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On the residues and Euler–Kronecker constants of cyclic number fields

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Abstract
For a number field K, the associated Dedekind zeta function $\zeta_K(s)$ has a simple pole at $s = 1$, and we denote its residue by R_K. Ihara introduced the Euler–Kronecker constant γ_K. Let ℓ be an odd prime. We establish lower and upper bounds for R_K and γ_K when K is a cyclic extension of degree ℓ over $\mathbb{Q}$. These bounds are stronger than those known under the Generalized Riemann Hypothesis (GRH) and are shown to be sharp. However, the trade-off is that they hold only almost surely. Finally, we compute the average of the Euler–Kronecker constants for cyclic fields K of degree ℓ.

Keywords Cyclic extension · Residue · Euler–Kronecker constant

Mathematics Subject Classification Primary:11M06

1 Introduction

Let K be a number field. We define the Dedekind zeta function associated with the field K:

$$\zeta_K(s) := \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} \text{ for } \Re(s) > 1,$$

where $\mathfrak{a}$ runs over non-zero integral ideals in the ring of integers O_K. It is well-known that $\zeta_K(s)$ has meromorphic continuation to the complex plan $\mathbb{C}$ with only one simple pole at $s = 1$ with residue R_K [12, Ch.7, §5].

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From the Laurent expansion of $\zeta_K(s)$ at $s = 1$

$$\zeta_K(s) = \frac{R_K}{(s-1)} + c_0 + c_1(s-1) + c_2(s-1)^2 + \cdots ,$$

we define the Euler–Kronecker constant γ_K to be c_0/R_K, which is also given by

$$\gamma_K :== \lim_{s\to 1}\left(\frac{\zeta_K'}{\zeta_K}(s) + \frac{1}{s-1}\right).$$

In this article, we study the value distributions of the residues R_K and Euler–Kronecker constants γ_K of cyclic fields of prime degree $\ell > 2$.

Let us review some known results for these invariants. For a quadratic field $K = \mathbb{Q}(\sqrt{D})$, the residue R_K is given by the L-value $L(1, \chi_D)$, where $\chi_D = (\frac{D}{\cdot})$ is the quadratic character.

Littlewood [11] showed the bound

$$(\frac{1}{2} + o(1))\frac{\zeta(2)}{e^\gamma \log\log|D|} \le L(1, \chi_D) \le (2 + o(1))e^\gamma \log\log|D|$$

under GRH, where γ is the Euler–Mascheroni constant. However, it was believed that the true sharp bounds would be those by replacing the numbers 2 and 1/2 in GRH bounds both by *one*.

We call a number field K of degree n an S_n-field if its Galois closure over $\mathbb{Q}$ is an S_n Galois extension. The first named author and Kim [2] showed that the bounds below

$$(1 + o(1))\frac{\zeta(n)}{e^\gamma \log\log|D_K|} \le R_K \le (1 + o(1))(e^\gamma \log\log|D_K|)^{n-1} \tag{1.1}$$

hold for S_3-, S_4-, and S_5-fields[1] except for a small number of fields when they are ordered by discriminant. Moreover, they showed that the bounds are sharp by generating infinitely many number fields with extreme residues.

We define our family of number fields by, for a prime $\ell > 2$,

$$F(\mathbb{Z}/\ell\mathbb{Z}, X) := \{K\colon \text{cyclic extension of degree } \ell \mid X < f(K) \le 2X\},$$

where $f(K)$ is the conductor of K, and let $N(\mathbb{Z}/\ell\mathbb{Z}, X) := \#F(\mathbb{Z}/\ell\mathbb{Z}, X)$. In Sect. 2 2, we see that $N(\mathbb{Z}/\ell\mathbb{Z}, X) \asymp_\ell X$.

We note that for K in $F(\mathbb{Z}/\ell\mathbb{Z}, X)$,

$$L(s, \rho_K) := \zeta_K(s)/\zeta(s) = \prod_{i=1}^{\ell-1} L\big(s, \chi_K^i\big),$$

[1] For S_5-fields, we assume the strong Artin conjecture.

where $\chi_K, \chi_K^2, \dots \chi_K^{\ell-1}$ are ℓ-th order primitive Dirichlet characters associated with the field K, and $R_K = L(1, \rho_K) = \prod_{i=1}^{\ell-1} L(1, \chi^i)$. We extend the results in [2] for the family $F(\mathbb{Z}/\ell\mathbb{Z}, X)$.

Theorem 1.1 *Let $\ell > 2$ be a prime. Then, except for $O_\ell(Xe^{-c' \frac{\log X}{\log\log X} \log\log\log X})$ fields in $F(\mathbb{Z}/\ell\mathbb{Z}, X)$ with some constant $c' > 0$, we have*

$$\frac{\zeta(\ell)}{e^\gamma \log\log f(K)} \left(1 + O\left(\frac{1}{(\log\log f(K))^{\frac{1}{2}}}\right)\right) \le R_K$$
$$\le (e^\gamma \log\log f(K))^{\ell-1} \left(1 + O\left(\frac{1}{(\log\log f(K))^{\frac{1}{2}}}\right)\right).$$

Moreover, we show that the upper bound and the lower bound in Theorem 1.1 are sharp.

Theorem 1.2 (1) *The number of cyclic extensions K in $F(\mathbb{Z}/\ell\mathbb{Z}, X)$ for which*

$$R_K = (e^\gamma \log\log f(K))^{\ell-1} \left(1 + O\left(\frac{1}{\log\log f(K)}\right)\right)$$

is

$$\gg_\ell Xe^{-\log \ell \frac{\log X}{\log\log X} - \log \ell \log\log\log X}.$$

(2) *The number of cyclic extensions K in $F(\mathbb{Z}/\ell\mathbb{Z}, X)$ for which*

$$R_K = \frac{\zeta(\ell)}{e^\gamma \log\log f(K)} \left(1 + O\left(\frac{1}{\log\log f(K)}\right)\right)$$

is

$$\gg_\ell Xe^{-\log \frac{\ell}{\ell-1} \frac{\log X}{\log\log X} - \log \ell \log\log\log X}.$$

It was Ihara [7] who introduced the Euler–Kronecker constant for the first time. In [7], it was shown that the main order terms of the upper bound and lower bound for γ_L under GRH are

$$2 \log\log \sqrt{|D_L|}, \qquad -2(n-1) \log\left(\frac{\log \sqrt{|D_L|}}{n-1}\right)$$

The first named author and Kim [1] improved the upper and lower bounds

$$\log\log |D_L| + O(\log\log\log D_L) \qquad -(n-1)\log\log D_L + O(\log\log\log |D_L|) \tag{1.2}$$

under Artin conjecture, GRH and a certain zero-density hypothesis [1, CONJECTURE 10.4].

We have extended (1.2) to our family almost surely.

Theorem 1.3 *Except for* $O\left(Xe^{-c''\frac{\log X}{\log\log X}\log\log\log X}\right)$ *number fields* K *in* $F(\mathbb{Z}/\ell\mathbb{Z}, X)$ *with some constant* $c'' > 0$, *we have*

$$\begin{aligned} &-(\ell-1)\log\log f(K) + O((\log\log f(K))^{\frac{1}{2}}) \\ &\leq \gamma_K \leq \log\log f(K) + O\Big((\log\log f(K))^{\frac{1}{2}}\Big). \end{aligned}$$

We also show that the bounds in Theorem 1.3 are sharp.

Theorem 1.4 (1) *The number of cyclic extensions* K *in* $F(\mathbb{Z}/\ell\mathbb{Z}, X)$ *for which*

$$\gamma_K = -(\ell-1)\log\log f(K) + O\Big((\log\log f(K))^{\frac{1}{2}}\Big)$$

is

$$\gg_\ell Xe^{-\log\ell\frac{\log X}{\log\log X} - \log\ell\log\log\log X}.$$

(2) *The number of cyclic extensions* K *in* $F(\mathbb{Z}/l\mathbb{Z}, X)$ *for which*

$$\gamma_K = \log\log f(K) + O\Big((\log\log f(K))^{\frac{1}{2}}\Big)$$

is

$$\gg_\ell Xe^{-\log\frac{\ell}{\ell-1}\frac{\log X}{\log\log X} - \log\ell\log\log\log X}.$$

As a first step toward understanding the distribution of Euler–Kronecker constants, we computed their average over our family. We have the following beautiful formula for the average of Euler–Kronecker constants.

Theorem 1.5 *Let* ℓ *be an odd prime. The average Euler–Kronecker constant* γ_ℓ *for the family* $F(\mathbb{Z}/\ell\mathbb{Z}, X)$ *is*

$$\gamma_\ell := \lim_{X\to\infty}\frac{1}{N(\mathbb{Z}/\ell\mathbb{Z}, X)}\sum_{K\in F(\mathbb{Z}/\ell\mathbb{Z},X)}\gamma_K = \gamma_{\mathbb{Q}} + (\ell-1)\sum_p \ell|TS_p|\frac{\log p}{p^\ell - 1},$$

where

$$|TS_p| = \begin{cases} \frac{\ell}{\ell^2+\ell-1} & \text{if } p = \ell, \\ \frac{p}{\ell(p+\ell-1)} & \text{if } p \equiv 1 \pmod{\ell}, \\ \frac{1}{\ell} & \text{otherwise}. \end{cases}$$

Furthermore, we have

$$\lim_{\ell\to\infty} \gamma_\ell = \gamma_{\mathbb{Q}}.$$

In Sect. 2, we recall a recent result on the count of cyclic fields of degree $\ell > 2$ satisfying certain local conditions. In Sect. 3, assuming a large zero-free region, we estimate the L-value $L(1, \rho_K)$ using a short partial Euler product. In Sect. 4, we prove Theorem 1.1. In Sect. 5, we express the logarithmic derivative of $L(s, \rho_K)$ at $s = 1$ as a short sum, again under a large zero-free region. In Sects. 6, 7, and 8, we prove Theorem 1.5, Theorem 1.3, and Theorems 1.2 and 1.4, respectively.

2 The count of cyclic fields of prime degree $\ell > 2$

In this section, we present results recently established by the first author and Oh [3]. Let K be a cyclic extension of prime degree $\ell > 2$. Then a prime p can be totally split, inert, or totally ramified in the field K. Let us denote the corresponding decomposition types of p by TS_p, IN_p and TR_p respectively. We call them the local conditions at prime p.

We introduce some notations first.

$$\mathfrak{C}_\ell = \frac{1}{\ell}\left(1 + \frac{\ell-1}{\ell}\right) \prod_{p \equiv 1 \pmod{\ell}} \left(1 + \frac{\ell-1}{p}\right)\left(1 - \frac{1}{p}\right) \prod_{p \not\equiv 1 \pmod{\ell}} \left(1 - \frac{1}{p}\right),$$

$$|TS_p| = \begin{cases} \frac{\ell}{\ell^2+\ell-1} & \text{if } p = \ell, \\ \frac{p}{\ell(p+\ell-1)} & \text{if } p \equiv 1 \pmod{\ell}, \\ \frac{1}{\ell} & \text{otherwise.} \end{cases}$$

$$|TR_p| = \begin{cases} \frac{\ell-1}{\ell^2+\ell-1} & \text{if } p = \ell, \\ \frac{\ell-1}{p+\ell-1} & \text{if } p \equiv 1 \pmod{\ell}, \\ 0 & \text{otherwise.} \end{cases}$$

and $|IN_p| = (\ell - 1)|TS_p|$.

