowledgments: The authors are grateful to the anonymous reviewers for their insightful comments that led to the improvement of the manuscript.

Funding information: Jaewook Ahn acknowledges support of the National Research Foundation (NRF) of Korea (RS-2024-00336346 and RS-2025-24523482). Dongkwang Kim is supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) grant funded by the Ministry of Education (RS-2024-00450030).

Author contribution: All authors have accepted responsibility for the entire content of this manuscript and consented to its submission to the journal, reviewed all the results, and approved the final version of the manuscript. All authors conceived the study, conducted the mathematical analysis, and interpreted the results. D. Kim prepared the manuscript with contributions from J. Ahn.

Conflict of interest: Authors state no conflict of interest.

References

[1] I. Tuval, L. Cisneros, C. Dombrowski, C. W. Wolgemuth, J. O. Kessler, and R. E. Goldstein, *Bacterial swimming and oxygen transport near contact lines*, Proc. Natl. Acad. Sci. **102** (2005), no. 7, 2277–2282.

[2] J. Shioi, C. V. Dang, and B. L. Taylor, *Oxygen as attractant and repellent in bacterial chemotaxis*, J. Bacteriol. **169** (1987), no. 7, 3118–3123.

[3] J. Lankeit and M. Winkler, *Depleting the signal: analysis of chemotaxis-consumption models – a survey*, Stud. Appl. Math. **151** (2023), no. 4, 1197–1229.

[4] J. Jiang, *Eventual smoothness and exponential stabilization of global weak solutions to some chemotaxis systems*, SIAM J. Math. Anal. **51** (2019), no. 6, 4604–4644.

[5] M. Winkler, *Global weak solutions in a three-dimensional chemotaxis–Navier–Stokes system*, Ann. Inst. H. Poincaré Anal. Non Linéaire **33** (2016), no. 5, 1329–1352.

[6] D. Kim, *Global solutions for chemotaxis-fluid systems with singular chemotactic sensitivity*, Discrete Contin. Dyn. Syst. Ser. B **28** (2023), no. 10, 5380–5395.

[7] J. Lankeit and G. Viglialoro, *Global existence and boundedness of solutions to a chemotaxis-consumption model with singular sensitivity*, Acta Appl. Math. **167** (2020), no. 1, 75–97.