Let $S = \{p_1, \ldots, p_k\}$ be a finite set of prime numbers, and let $\mathcal{LC} = \{LC_p | p \in S\}$ be a finite collection of local conditions, and let $N(\mathbb{Z}/\ell\mathbb{Z},\ X,\ \mathcal{LC})$ be the number of cyclic extensions $K/\mathbb{Q}$ of degree ℓ such that their conductor $f(K) \le X$ and they satisfy all the local conditions in $\mathcal{LC}$.

The first named author and Oh estimated $N(\mathbb{Z}/\ell\mathbb{Z},\ X,\ \mathcal{LC})$ as follows,[2]

[2] In [3, Theorem 1.1] the condition on the conductors is $f(K) \le X$. It is clear that our condition $X < f(K) \le 2X$ gives the same estimate.

Proposition 2.1 [3, Theorem 1.1] *For a given $\varepsilon > 0$, we have*

$$N(\mathbb{Z}/\ell\mathbb{Z},\ X,\ \mathcal{LC}) = \prod_{LC_p \in S} |LC_p| \mathfrak{C}_\ell X + O_{\ell,\varepsilon}\left(2^{|\mathcal{LC}|}(\ell-1)^I(\ell+1)^R \left[\left(\prod_{p\in S} p\right)^{\frac{\ell-1}{6}+\varepsilon} X^{\frac{\ell+2}{\ell+5}+\varepsilon} + \left(\prod_{p\in S} p\right)^{\frac{\ell-1}{4}} X^{\frac{1}{2}+\varepsilon} \right]\right), \tag{2.1}$$

where I is the number of primes p in S such that $LC_p = IN_p$, and R is the number of primes p in S such that $LC_p = TR_p$.

To prove Theorem 1.5, the following auxiliary lemma on character sums is necessary.

Let $D_\ell(X) := \{\chi\ :\ \ell\text{-th order primitive Dirichlet character} \mid f(\chi) \le X\}$, where $f(\chi)$ is the conductor of χ. There is an one to $(\ell-1)$ correspondence from $F(\mathbb{Z}/\ell\mathbb{Z}, X)$ to $D_\ell(X)$ as follows:

$$\begin{array}{ccc} F(\mathbb{Z}/\ell\mathbb{Z}, X) & \longrightarrow & D_\ell(X) \\ K, \zeta_K(s)/\zeta(s) = L(s,\chi_K)L(s,\chi_K^2)\cdots L(s,\chi_K^{\ell-1}) & \longrightarrow & \chi_K, \chi_K^2, \ldots, \chi_K^{\ell-1}, \end{array}$$

the corresponding field K and characters χ_K^i, $i = 1, 2, \ldots, \ell-1$ have the same conductor.

Let $\chi_{\rho_K}(n) := \chi_K(n) + \chi_K^2(n) + \cdots + \chi_K^{\ell-1}(n)$ for n in $\mathbb{N}$. By the one-to-one correspondence above, we have

$$\sum_{K\in F(\mathbb{Z}/\ell\mathbb{Z},X)} \chi_{\rho_K}(n) = \sum_{\chi\in D_\ell(X)} \chi(n).$$

We will apply the following proposition for the estimation of the sum $\sum_{K\in F(\mathbb{Z}/\ell\mathbb{Z},X)} \chi_{\rho_K}(n)$.

Proposition 2.2 [3, Lemma 3.1] *Let $n = p_1^{e_1}\cdots p_k^{e_k}$. We have*

$$\sum_{\chi\in D_\ell(X)} \chi(n) = (\ell-1)J(n)\prod_{i=1}^k \ell|TS_{p_i}|\mathfrak{C}_\ell X + O_{\ell,\varepsilon}\left((2\ell)^k \left[\prod_i^k p_i^{\frac{\ell-1}{6}+\varepsilon} X^{\frac{\ell+2}{\ell+5}+\varepsilon} + \prod_{i=1}^k p_i^{\frac{\ell-1}{4}} X^{\frac{1}{2}+\varepsilon}\right]\right), \tag{2.2}$$

where $J(n) = 1$ if n is a ℓ-th power full, and zero otherwise.

3 Product expression of $L(1, \rho_K)$ under a large zero-free region

To prove Theorem 1.1, we first show that under a certain large zero-free region $L(1, \rho_K)$ appears as a short Euler product. To get such an Euler product, we follow the argument in [2, proposition 3.1]. It is known that Dirichlet L-functions are entire and for $\Re(s) > 1$, we have

$$L(s, \rho_K) = \prod_p \prod_{i=1}^{\ell-1} \left(1 - \frac{\chi^i(p)}{p^s}\right)^{-1}, \text{ and } \log L(s, \rho_K) = \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi_{\rho_K}(n)}{n^s \log n},$$

where $\chi_{\rho_K}(n) = \chi_K(n) + \cdots + \chi_K^{\ell-1}(n)$.

Lemma 3.1 *Suppose that K is in $F(\mathbb{Z}/\ell\mathbb{Z}, X)$, and $L(s, \rho_K) \neq 0$ when $s = \sigma + it \in [\alpha, 1] \times [-x, x]$, where $x = (\log X)^\beta$ and $\beta(1-\alpha) > 4$. Then we have*

$$\log L(1, \rho_K) = \sum_{n \le x} \frac{\Lambda(n)\chi_{\rho_K}(n)}{n \log n} + O_\ell\left(\frac{(\log\log X)^2}{(\log X)^3}\right). \tag{3.1}$$

Proof We use Perron's formula [9, Theorem 7.2] to get

$$\frac{1}{2\pi i}\int_{c-ix}^{c+ix} \log L(1+s, \rho_K)\frac{x^s}{s}ds = \sum_{n \le x} \frac{\Lambda(n)\chi_{\rho_K}(n)}{n \log n} + O_\ell\Big(\frac{\log x}{x}\Big), \tag{3.2}$$

where $c = \frac{1}{\log x}$. Now, we move the contour using the Cauchy's Residue Theorem. Then, the left hand side of (3.2) is

$$\frac{1}{2\pi i}\left(\int_{c-ix}^{\alpha-1+c-ix} + \int_{\alpha-1+c-ix}^{\alpha-1+c+ix} + \int_{\alpha-1+c+ix}^{c+ix}\right) \log L(1+s, \rho_K)\frac{x^s}{s}ds + \log L(1, \rho_K).$$

Next, we follow the proof of [6, Lemma 8.1] to estimate $|\log L(s, \rho_K)|$ for $\alpha + c \le \Re(s) \le c + 1$.

We note that for $\Re(s) \ge \frac{3}{2}$, $\log L(s, \rho_K) = O_\ell(1)$, and for $\Re(s) \le 2$, consider the circles with centre $2 + it$ and radii $r := 2 - \sigma$ and $R := 2 - \alpha$. Then the smaller circle passes s, and by assumption, $\log L(s, \rho_K)$ is analytic inside the larger circle. For $\frac{1}{2} < Re(s) \le \frac{3}{2}$, we have, by [4, Lemma 1],

$$L(s, \rho_K) \ll_\ell f(K)^{\frac{1}{2}}(|s| + 1)^{\frac{\ell}{2}}.$$

Then, we obtain $\Re\Big(\log L(s,\rho_K)) = \log|L(s,\rho_K)| \ll_\ell \log f(K) + \log(|s|+1) \ll_\ell \log X + \log(|s|+1)$. By applying the Borel-Caratheodory theorem we get that

$$
\begin{aligned}
|\log L(s,\rho_K)| &\le \frac{2r}{R-r}\max_{|z-2-it|=R}\Re(\log L(z,\rho_K)) + \frac{R+r}{R-r}|\log L(2+it,\rho_K)| \\
&\ll_\ell \frac{1}{\sigma-\alpha}(\log f(K)+\log(|s|+1)) + \frac{1}{\sigma-\alpha} \\
&\ll_\ell \log x(\log f(K)+\log(|s|+1)) \\
&\ll_\ell \log\log X(\log X+\log(|s|+1))
\end{aligned}
$$

for $\alpha + c \le \Re(s) \le c+1$.

Thus, we have that

$$
\begin{aligned}
\int_{c-ix}^{\alpha-1+c-ix} \log L(1+s,\rho_K)\frac{x^s}{s}ds &\ll_\ell \log x\int_c^{\alpha-1+c}(\log f(K)+\log x)x^{\sigma-1}d\sigma \\
&\ll_\ell (\log f(K)+\log x)x^{-1} \ll_\ell \frac{1}{(\log X)^{\beta-1}},
\end{aligned}
$$

and, since $\beta(1-\alpha) > 4$,

$$
\begin{aligned}
&\int_{\alpha-1+c-ix}^{\alpha-1+c+ix} \log L(1+s,\rho_K)\frac{x^s}{s}ds \\
&\ll_\ell \log x\int_{-x}^{x}(\log f(K)+\log x)\frac{x^{\alpha-1}}{|\alpha-1+c|+|t|}dt \\
&\ll_\ell (\log x)^2(\log f(K)+\log x)x^{\alpha-1} \ll_\ell \frac{(\log\log X)^2}{(\log X)^3},
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{\alpha-1+c+ix}^{c+ix} \log L(1+s,\rho_K)\frac{x^s}{s}ds &\ll_\ell \log x\int_{\alpha-1+c}^{c}(\log f(K)+\log x)x^{\sigma-1}d\sigma \\
&\ll_\ell (\log f(K)+\log x)x^{-1} \ll_\ell \frac{1}{(\log X)^{\beta-1}}.
\end{aligned}
$$

By combining three inequalities above, we obtain that

$$
\begin{aligned}
&\frac{1}{2\pi i}\left(\int_{c-ix}^{\alpha-1+c-ix} + \int_{\alpha-1+c-ix}^{\alpha-1+c+ix} + \int_{\alpha-1+c+ix}^{c+ix}\right)\log L(1+s,\rho_K)\frac{x^s}{s}ds \\
&\quad\ll_\ell \frac{(\log\log X)^2}{(\log X)^3},
\end{aligned}
$$

and it completes the proof. □

We show that $L(1, \rho_K)$ can be estimated as a short Euler product when it satisfies the zero-free region condition in Lemma 3.1.

Lemma 3.2 *Suppose that K is in $F(\mathbb{Z}/\ell\mathbb{Z}, X)$, and $L(s, \rho_K) \neq 0$ when $s = \sigma + it \in [\alpha, 1] \times [-x, x]$, where $x = (\log X)^\beta$ and $\beta(1-\alpha) > 4$. Then, we have*

$$L(1, \rho_K) = \prod_{p \le x} \prod_{i=1}^{\ell-1} \left(1 - \frac{\chi^i(p)}{p}\right)^{-1} \left(1 + O_\ell\left(\frac{1}{\log\log X}\right)\right). \tag{3.3}$$

Proof First, we rewrite the sum

$$\sum_{n \le x} \frac{\Lambda(n)\chi_{\rho_K}(n)}{n \log n} = \sum_{p \le x} \sum_{i=1}^{\ell-1} \sum_{k \le \frac{\log x}{\log p}} \frac{1}{k} \left(\frac{\chi^i(p)}{p}\right)^k.$$

By the Taylor expansion of $\log(1-x)$, we find

$$\sum_{k \le \frac{\log x}{\log p}} \frac{1}{k} \left(\frac{\chi^i(p)}{p}\right)^k = -\log\left(1 - \frac{\chi^i(p)}{p}\right) - \sum_{k > \frac{\log x}{\log p}} \frac{1}{k} \left(\frac{\chi^i(p)}{p}\right)^k.$$

Here, we have

$$\left| \sum_{k > \frac{\log x}{\log p}} \frac{1}{k} \left(\frac{\chi^i(p)}{p}\right)^k \right| \le \sum_{k > \frac{\log x}{\log p}} \frac{1}{kp^k} \le \frac{1}{x \log x} \frac{\log p}{1 - p^{-1}}.$$

By Chebyshev's estimate,

$$\begin{aligned} \sum_{n \le x} \frac{\Lambda(n)\chi_{\rho_K}(n)}{n \log n} &= -\sum_{p \le x} \sum_{i=1}^{\ell-1} \log\left(1 - \frac{\chi^i(p)}{p}\right) + O_\ell\left(\frac{1}{x \log x} \sum_{p \le x} \frac{\log p}{1 - p^{-1}}\right) \\ &= -\sum_{p \le x} \sum_{i=1}^{\ell-1} \log\left(1 - \frac{\chi^i(p)}{p}\right) + O_\ell\left(\frac{1}{\log x}\right). \end{aligned}$$

By Lemma 3.1, we have

$$\log L(1, \rho_K) = \log\left(\prod_{p \le x} \prod_{i=1}^{\ell-1} \left(1 - \frac{\chi^i(p)}{p}\right)^{-1}\right) + O_\ell\left(\frac{1}{\log x}\right).$$

Thus, we have

$$L(1,\rho_K)=\exp\left(\log\left(\prod_{p\le x}\prod_{i=1}^{\ell-1}\left(1-\frac{\chi^i(p)}{p}\right)^{-1}\right)+O_\ell\left(\frac{1}{\log x}\right)\right)$$
$$=\prod_{p\le x}\prod_{i=1}^{\ell-1}\left(1-\frac{\chi^i(p)}{p}\right)^{-1}\left(1+O_\ell\left(\frac{1}{\log\log X}\right)\right).$$

□

4 Proof of Theorem 1.1

We follow the ideas in [2] for the proof of Theorem 1.1. We define $L(X)$ to be the set of L-functions associated with fields K in $F(\mathbb{Z}/\ell\mathbb{Z},X)$:

$$L(X):=\{L(s,\rho_K)=\prod_{i=1}^{\ell-1}L(s,\chi_K^i)\mid K\in F(\mathbb{Z}/\ell\mathbb{Z},X)\}.$$

In order to have the zero-free region required for Lemma 3.2, we appeal Kowalski-Michel's density theorem [10, Theorem 2]. Since the observation that most L-functions in $L(X)$ have a broad zero-free region near $s=1$ in crucial throughout the paper, we state the setting and the main result in [10] in detail.

Let $n\ge 1$ be a fixed integer. We assume we are given, for all $q\ge 1$, a finite set (possibly empty) $S(q)$ of cuspidal automorphic representations of $GL(n)/\mathbb{Q}$ which satisfy the following conditions:

(1) The forms $f\in S(q)$ satisfy the Ramanujan-Petersson conjecture at the finite places, namely all the local components f_p of f at a prime p are tempered.
(2) There exists $A>0$ such that for all $f\in S(q)$, the conductor $\mathrm{Cond}(f)$ of f satisfies
$$\mathrm{Cond}(f)\le q^A.$$
(3) There exists $d>0$ such that
$$|S(q)|\ll q^d$$
for all $q\ge 1$, the implied constant depending only on the family.
(4) All the f in $S(q)$ have the same component at ∞, hence the same gamma factor in the functional equation.

For a cuspidal automorphic representation f on $GL(n)/\mathbb{Q}$, we define

$$N(f;\alpha,T)=\left|\{\rho\in\mathbb{C}|\Re(\rho)\ge\alpha,|\Im(\rho)|\le T,L(f,\rho)=0\}\right|,$$

where the zeros are counted with multiplicity. Then, we have

Theorem 4.1 [10, Theorem 2] *Let $S(q)$ and $q \geq 1$ be as above. Let $\alpha \geq \frac{3}{4}$ and $T \geq 2$. Then there exists $c_0 > 0$, depending only on the family, such that*

$$\sum_{f \in S(q)} N(f; \alpha, T) \ll T^B q^{c_0 \frac{1-\alpha}{2\alpha-1}}$$

for all $q \geq 1$ and some $B \geq 0$ (depending only on the family). The implied constant depends on the family and the choicee of c_0, and one can choose any $c_0 > c_0{}'$, where

$$c_0{}' = \frac{5nA}{2} + d.$$

We can view a primitive Dirichlet character χ as a cuspidal representation of $GL(1)/\mathbb{Q}$. However, the isobaric sum $\chi + \chi^2 + \cdots + \chi^{\ell-1}$ is not cuspidal, we cannot apply [10, Theorem 2] directly to $L(X)$. Instead, we define

$$D_\ell(X) := \{\chi \mid \chi\text{: primitive Dirichlet character of } \ell\text{-th order, } X \leq f(\chi) \leq 2X\}.$$

Then, by [10, Theorem 2], we have

$$\sum_{\chi \in D_\ell(X)} N(\chi, \alpha, T) \ll T^B X^{\frac{7}{2}\frac{1-\alpha}{2\alpha-1}}.$$

This means that every L-function in $D_\ell(X)$ is zero-free in the region $[\alpha, 1] \times [-T, T]$ except for $O(T^B X^{\frac{7}{2}\frac{1-\alpha}{2\alpha-1}})$ L-functions.

Since $L(s, \rho_K) = \prod_{i=1}^{\ell-1} L(s, \chi_K^i)$ and each primitive Dirichlet characters χ_K appears as a component of $L(s, \rho_K)$ uniquely, still we can say that every L-function in $L(X)$ is zero-free in the region $[\alpha, 1] \times [-T, T]$ except for $O(T^B X^{\frac{7}{2}\frac{1-\alpha}{2\alpha-1}})$ L-functions. By taking $T = (\log X)^\beta$, except for such $O\left((\log X)^{\beta B} X^{\frac{7}{2}\frac{1-\alpha}{2\alpha-1}}\right)$ L-functions, we can apply Lemma 3.2 to L-functions in $L(X)$ to get

$$L(1, \rho_K) = \prod_{p \leq x} \prod_{i=1}^{\ell-1} \left(1 - \frac{\chi^i(p)}{p}\right)^{-1} \left(1 + O_\ell\left(\frac{1}{\log\log X}\right)\right).$$

Let $y = c_1 \log X$ for some constant $c_1 > 0$. To estimate $\prod_{p\le x}\prod_{i=1}^{\ell-1}(1-\frac{\chi^i(p)}{p})^{-1}$, we look at the product over primes bigger than y:

$$\begin{aligned}\prod_{y\le p\le x}\prod_{i=1}^{\ell-1}\left(1-\frac{\chi^i(p)}{p}\right)^{-1} &= \exp\left(-\sum_{y\le p\le x}\sum_{i=1}^{\ell-1}\log\left(1-\frac{\chi^i(p)}{p}\right)\right)\\ &= \exp\left(\sum_{y\le p\le x}\frac{\chi_{\rho_K}(p)}{p}+O_\ell\Big(\sum_{y\le p\le x}\frac{1}{p^2}\Big)\right)\\ &= \exp(\sum_{y\le p\le x}\frac{\chi_{\rho_K}(p)}{p})\Big(1+O_\ell\Big(\frac{1}{y\log y}\Big)\Big). \end{aligned} \tag{4.1}$$

The last equality in (4.1) follows, using Chebyshev's estimate, from

$$\sum_{y\le p}\frac{1}{p^2}\le\int_y^\infty\frac{1}{t^2}d\Big(\frac{2t}{\log t}\Big)\le\frac{2}{y\log y}.$$

We will show that the sum $\sum_{y\le p\le x}\frac{\chi_{\rho_K}(p)}{p}$ is almost surely negligible. Before proceeding, we introduce the following key lemma.

Lemma 4.2 [2, Lemma 4.3] *Let $y = c_1 \log X$ with some constant $c_1 > 0$. Assume that $r_1+\cdots+r_m+r_{m+1}+\cdots+r_u = 2r$ with $r_1=\cdots=r_m=1$ and $r_{m+1},\ldots,r_u\ge 2$, and $1\le u\le 2r$. Then we have*

$$\frac{1}{u!}\frac{1}{r_1!\cdots r_m!r_{m+1}!\cdots r_u!}\frac{y^u}{y^{m+r}}\frac{(\log y)^r}{(\log y)^u}\le\frac{1}{r!}. \tag{4.2}$$

Using Lemma 4.2, we prove the following proposition.

Proposition 4.3 *Assume $r\le c_2\frac{\log X}{\log\log X}$ for some $c_2>0$. Then*

$$\sum_{L(s,\rho_K)\in L(X)}\left(\sum_{y\le p\le x}\frac{\chi_{\rho_K}(p)}{p}\right)^{2r}\ll_\ell 2^{12r}(\ell-1)^{4r}\frac{(2r)!}{r!}\frac{X}{(y\log y)^r}. \tag{4.3}$$

Proof By the multinomial formula, we have

$$\begin{aligned}
&\sum_{L(s,\rho_K)\in L(X)} \left(\sum_{y\le p\le x} \frac{\chi_{\rho_K}(p)}{p} \right)^{2r} \\
&= \sum_{L(s,\rho_K)\in L(X)} \sum_{u=1}^{2r} \frac{1}{u!} \sum_{\substack{r_1+\cdots+r_u=2r \\ r_1,\ldots,r_u\ge 1}} \frac{(2r)!}{r_1!\ldots r_u!} \sum_{\substack{y\le p_1,\ldots p_u\le x \\ p_i\ne p_j}} \frac{\chi_{\rho_K}^{r_1}(p_1)\cdots\chi_{\rho_K}^{r_u}(p_u)}{p_1^{r_1}\cdots p_u^{r_u}} \\
&= \sum_{u=1}^{2r} \frac{1}{u!} \sum_{\substack{r_1+\cdots+r_u=2r \\ r_1,\ldots,r_u\ge 1}} \frac{(2r)!}{r_1!\ldots r_u!} \sum_{\substack{y\le p_1,\ldots p_u\le x \\ p_i\ne p_j}} \frac{1}{p_1^{r_1}\cdots p_u^{r_u}} \\
&\sum_{L(s,\rho_K)\in L(X)} \chi_{\rho_K}^{r_1}(p_1)\cdots\chi_{\rho_K}^{r_u}(p_u).
\end{aligned}$$

Suppose that $r_i \ge 2$ for all i. Since $L(X) \asymp_\ell X$, we have

$$\begin{aligned}
&\sum_{y\le p_1,\ldots p_u\le x p_i\ne p_j} \frac{1}{p_1^{r_1}\cdots p_u^{r_u}} \sum_{L(s,\rho_K)\in L(X)} \chi_{\rho_K}^{r_1}(p_1) \\
&\quad \cdots \chi_{\rho_K}^{r_u}(p_u) \ll (\ell-1)^{2r} X \left(\sum_{y\le p\le x} \frac{1}{p^{r_1}} \right) \cdots \left(\sum_{y\le p\le x} \frac{1}{p^{r_u}} \right) \\
&\quad \ll (\ell-1)^{2r} 2^{2r} X \frac{1}{y^{r_1-1}\log y} \cdots \frac{1}{y^{r_u-1}\log y} \\
&\quad = (\ell-1)^{2r} 2^{2r} X \left(\frac{1}{y\log y} \right)^r \left(\frac{\log y}{y} \right)^{r-u}.
\end{aligned}$$

Since $r_i \ge 2$ for all $i = 1, \ldots, u$, we have $u \le r$. Then we have

$$\frac{r!}{u! r_1! \cdots r_u!} \le \frac{r!}{u!} = r(r-1)\cdots(r-u+1) \le r^{r-u} \ll \left(\frac{y}{\log y} \right)^{r-u}$$

because $r \le c_2 \frac{\log X}{\log\log X}$. Equivalently, we have

$$\frac{1}{u! r_1! \cdots r_u!} \left(\frac{\log y}{y} \right)^{r-u} \ll \frac{1}{r!}.$$

Hence

$$\frac{(2r)!}{u!r_1!\cdots r_u!}\sum_{\substack{y\le p_1,\ldots p_u\le x\\ p_i\ne p_j}}\frac{1}{p_1^{r_1}\cdots p_u^{r_u}}\sum_{L(s,\rho_K)\in L(X)}\chi_{\rho_K}^{r_1}(p_1)\cdots\chi_{\rho_K}^{r_u}(p_u)$$
$$\ll(\ell-1)^{2r}2^{2r}X\frac{(2r)!}{r!}\frac{1}{(y\log y)^r}. \tag{4.4}$$

for $r_i\ge 2$ for all $i=1,\ldots,u$.