[8] Y. Tao and M. Winkler, *Global existence and boundedness in a Keller-Segel-Stokes model with arbitrary porous medium diffusion*, Discrete Contin. Dyn. Syst. **32** (2012), no. 5, 1901–1914.
[9] M. D. Francesco, A. Lorz, and P. A. Markowich, *Chemotaxis-fluid coupled model for swimming bacteria with nonlinear diffusion: global existence and asymptotic behavior*, Discrete Contin. Dyn. Syst. **28** (2010), no. 4, 1437–1453.
[10] Y. Tao and M. Winkler, *Locally bounded global solutions in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion*, Ann. Inst. H. Poincaré Anal. Non Linéaire **30** (2013), no. 1, 157–178.
[11] M. Winkler, *Global existence and stabilization in a degenerate chemotaxis-Stokes system with mildly strong diffusion enhancement*, J. Differential Equations **264** (2018), no. 10, 6109–6151.
[12] Y. Tian and Z. Xiang, *Global boundedness to a 3D chemotaxis-Stokes system with porous medium cell diffusion and general sensitivity*, Adv. Nonlinear Anal. **12** (2023), no. 1, 23–53.
[13] M. Jiang, D. Liu, R. Huang, M. Jiang, D. Liu, and R. Huang, *Boundedness and stabilization in a quasilinear chemotaxis model with nonlocal growth term and indirect signal production*, Commun. Anal. Mech. **17** (2025), no. 2, 387–412.
[14] D. Yan and C. Liu, *Global boundedness and stability of a quasilinear two-species chemotaxis-competition model with nonlinear sensitivities*, Adv. Nonlinear Anal. **14** (2025), no. 1, 20250129.
[15] Y. Wang and X. Cao, *Global classical solutions of a 3D chemotaxis-Stokes system with rotation*, Discrete Contin. Dyn. Syst. Ser. B **20** (2015), no. 9, 3235–3254.
[16] S. Zhou, *Boundedness in chemotaxis–Stokes system with rotational flux term*, Nonlinear Anal. Real World Appl. **45** (2019), 299–308.
[17] S. Ishida, *Global existence and boundedness for chemotaxis-Navier-Stokes systems with position-dependent sensitivity in 2D bounded domains*, Discrete Contin. Dyn. Syst. **35** (2015), no. 8, 3463–3482.
[18] M. Winkler, *Boundedness and large time behavior in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion and general sensitivity*, Calc. Var. Partial Differential Equations **54** (2015), no. 4, 3789–3828.
[19] X. Cao and J. Lankeit, *Global classical small-data solutions for a three-dimensional chemotaxis Navier–Stokes system involving matrix-valued sensitivities*, Calc. Var. Partial Differential Equations **55** (2016), no. 4, 107.
[20] M. Winkler, *Global mass-preserving solutions in a two-dimensional chemotaxis-Stokes system with rotational flux components*, J. Evol. Equ. **18** (2018), no. 3, 1267–1289.
[21] F. Heihoff, *Global mass-preserving solutions for a two-dimensional chemotaxis system with rotational flux components coupled with a full Navier–Stokes equation*, Discrete Contin. Dyn. Syst. Ser. B **25** (2020), no. 12, 4703–4719.
[22] M. Winkler, *Can rotational fluxes impede the tendency toward spatial homogeneity in nutrient Taxis(-Stokes) systems?* Int. Math. Res. Not. **2021** (2021), no. 11, 8106–8152.
[23] M. Winkler, *Chemotaxis-Stokes interaction with very weak diffusion enhancement: blow-up exclusion via detection of absorption-induced entropy structures involving multiplicative couplings*, Adv. Nonlinear Stud. **22** (2022), no. 1, 88–117.
[24] T. Black, *Very mild diffusion enhancement and singular sensitivity: existence of bounded weak solutions in a two-dimensional chemotaxis-Navier–Stokes system*, J. Differential Equations **443** (2025), 113555.
[25] T. Black and M. Winkler, *Global weak solutions and absorbing sets in a chemotaxis-Navier–Stokes system with prescribed signal concentration on the boundary*, Math. Models Methods Appl. Sci. **32** (2022), no. 01, 137–173.
[26] M. Braukhoff, *Global (weak) solution of the chemotaxis-Navier–Stokes equations with non-homogeneous boundary conditions and logistic growth*, Ann. Inst. H. Poincaré Anal. Non Linéaire **34** (2017), no. 4, 1013–1039.
[27] J. Ahn, K. Kang, and J. Lee, *Regular solutions of chemotaxis-consumption systems involving tensor-valued sensitivities and Robin type boundary conditions*, Math. Models Methods Appl. Sci. **33** (2023), no. 11, 2337–2360.
[28] S.-O. Yang and J. Ahn, *Long time asymptotics of small mass solutions for a chemotaxis-consumption system involving prescribed signal concentrations on the boundary*, Nonlinear Anal. Real World Appl. **79** (2024), 104129.
[29] Y. Wang and M. Winkler, *Finite-time blow-up in a repulsive chemotaxis-consumption system*, Proc. Roy. Soc. Edinburgh Sect. A **153** (2023), no. 4, 1150–1166.
[30] J. Ahn and M. Winkler, *A critical exponent for blow-up in a two-dimensional chemotaxis-consumption system*, Calc. Var. Partial Differential Equations **62** (2023), no. 6, 180.
[31] J. Ahn, K. Kang, and D. Kim, *Global boundedness and blow-up in a repulsive chemotaxis-consumption system in higher dimensions*, J. Differential Equations **443** (2025), 113503.
[32] S. Kohatsu, H. Sato, and Y. Tanaka, *Finite-time blow-up in a two-dimensional repulsive chemotaxis-consumption system with superlinear density-dependent sensitivity*, J. Math. Anal. Appl. **548** (2025), no. 1, 129375.
[33] M. Winkler, *Small-mass solutions in the two-dimensional Keller–Segel system coupled to the Navier–Stokes equations*, SIAM J. Math. Anal. **52** (2020), no. 2, 2041–2080.
[34] M. Winkler, *A three-dimensional Keller–Segel–Navier–Stokes system with logistic source: global weak solutions and asymptotic stabilization*, J. Funct. Anal. **276** (2019), no. 5, 1339–1401.
[35] K. Fujie and T. Senba, *Global existence and boundedness of radial solutions to a two dimensional fully parabolic chemotaxis system with general sensitivity*, Nonlinearity **29** (2016), no. 8, 2417–2450.
[36] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman Advanced Pub. Program, Boston, 1985.
[37] S. Ishida, K. Seki, and T. Yokota, *Boundedness in quasilinear Keller–Segel systems of parabolic–parabolic type on non-convex bounded domains*, J. Differential Equations **256** (2014), no. 8, 2993–3010.