Now, suppose that for some i, $r_i=1$. Then we may assume that $r_1+\cdots+r_m+r_{m+1}+\cdots+r_u=2r$ with $r_1=\cdots=r_m=1$ and $r_{m+1},\ldots,r_u\ge 2$. If some p_i is totally ramified, then $\chi_{\rho_K}(p)=0$. Thus, we can assume that $p_1,\ldots,p_u$ are unramified. Fix local conditions at primes $p_2,\ldots,p_u$. Let $\mathcal{LC}_{p_1}(LC_2,\ldots,LC_u)$ denote the finite collection of all the local conditions at the prime p_1 with the local conditions at primes. Then, we have

$$\sum_{L(s,\rho_K)\in L(X)}\chi_{\rho_K}(p_1)\cdots\chi_{\rho_K}(p_m)\chi_{\rho_K}^{r_{m+1}}(p_{m+1})\cdots\chi_{\rho_K}^{r_u}(p_u)$$
$$=\sum_{LC_2,\ldots,LC_u}\sum_{K\in F(\mathbb{Z}/\ell\mathbb{Z},X,\mathcal{LC}_{p_1}(LC_2,\ldots,LC_u))}\chi_{\rho_K}$$
$$\times s(p_1)\chi_{\rho_K}(p_2)\cdots\chi_{\rho_K}(p_m)\chi_{\rho_K}^{r_{m+1}}(p_{m+1})\cdots\chi_{\rho_K}^{r_u}(p_u).$$

Note that $\chi_{\rho_K}(p_2)\cdots\chi_{\rho_K}(p_m)\chi_{\rho_K}^{r_{m+1}}(p_{m+1})\cdots\chi_{\rho_K}^{r_u}(p_u)$ is a constant if $K\in F(\mathbb{Z}/\ell\mathbb{Z},X,\mathcal{LC}_{p_1}(LC_2,\ldots,LC_u))$. Therefore, this fact and Proposition 2.1 implies that the main term of the inner sum vanishes. The contribution from the error term in Proposition 2.1 is at most

$$\ll 2^{u-1}(\ell-1)^{2r}2^u(\ell-1)^u\left[\left(\prod_{i=1}^u p_i\right)^{\frac{\ell-1}{6}+\varepsilon}X^{\frac{\ell+2}{\ell+5}+\varepsilon}+\left(\prod_{i=1}^u p_i\right)^{\frac{\ell-1}{4}}X^{\frac12+\varepsilon}\right]$$
$$\ll 2^{2u}(\ell-1)^{2r+u}\left[\left(\prod_{i=1}^u p_i\right)^{\frac{\ell-1}{6}+\varepsilon}X^{\frac{\ell+2}{\ell+5}+\varepsilon}+\left(\prod_{i=1}^u p_i\right)^{\frac{\ell-1}{4}}X^{\frac12+\varepsilon}\right]$$

So we find that

$$\frac{(2r)!}{u!r_1!\cdots r_u!}\sum_{\substack{y\le p_1,\ldots p_u\le x\\ p_i\ne p_j}}\frac{1}{p_1^{r_1}\cdots p_u^{r_u}}\sum_{L(s,\rho_K)\in L(X)}\chi_{\rho_K}$$
$$(p_1)\cdots\chi_{\rho_K}(p_m)\chi_{\rho_K}^{r_{m+1}}(p_{m+1})\cdots\chi_{\rho_K}^{r_u}(p_u). \tag{4.5}$$

is

$$\ll 2^{4r}(\ell-1)^{4r}\frac{(2r)!}{u!r_1!\cdots r_u!}\Bigg(X^{\frac{\ell+2}{\ell+5}+\varepsilon}\Bigg(\sum_{y\le p_1\le x}p_1^{\frac{\ell-7}{6}+\varepsilon}\cdots\sum_{y\le p_m\le x}p_m^{\frac{\ell-7}{6}+\varepsilon}$$

$$\sum_{y\le p_{m+1}\le x}p_{m+1}^{\frac{\ell-1}{6}-r_{m+1}+\varepsilon}\cdots\sum_{y\le p_u\le x}p_u^{\frac{\ell-1}{6}-r_u+\varepsilon}\Bigg)$$

$$+X^{\frac12+\varepsilon}\Bigg(\sum_{y\le p_1\le x}p_1^{\frac{\ell-5}{4}}\cdots\sum_{y\le p_m\le x}p_m^{\frac{\ell-5}{4}}\sum_{y\le p_{m+1}\le x}p_{m+1}^{\frac{\ell-1}{4}-r_{m+1}}\cdots\sum_{y\le p_u\le x}p_u^{\frac{\ell-1}{4}-r_u}\Bigg)\Bigg).$$

By Chebyshev's estimate, we have

$$\sum_{y\le p\le x}p^{\frac{\ell-7}{6}+\varepsilon}\ll_\ell 2\frac{x^{\frac{\ell-1}{6}+\varepsilon}}{\log x},\ \text{and}\ \sum_{y\le p\le x}p^{\frac{\ell-5}{4}}\ll_\ell 2\frac{x^{\frac{\ell-1}{4}}}{\log x},$$

for $r\ge 2$

$$\sum_{y\le p\le x}p^{\frac{\ell-1}{6}-r+\varepsilon}\ll_\ell 2x^{1-r}\frac{x^{\frac{\ell-1}{6}+\varepsilon}}{\log x},\ \text{and}\ \sum_{y\le p\le x}p^{\frac{\ell-1}{4}-r}\ll_\ell 2x^{1-r}\frac{x^{\frac{\ell-1}{4}}}{\log x}.$$

Then (4.5) becomes

$$\begin{aligned}
&\ll_\ell 2^{6r}(\ell-1)^{4r}\frac{(2r)!}{u!r_1!\cdots r_u!}\left(X^{\frac{\ell+2}{\ell+5}+\varepsilon}\left(\frac{x^{\frac{\ell-1}{6}+\varepsilon}}{\log x}\right)^u+X^{\frac12+\varepsilon}\left(\frac{x^{\frac{\ell-1}{4}}}{\log x}\right)^u\right)\\
&\ll_\ell 2^{6r}(\ell-1)^{4r}\frac{(2r)!}{r!}y^m\left(\frac{y}{\log y}\right)^{r-u}\left(X^{\frac{\ell+2}{\ell+5}+\varepsilon}\left(\frac{x^{\frac{\ell-1}{6}+\varepsilon}}{\log x}\right)^u+X^{\frac12+\varepsilon}\left(\frac{x^{\frac{\ell-1}{4}}}{\log x}\right)^u\right)\\
&\ll_\ell 2^{6r}(\ell-1)^{4r}\frac{(2r)!}{r!}y^r\left(X^{\frac{\ell+2}{\ell+5}+\varepsilon}\left(x^{\frac{\ell-1}{6}+\varepsilon}\right)^u+X^{\frac12+\varepsilon}\left(x^{\frac{\ell-1}{4}}\right)^u\right)\\
&\ll_\ell 2^{6r}(\ell-1)^{4r}\frac{(2r)!}{r!}\left(X^{\frac{\ell+2}{\ell+5}+\varepsilon}(\log X)^{(\frac{\ell-1}{6}+\varepsilon)\beta u+r}+X^{\frac12+\varepsilon}(\log X)^{\frac{\ell-1}{4}\beta u+r}\right)\\
&\ll_\ell 2^{6r}(\ell-1)^{4r}\frac{(2r)!}{r!}\left(X^{\frac{\ell+2}{\ell+5}+\varepsilon}(\log X)^{2r((\frac{\ell-1}{6}+\varepsilon)\beta+1)}+X^{\frac12+\varepsilon}(\log X)^{2r(\frac{\ell-1}{4}\beta+1)}\right).
\end{aligned}$$

The second and third inequalities follow from Lemma 4.2 and $m\le u$, respectively. Now, if we take $c_2<\min(\frac{3/(\ell+5)-\varepsilon}{((\ell-1)/3+2\varepsilon)\beta+3},\frac{1/2-\varepsilon}{(\ell-1)\beta/2+3})$, then we have

$$(X^{\frac{\ell+2}{\ell+5}+\varepsilon}(\log X)^{2r((\frac{\ell-1}{6}+\varepsilon)\beta+1)}+X^{\frac12+\varepsilon}(\log X)^{2r(\frac{\ell-1}{4}\beta+1)})\ll\frac{X}{(y\log y)^r}.$$

Hence, we have

$$\frac{1}{u!}\frac{(2r)!}{r_1!\cdots r_u!}\sum_{\substack{y\le p_1,\ldots p_u\le x\\ p_i\ne p_j}}\frac{1}{p_1^{r_1}\cdots p_u^{r_u}}\sum_{L(s,\rho_K)\in L(X)}\chi_{\rho_K}^{r_1}(p_1)\cdots\chi_{\rho_K}^{r_u}(p_u) \tag{4.6}$$

$$\ll_\ell 2^{6r}(\ell-1)^{4r}\frac{(2r)!}{r!}X\frac{2^{4r}}{(y\log y)^r}.$$

for $r_1+\cdots+r_m+r_{m+1}+\cdots+r_u=2r$ with $r_1=\cdots=r_m=1$ and $r_{m+1},\ldots,r_u\ge 2$.
Together with (4.4) and (4.6), we finally get

$$\sum_{u=1}^{2r}\sum_{\substack{r_1+\cdots+r_u=2r\\ r_1,\ldots,r_u\ge 1}}\frac{1}{u!}\frac{(2r)!}{r_1!\cdots r_u!}\sum_{\substack{y\le p_1,\ldots p_u\le x\\ p_i\ne p_j}}\frac{1}{p_1^{r_1}\cdots p_u^{r_u}}\sum_{L(s,\rho_K)\in L(X)}\chi_{\rho_K}^{r_1}(p_1)\cdots\chi_{\rho_K}^{r_u}(p_u)$$

$$\ll_\ell 2^{8r}(\ell-1)^{4r}\frac{(2r)!}{r!}X\frac{2^{4r}}{(y\log y)^r}.$$

because the number of compositions of $2r$ is 2^{2r-1}. □

Now, we can say that the sum $\sum_{y\le p\le x}\frac{\chi_{\rho_K}(p)}{p}$ is almost surely negligible.

Corollary 4.4 *Except for $O_\ell(Xe^{-c'\frac{\log X}{\log\log X}\log\log\log X})$ L-functions in $L(X)$ we have*

$$\left|\sum_{y\le p\le x}\frac{\chi_{\rho_K}(p)}{p}\right|<\frac{1}{(\log\log X)^{\frac{1}{2}}}. \tag{4.7}$$

Proof Take $r=c_2\frac{\log X}{\log\log X}$. By Proposition 4.3, the number of L-functions in $L(X)$ with $|\sum_{y\le p\le x}\frac{\chi_{\rho_K}(p)}{p}|>\frac{1}{(\log\log X)^{\frac{1}{2}}}$ is

$$\ll_\ell 2^{12r}(\ell-1)^{4r}r^rX\frac{1}{(\log X)^r}$$

$$\ll_\ell X\exp\left(\frac{c_2\log X}{\log\log X}(12\log 2+4\log(\ell-1)+\log c_2-\log\log\log X)\right)$$

$$\ll_\ell Xe^{-c'\frac{\log X}{\log\log X}\log\log\log X},$$

where the first inequality follows from Stirling formula. □

Since except for $O(Xe^{-c'\frac{\log X}{\log\log X}\log\log\log X})$ L-functions in $L(X)$ have the zero-free region we require, by Lemma 3.2, (4.1), and Corollary 4.4, those L-functions with the zero-free region satisfy that

$$L(1,\rho_K) = \prod_{p\le y}\prod_{i=1}^{l-1}\left(1-\frac{\chi^i(p)}{p}\right)^{-1}\left(1+O_\ell\left(\frac{1}{\log\log X}\right)\right) \tag{4.8}$$
$$\cdot\left(1+O_\ell\left(\frac{1}{(\log\log X)^{\frac{1}{2}}}\right)\right)\left(1+O_\ell\left(\frac{1}{\log X\log\log X}\right)\right)$$
$$=\prod_{p\le y}\prod_{i=1}^{l-1}\left(1-\frac{\chi^i(p)}{p}\right)^{-1}\left(1+O_\ell\left(\frac{1}{(\log\log X)^{\frac{1}{2}}}\right)\right).$$

We note that

$$\prod_{i=1}^{\ell-1}\left(1-\frac{\chi^i(p)}{p}\right)^{-1} = \begin{cases}\left(1-\frac{1}{p}\right)^{-(\ell-1)} & \text{if } p \text{ is totally split,}\\ \frac{(1-1/p^\ell)^{-1}}{(1-1/p)^{-1}} & \text{if } p \text{ is inert.}\end{cases}$$

Since $\frac{(1-x)^\ell}{1-x^\ell}\le 1$ for $0<x<1$, we have the inequalities

$$\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^\ell}\right)^{-1}\le\prod_{i=1}^{\ell-1}\left(1-\frac{\chi^i(p)}{p}\right)^{-1}\le\left(1-\frac{1}{p}\right)^{-(\ell-1)}.$$

So we have

$$\prod_{p\le y}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^\ell}\right)^{-1}\left(1+O_\ell\left(\frac{1}{(\log\log X)^{\frac{1}{2}}}\right)\right)\le L(1,\rho_K)$$
$$\le\prod_{p\le y}\left(1-\frac{1}{p}\right)^{-(\ell-1)}\left(1+O_\ell\left(\frac{1}{(\log\log X)^{\frac{1}{2}}}\right)\right).$$

Applying Mertens' Theorem

$$\prod_{p\le y}\left(1-\frac{1}{p}\right)=\frac{e^{-\gamma}}{\log y}\left(1+O\left(\frac{1}{\log y}\right)\right)$$

and the partial Euler product approximation of $\zeta(\ell)$

$$\prod_{p\le y}\left(1-\frac{1}{p^\ell}\right)^{-1}=\zeta(\ell)\left(1+O\left(\frac{1}{y^{\ell-1}}\right)\right)$$

to the product above, we have obtained

$$\frac{\zeta(l)}{e^{\gamma}\log\log X}\left(1+O\left(\frac{1}{(\log\log X)^{\frac{1}{2}}}\right)\right)\le L(1,\rho_K)$$
$$\le\left(e^{\gamma}\log\log X\right)^{l-1}\left(1+O\left(\frac{1}{(\log\log X)^{\frac{1}{2}}}\right)\right).$$

Since $f(K)\asymp X$ for K in $F(\mathbb{Z}/\ell\mathbb{Z},X)$, we have

$$\frac{\zeta(l)}{e^{\gamma}\log\log f(K)}\left(1+O\left(\frac{1}{(\log\log f(K))^{\frac{1}{2}}}\right)\right)\le L(1,\rho_K)$$
$$\le\left(e^{\gamma}\log\log f(K)\right)^{l-1}\left(1+O\left(\frac{1}{(\log\log f(K))^{\frac{1}{2}}}\right)\right),$$

and Theorem 1.1 follows.

5 Formula for $\frac{L'}{L}(1,\rho_K)$

In this section, we show that when the L-function $L(1,\rho_K)$ satisfies the zero-free region condition given in Lemma 3.1, the logarithmic derivative of $L(s,\rho_K)$ at $s=1$ can be accurately approximated by a short sum. To deal with L-functions that may not satisfy the required zero-free region condition, we also establish an upper bound for their logarithmic derivatives. The arguments presented here are standard and well known. Nevertheless, for the sake of completeness, we provide the full details. We begin with the following two lemmas.

Lemma 5.1 [5, Lemma 2] *Assume $f(z)$ is a holomorphic in $|z-z_0|<r$ and satisfies*

$$\Re(f(z)-f(z_0))\le M.$$

Then there is an absolute constant B>0 so that for $|z-z_0|=r_0<r$ we get

$$|f'(z)|<\frac{BMr}{(r-r_0)^2}.$$

Lemma 5.2 [8, Theorem 5.10] *There exists an absolute constant $c>0$ such that for any primitive Dirichlet character χ modulo q, $L(s,\chi)$ has at most one zero in the region*

$$\sigma\ge 1-\frac{c}{\log q(|t|+3)}. \tag{5.1}$$

The exceptional zero may occur only if χ is real. In particular, for cyclic Galois extension K of degree $\ell>2$, $L(s,\rho_K)$ has no exceptional zero.

Using Lemmas 5.1 and 5.2, we give an unconditional bound for $\frac{L'}{L}(1,\rho_K)$.

Lemma 5.3 *We have*

$$\frac{L'}{L}(1,\rho_K) \ll_\ell (\log f(K))^3 . \tag{5.2}$$

Proof Let $\sigma_0 = 1 - \frac{c}{\log q_\chi(|t_0|+3)}$ for some $t_0 > 0$, where c is the constant in Lemma 5.2. By the known convexity bound for Dirichlet L-function [8, Theorem 5.23], there is an absoulte constant $C > 0$ such that

$$| L(1/2,\rho_K) | \ll \left(C^{\frac{4}{\ell-1}} f(K)(|t|+1) \right)^{\frac{\ell-1}{4}} .$$

Hence, Phragmén-Lindelöf principle implies that

$$| L(s,\rho_K) | \ll \left(C^{\frac{4}{\ell-1}} f(K)(|t|+1) \right)^{(\ell-1)\frac{3-2\sigma}{8}}$$

for $\frac{1}{2} < \sigma \leq \frac{3}{2}$, where $s = \sigma + it$. In particular, we have

$$\Re(\log L(s)) = \log |L(s)| \leq (\ell-1)\frac{3-\sigma}{8}\log f(K)(|t|+1) + \frac{3-2\sigma}{2}\log C$$

for $\frac{1}{2} < \sigma \leq \frac{3}{2}$.

Then there is a constant $C' > 0$ such that

$$\Re(\log L(s,\rho_K) - \log L(3/2,\rho_K)) \leq (\ell-1)\frac{3-\sigma}{8}\log f(K)(|t|+1) + \frac{3-2\sigma}{2}\log C'$$

for $\frac{1}{2} < \sigma \leq \frac{3}{2}$.

Let $r = \frac{3}{2} - \frac{\sigma_0+1}{2}$ and let $r_0 = \frac{3}{2} - \frac{\sigma_0+3}{4}$. By lemma 5.1 there is a constant $A > 0$ such that for $|s - \frac{3}{2}| = r_0 \leq r$,

$$\begin{aligned}\frac{L'}{L}(s) &< \frac{8(2-\sigma_0)}{(1-\sigma_0)^2} A \left((\ell-1)\frac{3-\sigma}{8}\log f(K)(|t|+1) + \frac{3-2\sigma}{2}\log C' \right) \\ &\ll_\ell (\log f(K)(|t_0|+3))^3\end{aligned}$$

for all $s \in [\sigma_0, 1] \times [-t_0, t_0]$. Taking $t_0 = 5$ we finish the proof. □

From now on, we assume that $L(s,\rho_K)$ has the zero-free region given in Lemma 3.1.

Lemma 5.4 *Suppose that $L(s, \rho_K) \neq 0$ when $s = \sigma + it \in [\alpha, 1] \times [-x, x]$, where $x = (\log X)^{\beta}$ and $\beta(1-\alpha) > 4$. Then we have, for $y = x^{\eta}$ with any $0 < \eta < 1$,*

$$\begin{aligned}\frac{L'}{L}(1, \rho_K) = &- \sum_{p \le y} \frac{\chi_{\rho_K}(p)\log p}{p} - \sum_{y < p \le x^2} \frac{\chi_{\rho_K}(p)\log p}{p} e^{-p/x} \\ &- \sum_{\substack{p^k \le x \\ k \ge 2}} \frac{\chi_{\rho_K}(p^k)\log p}{p^k} e^{-p^k/x} \\ &+ O_{\ell,\alpha,\beta}\left(\frac{1}{\sqrt{\log X}}\right). \end{aligned} \tag{5.3}$$

Proof Using the Mellin inversion of $\Gamma(s)$, we have

$$\frac{-1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{L'}{L}(1+s, \rho_K)\Gamma(s)x^s ds = \sum_p \sum_{i=1}^{\ell-1}\sum_{k=1}^{\infty} \log p \frac{\chi^i(p^k)}{p^k} e^{-p^k/x}. \tag{5.4}$$

For $k \ge 2$ we have

$$\begin{aligned}\sum_p \sum_{i=1}^{\ell-1}\sum_{k=2}^{\infty} \log p \frac{\chi^i(p^k)}{p^k} e^{-p^k/x} = &\sum_{\substack{p^k \le x \\ k \ge 2}} \frac{\chi_{\rho_K}(p^k)\log p}{p^k} e^{-p^k/x} \\ &+ \sum_{\substack{x < p^k \le x^2 \\ k \ge 2}} \frac{\chi_{\rho_K}(p^k)\log p}{p^k} e^{-p^k/x} \\ &+ O_\ell\left(\sum_{p^k \ge x^2, k \ge 2} \frac{\log p}{p^k} e^{-p^k/x}\right). \end{aligned}$$

For the error term above, we can handle it as below.

$$\begin{aligned}\sum_{\substack{p^k \ge x^2 \\ k \ge 2}} \frac{\log p^k}{p^k} e^{-p^k/x} &\le \sum_{n \ge x^2} \frac{\log n}{n} e^{-n/x} \\ &\le \int_{x^2}^{\infty} \frac{\log t}{t} e^{-t/x} dt \le \frac{2\log x}{x^2}\int_{x^2}^{\infty} e^{-t/x} dt \\ &\ll \frac{\log x}{x} e^{-x}. \end{aligned}$$

The second term is bounded by

$$\ll_\ell \sum_{\sqrt{x}<p\le x} \sum_{2\le k\le \frac{2\log x}{\log p}} \frac{\log p}{p^k} \ll_\ell \sum_{\sqrt{x}<p\le x} \frac{\log p}{p^2} \ll_\ell \frac{1}{x^{1/2}}.$$

For those terms with $k=1$, we apply the following two inequalities

$$\sum_{p\le y} \frac{\log p}{p}(1-e^{-p/x}) \ll \frac{1}{x}\sum_{p\le y}\log p \ll \frac{y}{x} = \frac{1}{x^{1-\eta}},$$

and

$$\begin{aligned}\sum_{p>x^2} \frac{\log p}{p} e^{-p/x} &= \int_{x^2}^{\infty} \frac{e^{-t/x}}{t} d\theta(t) \\ &= -\frac{\theta(x^2)}{x^2}e^{-x} - \int_{x^2}^{\infty} \theta(t)\left(-\frac{e^{-t/x}}{t^2} - \frac{e^{-t/x}}{tx}\right) dt \ll e^{-x},\end{aligned}$$

where we used Chebyshev's estimate $\theta(t)\le 2t$. Thus, (5.4) is equal

$$\begin{aligned}&\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{L'}{L}(1+s,\rho_K)\Gamma(s)x^s ds \\ &= -\sum_{p\le y} \frac{\chi_{\rho_K}(p)\log p}{p} - \sum_{y<p\le x^2} \frac{\chi_{\rho_K}(p)\log p}{p} e^{-p/x} \\ &\quad - \sum_{\substack{p^k\le x^2 \\ k\ge 2}} \frac{\chi_{\rho_K}(p^k)\log p}{p^k} e^{-p^k/x} + O_\ell\left(\frac{1}{x^{1-\eta}} + \frac{1}{x^{1/2}}\right). \qquad (5.5)\end{aligned}$$

Let T be a fixed positive real number sufficiently smaller than x. Then we move the contour to obtain

$$\begin{aligned}&\frac{1}{2\pi i}\int_{2-iT}^{2+iT} \frac{L'}{L}(1+s,\rho_K)\Gamma(s)x^s ds \\ &= \frac{L'}{L}(1,\rho_K) + \frac{1}{2\pi i}\int_{2-iT}^{\frac{\alpha+1}{2}-1-iT} \frac{L'}{L}(1+s,\rho_K)\Gamma(s)x^s ds \\ &\quad + \frac{1}{2\pi i}\int_{\frac{\alpha+1}{2}-1-iT}^{\frac{\alpha+1}{2}-1+iT} \frac{L'}{L}(1+s,\rho_K)\Gamma(s)x^s ds \\ &+ \frac{1}{2\pi i}\int_{\frac{\alpha+1}{2}-1+iT}^{2+iT} \frac{L'}{L}(1+s,\rho_K)\Gamma(s)x^s ds.\end{aligned}$$

From the proof of Lemma 5.3, we have, for $\frac{1}{2} \le \Re(s) \le \frac{3}{2}$,

$$L(s, \rho_K) \ll_\ell (f(X)(|s|+1))^{\frac{\ell-1}{4}}.$$

Then, for $\frac{1}{2} < \Re(s) \le \frac{3}{2}$

$$\Re(\log L(s, \rho_K)) = \log |L(s, \rho_K)| \ll_\ell \log(f(K)(|s|+1)),$$

and for $\Re(s) > \frac{3}{2}$, trivially we have

$$\log |\, L(s, \rho_K)\, | = O_\ell(1).$$

Thus, we have

$$\Re(\log L(s, \rho_K) - \log L(2+it, \rho_K)) \ll_\ell \log(f(K)(|s|+1))$$

for $\Re(s) > \frac{1}{2}$.

By assumption on the zero-free region, for any $\varepsilon > 0$, $\log L(s, \rho_K)$ is holomorphic in $|s-2-it| < r := 2-\alpha$ for $|t| \le T$. Applying Lemma 5.1 with $r_0 = 2 - \frac{\alpha+1}{2}$ to $\log L(s, \rho_K)$, we have

$$|\frac{L'}{L}(s, \rho_K)| \ll_\ell \log f(K)(|s|+1) \tag{5.6}$$

for $|s-2-it| \le 2 - \frac{\alpha+1}{2}$ with $|t| \le T$, and we conclude that (5.6) holds for all $s \in [\frac{\alpha+1}{2}, 1] \times [-T, T]$.

Hence, by (5.6) and the estimate $\Gamma(\frac{\alpha+1}{2} - 1 + it) \ll \frac{1}{1-\frac{\alpha+1}{2}}$,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\frac{\alpha+1}{2}-1-iT}^{\frac{\alpha+1}{2}-1+iT} \frac{L'}{L}(1+s, \rho_K)\Gamma(s)x^s ds &\ll_\ell \frac{x^{\frac{\alpha+1}{2}-1}}{1-\frac{\alpha+1}{2}} \int_0^T \log\left(f(K)(t+2)\right)dt \\ &\ll_{\ell,\alpha} x^{\frac{\alpha+1}{2}-1} T \log\left(f(K)(T+2)\right). \end{aligned}$$

For $s = \sigma \pm iT$ with $\frac{\alpha+1}{2} - 1 \le \sigma \le 2$, by (5.6) and Stirling formula, we have

$$\frac{L'}{L}(1+s, \rho_K) \ll_\ell \log f(K)(|T|+2), \quad \Gamma(s) \ll T^{-1} e^{-\frac{T}{2}}.$$

Then

$$\frac{1}{2\pi i} \int_{\frac{\alpha}{2}-1+\pm iT}^{2\pm iT} \frac{L'}{L}(1+s, \rho_K)\Gamma(s)x^s ds \ll_\ell T^{-1} e^{-\frac{T}{2}} \log(f(K)(T+2))x^2.$$

Thus, we combine the computations above to get

$$\begin{aligned}&\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{L'}{L}(1+s,\rho_K)\Gamma(s)x^s ds\\&\quad=\frac{L'}{L}(1,\rho_K)+O_{\ell,\alpha}(T^{-1}e^{-\frac{T}{2}}\log(f(K)(T+2))x^2\\&\qquad+x^{\frac{\alpha+1}{2}-1}T\log(f(K)(T+2))).\end{aligned}$$

Since

$$\frac{1}{2\pi i}\int_{2\pm iT}^{2\pm i\infty}\frac{L'}{L}(1+s,\rho_K)\Gamma(s)x^s ds\ll x^2\int_T^\infty e^{-t/2}dt\ll x^2e^{-T/2}.$$

by the Stirling formula $\Gamma(s)\ll e^{-|t|/2}$, we have

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(2)}\frac{L'}{L}(1+s,\rho_K)\Gamma(s)x^s ds\\&\quad=\frac{L'}{L}(1,\rho_K)+O_{\ell,\alpha}\left(x^2e^{-\frac{T}{2}}\left[T^{-1}\log(f(K)(T+2))+1\right]\right)\\&\qquad+x^{\frac{\alpha+1}{2}-1}T\log(f(K)(T+2))).\end{aligned}\tag{5.7}$$

Let us take $T=x^{\frac{1}{2\beta}}$. Since $\beta(1-\alpha)>4$, (5.7) is

$$\frac{1}{2\pi i}\int_{(2)}\frac{L'}{L}(1+s,\rho_K)\Gamma(s)x^s ds=\frac{L'}{L}(1,\rho_K)+O_{\ell,\alpha,\beta}\left(\frac{1}{\sqrt{\log X}}\right).\tag{5.8}$$

Combining (5.5) and (5.8) completes the proof. □

To prove Theorem 1.3, only weaker result than Lemma 5.4 is sufficient. By slightly modifying the proof of Lemma 5.4 we can easily have the following result.

Lemma 5.5 *Assume the condition as in Lemma 5.4. Then we obtain*

$$\frac{L'}{L}(1,\rho_K)=-\sum_{p\le x}\frac{\chi_{\rho_K}(p)\log p}{p}+O_\ell(1).\tag{5.9}$$

Proof The proof of Lemma 5.4 shows that

$$\sum_p\sum_{i=1}^{l-1}\sum_{k=2}^{\infty}\log p\frac{\chi^i(p^k)}{p^k}e^{-p^k/x}=\sum_{i=1}^{l-1}\sum_{\substack{p^k\le x^2\\k\ge 2}}\frac{\chi^i(p^k)\log p}{p^k}e^{-p^k/x}+O_\ell(1).$$

For the first summation we have

$$\sum_{i=1}^{l-1}\sum_{\substack{p^k\le x^2\\ k\ge 2}}\frac{\chi^i(p^k)\log p}{p^k}e^{-p^k/x}\ll_\ell \sum_{p\le x}\sum_{k\ge 2}\frac{\log p}{p^k}\ll\sum_{p\le x}\frac{\log p}{p^2}\ll 1.$$

Thus we obtain

$$\sum_p\sum_{i=1}^{l-1}\sum_{k=2}^{\infty}\log p\frac{\chi^i(p^k)}{p^k}e^{-p^k/x}\ll_\ell 1.$$

We note that

$$\sum_p\sum_{i=1}^{l-1}\log p\frac{\chi^i(p)}{p}e^{-p/x}=\sum_{p\le x}\frac{\chi_{\rho_K}(p)\log p}{p}-\sum_{p\le x}\frac{\chi_{\rho_K}(p)\log p}{p}\left(1-e^{-p/x}\right)$$
$$+\sum_{p>x}\frac{\chi_{\rho_K}(p)\log p}{p}e^{-p/x}.$$

From the following two inequalities

$$\sum_{p\le x}\frac{\chi_{\rho_K}(p)\log p}{p}\left(1-e^{-p/x}\right)\ll_\ell\frac{1}{x}\sum_{p\le x}\log p\ll 1$$

and

$$\sum_{p>x}\frac{\chi_{\rho_K}(p)\log p}{p}e^{-p/x}\ll_\ell x\sum_{p>x}\frac{\log p}{p^2}\ll x\int_x^\infty\frac{1}{t^2}dt=1,$$

the claim follows. □

6 Proof of Theorem 1.5

In order to prove Theorem 1.5 it suffice to estimate

$$\frac{1}{N(\mathbb{Z}/\ell\mathbb{Z},X)}\sum_{K\in F(\mathbb{Z}/\ell\mathbb{Z},X)}\frac{L'}{L}(1,\rho_K)$$
$$=\frac{1}{N(\mathbb{Z}/\ell\mathbb{Z},X)}\left(\sum_{K\in F(\mathbb{Z}/\ell\mathbb{Z},X)}{}^{'}+\sum_{K\in F(\mathbb{Z}/\ell\mathbb{Z},X)}{}^{''}\right)\frac{L'}{L}(1,\rho_K),$$

where $\sum'$ runs over the corresponding L-function of K has the zero-free region in Lemma 5.4, and $\sum''$ runs over the corresponding L-function of K which might not have the zero-free region.

We take α such that $\alpha > \frac{9}{11}$ so that $\frac{7}{2} \cdot \frac{1-\alpha}{2\alpha-1} < 1$. Then, as explained in Section 4, except for the $O((\log X)^{\beta B} X^{\frac{7}{2}\cdot\frac{1-\alpha}{2\alpha-1}})$ L-functions in $L(X)$, every L-function in $L(X)$ is zero-free in $[\alpha, 1] \times [-(\log X)^{\beta}, (\log X)^{\beta}]$, where $\beta(1-\alpha) > 4$.

Using the trivial bound in Lemma 5.3, we can estimate the second sum as follows:

$$\sum''_{K\in F(\mathbb{Z}/\ell\mathbb{Z},X)} \frac{L'}{L}(1,\rho_K) \ll_\ell (\log X)^{3+\beta B} X^{(\frac{5\ell}{2}+1)\frac{1-\alpha}{2\alpha-1}}. \tag{6.1}$$

Now, it is left to compute the first sum. Let $x = (\log X)^{\beta}$ and let $y = x^{\eta}$ for any $0 < \eta < 1$. By Lemma 5.4, the computation of the first sum is reduced into the estimations of the following three summations:

$$\sum'_{K\in F(\mathbb{Z}/\ell\mathbb{Z},X)} \sum_{p\le y} \frac{\chi_{\rho_K}(p)\log p}{p}, \quad \sum'_{K\in F(\mathbb{Z}/\ell\mathbb{Z},X)} \sum_{y\le p\le x^2} \frac{\chi_{\rho_K}(p)\log p}{p} e^{-p/x},$$

and

$$\sum'_{K\in F(\mathbb{Z}/\ell\mathbb{Z},X)} \sum_{\substack{p^k\le x\\ k\ge 2}} \frac{\chi_{\rho_K}(p^k)\log p}{p^k} e^{-p^k/x}.$$

By Proposition 2.2 and Chevbyshev's estimate, the first summation satisfies

$$\begin{aligned}\sum'_{K\in F(\mathbb{Z}/\ell\mathbb{Z},X)} \sum_{p\le y} \frac{\chi_{\rho_K}(p)\log p}{p} &\ll_{\ell,\varepsilon} X^{\frac{\ell+2}{\ell+5}+\varepsilon} \sum_{p\le(\log X)^{\beta\eta}} p^{\frac{\ell-7}{6}+\varepsilon}\log p\\ &\ll_{\ell,\varepsilon} X^{\frac{\ell+2}{\ell+5}+\varepsilon}(\log X)^{\beta\eta(\frac{\ell-1}{6}+\varepsilon)},\end{aligned}$$

and the second summation satisfies

$$\begin{aligned}\sum'_{K\in F(\mathbb{Z}/\ell\mathbb{Z},X)} \sum_{y\le p\le x^2} \frac{\chi_{\rho_K}(p)\log p}{p} e^{-p/x} &\ll_{\ell,\varepsilon} X^{\frac{\ell+2}{\ell+5}+\varepsilon} \sum_{y\le p\le x^2} p^{\frac{\ell-7}{6}+\varepsilon}\log p\\ &\ll_{\ell,\varepsilon} X^{\frac{\ell+2}{\ell+5}+\varepsilon}(\log X)^{(\frac{\ell-1}{3}+2\varepsilon)\beta}.\end{aligned}$$

Now, we estimate the third summation. Since $\chi_{\rho_K}(p^k) = \chi_K(p^k) + \chi_K^2(p^k) + \cdots + \chi_K^{\ell-1}(p^k)$ where χ_K is a primitive Dirichlet characters associated with the field K, Proposition 2.2 says, for $p^k < x$,

$$\sum'_{K\in F(\mathbb{Z}/\ell\mathbb{Z},X)} \chi_{\rho_K}(p^k) = (\ell-1)J(p^k)\ell|TS_p|\mathfrak{C}_\ell X + O_{\ell,\varepsilon}\left(p^{\frac{\ell-1}{6}+\varepsilon} X^{\frac{\ell+2}{\ell+5}+\varepsilon}\right). \tag{6.2}$$

Then, the contribution of the error term in (6.2) to the third summation is

$$X^{\frac{\ell+2}{\ell+5}+\varepsilon} \sum_{p^k \le x, k \ge 2} p^{\frac{\ell-1}{6}-k+\varepsilon} \log p \ll_\ell X^{\frac{\ell+2}{\ell+5}+\varepsilon} \sum_{p \le \sqrt{x}} p^{\frac{\ell-1}{6}-2+\varepsilon} \log p$$
$$\ll_{\ell,\varepsilon} X^{\frac{\ell+2}{\ell+5}+\varepsilon} (\log X)^{\beta(\frac{\ell-7}{6}+\varepsilon)}.$$

Now, we compute the contribution of the main term in (6.2) to the third summation. For the prime powers p^k in the range $p^k < x$, we have $e^{-\frac{p^k}{x}} = 1 + O\left(\frac{p^{2k}}{x^2}\right)$, which is plugged into the third summation; then it becomes

$$(\ell-1)\mathfrak{C}_\ell X \sum_{p^{\ell m} \le x} \frac{\ell |TS_p| \log p}{p^{\ell m}} \left(1 + O\left(\frac{p^{2\ell m}}{x^2}\right)\right),$$

where the error term contribution is $\ll_\ell \frac{X}{x} \sum_{p \le x^{1/\ell}} \log p \sum_{m \le \frac{\log x}{\ell \log p}} 1 \ll_\ell \frac{X \log\log X}{(\log X)^{\beta(1-\frac{1}{\ell})}}$.

Then the main term contribution is

$$(\ell-1)\mathfrak{C}_\ell X \sum_{p \le x^{1/\ell}} \ell |TS_p| \frac{\log p}{p^\ell - 1} + O_\ell\left(\frac{X}{x} \sum_{p \le x^{1/\ell}} \frac{\log p}{p^\ell - 1}\right) \tag{6.3}$$
$$= (\ell-1)\mathfrak{C}_\ell X \sum_{p \le x^{1/\ell}} \ell |TS_p| \frac{\log p}{p^\ell - 1} + O_\ell\left(\frac{X}{x}\right)$$
$$= (\ell-1)\mathfrak{C}_\ell X \sum_{p} \ell |TS_p| \frac{\log p}{p^\ell - 1} + O_{\ell,\varepsilon}\left(\frac{X}{(\log X)^{\beta(1-1/\ell)}}\right).$$

After combining (6.1) and (6.3) with error terms we conclude that

$$\sum_{K \in F(\mathbb{Z}/\ell\mathbb{Z}, X)} \frac{L'}{L}(1, \rho_K) = (\ell-1)\mathfrak{C}_\ell X \sum_p \ell |TS_p| \frac{\log p}{p^\ell - 1} + O_{\ell,\varepsilon}\left(\frac{X \log\log X}{(\log X)^{\beta(1-1/\ell)}}\right),$$

and Theorem 1.5 follows.

7 Proof of Theorem 1.3

Now, we find the upper and lower bounds of the Euler–Kronecker constant for the number fields $K \in F(\mathbb{Z}/\ell\mathbb{Z}, X)$. For this purpose, we need the following technical lemma, which is an analogue of Proposition 4.3.

Proposition 7.1 *Let* $y = c_1 \log X$, $r \le c_2 \frac{\log X}{\log\log X}$ *for some constants* $c_1, c_2 > 0$. *Then we have*

$$\sum_{L(s,\rho_K)\in L(X)} \left(\sum_{y\le p\le x} \frac{\chi_{\rho_K}(p)\log p}{p} \right)^{2r} \ll_\ell 2^{6r-1}(\ell-1)^{4r}\frac{(2r)!}{r!}\left(\frac{\log y}{y}\right)^r X. \tag{7.1}$$

Proof We use the similar argument of the proof of Proposition 4.3. By the multinomial formula, we obtain

$$\sum_{L(s,\rho_K)\in L(X)} \left(\sum_{y\le p\le x} \frac{\chi_{\rho_K}(p)\log p}{p} \right)^{2r} = \sum_{u=1}^{2r}\frac{1}{u!}\sum_{r_1+\cdots+r_u=2r \, r_1,\ldots,r_u\ge 1}\frac{(2r)!}{r_1!\cdots r_u!}$$
$$\cdot \sum_{\substack{y\le p_1,\ldots,p_u\le x\\ p_i\ne p_j}} \frac{(\log p_1)^{r_1}\cdots(\log p_u)^{r_u}}{p_1^{r_1}\cdots p_u^{r_u}} \sum_{L(s,\rho_K)\in L(X)} \chi_{\rho_K}^{r_1}(p_1)\cdots\chi_{\rho_K}^{r_u}(p_u).$$

Suppose that $r_i \ge 2$ for all i. Then Chebyshev's estimate shows that

$$\sum_{y\le p\le x}\frac{(\log p)^{r_i}}{p^{r_i}} \le \int_y^\infty \frac{(\log t)^{r_i-1}}{t^{r_i}} d\left(\sum_{p\le t}\log p\right) \ll \frac{2(\log y)^{r_i-1}}{y^{\frac{1}{2}}}\int_y^\infty \frac{1}{t^{r_i-1/2}}dt$$
$$\ll 2\left(\frac{\log y}{y}\right)^{r_i-1}.$$

Then, by using the fact that $L(X) \asymp_\ell X$ and the above inequality, we have

$$\sum_{\substack{y\le p_1,\ldots,p_u\le x\\ p_i\ne p_j}} \frac{(\log p_1)^{r_1}\cdots(\log p_u)^{r_u}}{p_1^{r_1}\cdots p_u^{r_u}} \sum_{L(s,\rho_K)\in L(X)} \chi_{\rho_K}^{r_1}(p_1)\cdots\chi_{\rho_K}^{r_u}(p_u)$$
$$\ll_\ell (\ell-1)^{2r} X \left(\sum_{y\le p\le x}\frac{(\log p)^{r_1}}{p^{r_1}}\right)\cdots\left(\sum_{y\le p\le x}\frac{(\log p)^{r_u}}{p^{r_u}}\right)$$
$$\ll 2^{2r}(\ell-1)^{2r}X\left(\frac{\log y}{y}\right)^r\left(\frac{\log y}{y}\right)^{r-u}.$$

Then, by the similar argument of proof of Proposition 4.3 we obtain

$$\frac{1}{u!}\frac{(2r)!}{r_1!\cdots r_u!}\sum_{\substack{y\le p_1,\ldots,p_u\le x\\ p_i\ne p_j}} \frac{(\log p_1)^{r_1}\cdots(\log p_u)^{r_u}}{p_1^{r_1}\cdots p_u^{r_u}} \sum_{L(s,\rho_K)\in L(X)} \chi_{\rho_K}^{r_1}(p_1)\cdots\chi_{\rho_K}^{r_u}(p_u)$$
$$\ll_\ell 2^{2r}(\ell-1)^{2r}\frac{(2r)!}{r!}X\left(\frac{\log y}{y}\right)^r. \tag{7.2}$$

Next, suppose that $r_i = 1$ for some i. Then we may assume that $r_1+\cdots+r_m+r_{m+1}+\cdots+r_u = 2r$ with $r_1=\cdots=r_m=1$ and $r_{m+1},\ldots,r_u\ge 2$. If one of the p_i

is ramified, say p_1, then $\chi_{\rho_K}(p_1) = 0$. Thus we may also assume that $p_1, \ldots, p_u$ are all unramified. Then, the same argument in the proof of Propsition 4.3 shows that the main term vanishes, and so

$$\begin{aligned} &\frac{1}{u!}\frac{(2r)!}{r_1!\cdots r_u!}\sum_{\substack{y\le p_1,\ldots,p_u\le x\\ p_i\neq p_j}}\frac{(\log p_1)^{r_1}\cdots(\log p_u)^{r_u}}{p_1^{r_1}\cdots p_u^{r_u}}\sum_{L(s,\rho_K)\in L(X)}\chi_{\rho_K}^{r_1}(p_1)\cdots\chi_{\rho_K}^{r_u}(p_u)\\ &\quad\ll 2^{2r}(\ell-1)^{4r}\frac{1}{u!}\frac{(2r)!}{r_1!\cdots r_u!}\Bigg(X^{\frac{\ell+2}{\ell+5}+\varepsilon}\Bigg(\sum_{y\le p_1\le x}p_1^{\frac{\ell-7}{6}+\varepsilon}\log p_1\\ &\quad\cdots\sum_{y\le p_m\le x}p_m^{\frac{\ell-7}{6}+\varepsilon}\log p_m\sum_{y\le p_{m+1}\le x}p_{m+1}^{\frac{\ell-1}{6}-r_{m+1}+\varepsilon}\\ &\quad(\log p_{m+1})^{r_m}\cdots\sum_{y\le p_u\le x}p_u^{\frac{\ell-1}{6}-r_u+\varepsilon}(\log p_u)^{r_u}\Bigg)+X^{\frac{1}{2}+\varepsilon}\Bigg(\sum_{y\le p_1\le x}p_1^{\frac{\ell-5}{4}}\log p_1\\ &\quad\cdots\sum_{y\le p_m\le x}p_m^{\frac{\ell-5}{4}}\log p_m\\ &\sum_{y\le p_{m+1}\le x}p_{m+1}^{\frac{\ell-1}{4}-r_{m+1}}(\log p_{m+1})^{r_{m+1}}\cdots\sum_{y\le p_u\le x}p_u^{\frac{\ell-1}{4}-r_u}(\log p_u)^{r_u}\Bigg)\Bigg). \end{aligned} \tag{7.3}$$

We have the following inequalities, for $r \geq 2$,

$$\begin{aligned} \sum_{y\le p\le x}p^{\frac{\ell-7}{6}+\varepsilon}\log p &\ll \int_1^x t^{\frac{l-7}{6}+\varepsilon}d\Bigg(\sum_{p\le t}\log p\Bigg)\ll_\ell 2x^{\frac{l-1}{6}+\varepsilon},\ \sum_{y\le p\le x}p^{\frac{\ell-1}{6}+\varepsilon-r}(\log p)^r\\ &\ll \sum_{y\le p\le x}p^{\frac{\ell-7}{6}+\varepsilon}\log p\ll_\ell 2x^{\frac{\ell-1}{6}+\varepsilon},\ \sum_{y\le p\le x}p^{\frac{\ell-5}{4}}\log p\ll\int_1^x t^{\frac{\ell-5}{4}}d\Bigg(\sum_{p\le t}\log p\Bigg)\\ &\ll_\ell 2x^{\frac{\ell-1}{4}},\ \sum_{y\le p\le x}p^{\frac{\ell-1}{4}-r}(\log p)^r\\ &\ll \sum_{y\le p\le x}p^{\frac{\ell-5}{4}}\log p\ll_\ell 2x^{\frac{\ell-1}{4}}. \end{aligned}$$

Then (7.3) is

$$\begin{aligned} &\ll_\ell 2^{4r}(\ell-1)^{4r}\frac{1}{u!}\frac{(2r)!}{r_1!\cdots r_u!}\left(X^{\frac{\ell+2}{\ell+5}+\varepsilon}x^{(\frac{\ell-1}{6}+\varepsilon)u}+X^{\frac{1}{2}+\varepsilon}x^{\frac{\ell-1}{4}u}\right)\\ &\ll 2^{4r}(\ell-1)^{4r}\frac{(2r)!}{r!}y^m\left(\frac{y}{\log y}\right)^{r-u}\left(X^{\frac{\ell+2}{\ell+5}+\varepsilon}x^{(\frac{\ell-1}{6}+\varepsilon)u}+X^{\frac{1}{2}+\varepsilon}x^{\frac{\ell-1}{4}u}\right)\\ &\ll 2^{4r}(\ell-1)^{4r}\frac{(2r)!}{r!}(y\log y)^r\left(X^{\frac{\ell+2}{\ell+5}+\varepsilon}(\log X)^{(\frac{\ell-1}{6}+\varepsilon)\beta u}+X^{\frac{1}{2}+\varepsilon}(\log X)^{\frac{\ell-1}{4}\beta u}\right)\\ &\ll 2^{4r}(\ell-1)^{4r}\frac{(2r)!}{r!}(\log y)^r\left(X^{\frac{\ell+2}{\ell+5}+\varepsilon}(\log X)^{(\frac{\ell-1}{3}\beta+2\beta\varepsilon+1)r}+X^{\frac{1}{2}+\varepsilon}(\log X)^{(\frac{\ell-1}{2}\beta+1)r}\right). \end{aligned}$$

Thus, in this case we have

$$\frac{1}{u!}\frac{(2r)!}{r_1!\cdots r_u!}\sum_{\substack{y\le p_1,\dots,p_u\le x\\ p_i\neq p_j}}\frac{(\log p_1)^{r_1}\cdots(\log p_u)^{r_u}}{p_1^{r_1}\cdots p_u^{r_u}}\sum_{L(s,\rho_K)\in L(X)}\chi_{\rho_K}^{r_1}(p_1)\cdots\chi_{\rho_K}^{r_u}(p_u)$$
$$\ll_\ell 2^{4r}(\ell-1)^{4r}\frac{(2r)!}{r!}(\log y)^r\left(X^{\frac{\ell+2}{\ell+5}+\varepsilon}(\log X)^{(\frac{\ell-1}{3}\beta+2\beta\varepsilon+1)r}+X^{\frac{1}{2}+\varepsilon}(\log X)^{(\frac{\ell-1}{2}\beta+1)r}\right).$$

If we take $c_2<\min\left(\frac{1}{((\ell-1)\beta/3+2\beta\varepsilon+2)}(\frac{3}{\ell+5}-\varepsilon),\ \frac{1}{(\ell-1)\beta/2+2}(\frac{1}{2}-\varepsilon)\right)$, then

$$\frac{1}{u!}\frac{(2r)!}{r_1!\cdots r_u!}\sum_{\substack{y\le p_1,\dots,p_u\le x\\ p_i\neq p_j}}\frac{(\log p_1)^{r_1}\cdots(\log p_u)^{r_u}}{p_1^{r_1}\cdots p_u^{r_u}}\sum_{L(s,\rho_K)\in L(X)}\chi_{\rho_K}^{r_1}(p_1)\cdots\chi_{\rho_K}^{r_u}(p_u)$$
$$\ll_\ell 2^{4r}(\ell-1)^{4r}\frac{(2r)!}{r!}X\left(\frac{\log y}{y}\right)^r. \tag{7.4}$$

Combining (7.2) and (7.4) we get the desired one

$$\sum_{L(s,\rho_K)\in L(X)}\left(\sum_{y\le p\le x}\frac{\chi_{\rho_K}(p)\log p}{p}\right)^{2r}\ll 2^{6r-1}(\ell-1)^{4r}\frac{(2r)!}{r!}\left(\frac{\log y}{y}\right)^r X.$$

□

Note that we have, for $r\le c_2\frac{\log X}{\log\log X}$, some constant $c, c_2>0$,

$$\sum_{L(s,\rho_K)\in L(X)}\left(\sum_{y\le p\le x}\frac{\chi_{\rho_K}(p)\log p}{p}\right)^{2r}\ll\left(c(\ell-1)^4 r\frac{\log y}{y}\right)^r X$$

by the Stirling formula and Proposition 7.1. Take $r=c_2\frac{\log X}{\log\log X}$, and then by the proof of Corollary 4.4, except for $O(Xe^{-c'\frac{\log X}{\log\log X}\log\log\log X})$ L-functions in $L(X)$, we have that

$$\sum_{y\le p\le x}\frac{\chi_{\rho_K}(p)\log p}{p}<(\log\log f(K))^{\frac{1}{2}}.$$

For such a L-function $L(s,\rho_K)$, by Lemma 5.5,

$$\frac{L'}{L}(1,\rho_K)=-\sum_{p\le y}\frac{\chi_{\rho_K}(p)\log p}{p}+O_\ell((\log\log f(K))^{\frac{1}{2}}).$$

Since $-1 \le \chi_{\rho_K}(p) \le \ell - 1$ for all primes p, Mertens' theorem shows that the upper and lower bounds for $\frac{L'}{L}(1, \rho_K)$ are

$$\log\log f(K) + O((\log\log f(K))^{\frac{1}{2}}), \qquad -(\ell-1)\log\log f(K) + O((\log\log f(K))^{\frac{1}{2}}).$$

8 Proof of Theorems 1.2 and 1.4

Since we know when $L(1, \rho_K)$ and $\frac{L'}{L}(1, \rho_K)$ respectively attain their maximum and minimum, Theorem 1.2 and Theorem 1.4 directly follow from Proposition 2.1 .

Indeed, let $y = c \log X$ for some constant $c > 0$. By Proposition 2.1 we have

$$N(\mathbb{Z}/\ell\mathbb{Z},\ X,\ \mathcal{LC}) = \prod_{LC_p \in S} |LC_p| \mathfrak{C}_\ell X + O_{\ell,\varepsilon}\Big(2^{|\mathcal{LC}|}(\ell-1)^I(\ell+1)^R[(\prod_{p\in S} p)^{\frac{\ell-1}{6}+\varepsilon} X^{\frac{\ell+2}{\ell+5}+\varepsilon} + (\prod_{p\in S} p)^{\frac{\ell-1}{4}} X^{\frac{1}{2}+\varepsilon}]\Big). \tag{8.1}$$

Let $\mathcal{LC}_1 = \{TS_p : p \le y\}$ and let $\mathcal{LC}_2 = \{IN_p : p \le y\}$. Then, (8.1) implies that the main term of $N(\mathbb{Z}/\ell\mathbb{Z},\ X,\ \mathcal{LC}_1)$ is

$$\gg_\ell X \exp\left(-\log \ell \frac{\log X}{\log\log X}\right) \prod_{p \le y}\left(1 - \frac{1}{p}\right)^{\ell-1} \gg_\ell \frac{X}{(\log\log X)^{\ell-1}} \exp\left(-\log \ell \frac{\log X}{\log\log X}\right),$$

and the main term of $N(\mathbb{Z}/l\mathbb{Z},\ X,\ \mathcal{LC}_2)$ is

$$\gg_\ell \frac{X}{(\log\log X)^{\ell-1}} \exp\left(-\log \frac{\ell}{\ell-1} \frac{\log X}{\log\log X}\right)$$

because $|IN_p| = (\ell-1)|TS_p|$. Thus, the number of cyclic extensions K in $F(\mathbb{Z}/\ell\mathbb{Z}, X)$ for which

$$L(1, \rho_K) = (e^\gamma \log\log f(K))^{\ell-1}\left(1 + O\left(\frac{1}{\log\log f(K)}\right)\right)$$

and

$$\frac{L'}{L}(1, \rho_K) = -(\ell-1)\log\log f(K) + O((\log\log f(K))^{\frac{1}{2}})$$

is

$$\gg_\ell X e^{-\log \ell \frac{\log X}{\log\log X} - \log \ell \log\log\log X} - X e^{-c' \frac{\log X}{\log\log X}\log\log\log X}$$
$$\asymp X e^{-\log \ell \frac{\log X}{\log\log X} - \log \ell \log\log\log X},$$

and the number of cyclic extension K in $F(\mathbb{Z}/\ell\mathbb{Z}, X)$ for which

$$L(1, \rho_K) = \frac{\zeta(\ell)}{e^{\gamma} \log\log f(K)} \left(1 + O\left(\frac{1}{\log\log f(K)}\right)\right)$$

and

$$\frac{L'}{L}(1, \rho_K) = \log\log f(K) + O((\log\log f(K))^{\frac{1}{2}})$$

is

$$\gg_\ell Xe^{-\log\frac{\ell}{\ell-1}\frac{\log X}{\log\log X} - \log\ell \log\log\log X} - Xe^{-c'\frac{\log X}{\log\log X}\log\log\log X}$$
$$\asymp Xe^{-\log\frac{\ell}{\ell-1}\frac{\log X}{\log\log X} - \log\ell\log\log\log X}.$$

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Declarations

Conflict of interest The authors declare no Conflict of interest.

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