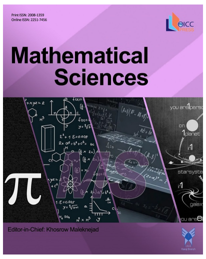

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Efficient Algorithms for the Solution of Linear and Nonlinear Caputo-Tempered Variable Order Partial Differential Equations

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Abstract:
This study introduces a new wavelet framework, referred to as the tempered fractional Gegenbauer wavelet (TFGW), for the numerical solution of tempered variable-order differential equations. We construct the TFGW operational matrices for tempered variable-order integration and develop the TFGW method to efficiently solve Caputo-tempered variable-order ordinary and boundary value problems. To further enhance computational efficiency for partial differential equations involving both time-fractional and spatial variable-order derivatives, we propose the L1-TFGW method based on the L1 approximation and the fast TFGW method, which incorporates a fast algorithm for fractional time derivatives in combination with the TFGW approach for spatial operators. For nonlinear problems, a fast-quasi TFGW method is devised by coupling quasilinearization with the fast TFGW strategy. The orthonormality of TFGW is established, and corresponding operational matrices are derived, including those tailored for boundary value problems. Error analyses are provided, and extensive numerical simulations demonstrate the accuracy, efficiency, and robustness of the proposed methods. The results confirm that the TFGW-based techniques offer a reliable and effective computational framework for linear and nonlinear Caputo-tempered variable-order models. To the best of our knowledge, this is the first work to introduce such wavelet-based fast algorithms for tempered fractional and spatial variable-order differential equations, providing a valuable tool for scientists and engineers dealing with complex multiscale fractional dynamics.

Keywords: Fast algorithm; Tempered fractional Gegenbauer wavelet; Tempered variable order differential equations; Operational matrices; Error analysis; Quasilinearization.

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1. Introduction

Fractional differential equations describe systems with non-integer order derivatives, helping us understand complex things better in physics, engineering, biology, and finance. Unlike classical equations, fractional derivatives capture long-range dependencies and nonlocal effects. Solving these equations is crucial for accurately modeling real-world processes and gaining insights into systems with intricate behaviors and memory effects. There are many definitions for fractional integrals and derivatives in the literature with singular and nonsingular kernels such as Hadamard [1], Katugampola [2, 3], Caputo [4], Reisz [5], Riemann–Liouville [6] and Tempered [7, 8]. It is difficult to find exact solutions to fractional differential equations, therefore numerous numerical and analytical methods have been developed, such as the shifted Legendre polynomial method, which has been applied to fractional advection–diffusion equations [9]. Multi-term time-fractional advection–diffusion models have been solved using a combination of Lagrange squared interpolation for temporal discretization and shifted Legendre polynomials for spatial discretization [10]. Convergence analyzes have been performed for both classical and tempered fractional Black–Scholes models [11, 12]. For nonlinear problems, efficient techniques have been developed for fractional Burgers–Fisher models [13, 14]. Meshless Kansa RBF methods have also been employed to study the fractional Rayleigh–Stokes equation [15].

Variable order differential equations represent a distinctive category within applied mathematics, wherein the equation's order dynamically changes based on specific parameters or functions involved. This departure from the conventional fixed-order differentials offers a more adaptable modeling framework, particularly suited for complex systems characterized by evolving dynamics. Widely applicable across disciplines like physics, engineering, and biology, variable order equations enable a more precise depiction of nonlinear phenomena or systems experiencing abrupt changes. Their study demands advanced mathematical techniques for analysis and solution, rendering them a compelling focal point within applied mathematics research.

It is somewhat surprising that tempered fractional differential equations have received relatively less attention compared to other types of fractional differential equations. We have identified and reviewed several papers from the literature related to tempered calculus. Authors discussed the Ulam-type stability and existence of the approximate solution for the variable order ψ tempered differential equation in [16]. In [17], authors developed a method, with the help of the generalized Lagrange interpolation polynomials, for the solution of tempered fractional differential equations. Legendre polynomial based method is developed in [18] for the solution of multiorder tempered fractional differential equations. A fast finite difference scheme is constructed in [19] by using SoE-approximations [20, 21] for a tempered fractional Burgers equation in porous media. A numerical method of order 2 is proposed in [22] for the approximate solution of tempered fractional diffusion equations. In [23], authors utilized the spectral collocation method for the solution of tempered fractional differential equations.

Wavelets serve as crucial mathematical tools in applied mathematics, finding widespread applications in signal and data analysis across various domains such as image processing, compression, and data analysis. Unlike Fourier analysis, which decomposes signals into sinusoidal functions, wavelets offer localized representations in both time and frequency domains, making them adept at capturing localized features and abrupt changes in signals. Moreover, their utility extends to solving differential equations, image reconstruction, and addressing inverse problems, making substantial contributions to computational mathematics and scientific computing. In the realm of applied mathematics, wavelets play a pivotal role in driving innovation and research, facilitating deeper insights into complex phenomena and enhancing our capacity to analyze and interpret real-world data through their inherent adaptability and efficiency. Multiple wavelets are employed in solving fractional differential equations. Among these wavelets are the Haar wavelet [24, 25], Legendre wavelet [26], Chebyshev wavelet [27], Gegenbauer wavelet [28, 29], Daubechies wavelet [30], Sine-cosine wavelet [31] and CAS wavelet [32].

In this paper, there are three main objectives: the first objective is to introduce the TFGW and the TFGW operational matrices method for the solution of Caputo-tempered variable order differential equation. For this purpose, we have constructed the TFGW matrix and its operational matrices of tempered variable order integration. The second main objective is to introduce the fast TFGW method for the solution of Caputo-tempered partial differential equations featuring both time fractional and spatial variable order derivatives. The purpose of this method is to reduce the computational cost and enhance the efficiency of the method. The third main objective is to introduce the fast-quasi TFGW method for the solution of nonlinear Caputo-tempered partial differential equations with time fractional and space variable order derivatives. We have also performed the supporting analysis of the proposed methods such as error analysis, procedure of implementation and numerical simulation.

The structure of this paper is outlined as follows: Section 2 provides essential definitions and results from fractional calculus, which will serve as the foundation for subsequent sections. Section 3 focuses on function approximations, specifically the TFGW and its operational matrices. In Section 4, we introduce a numerical method designed for solving Caputo-tempered variable order differential equations. Section 5 delves into the development of the fast TFGW method. In section 6, we work on the development of L1- TFGW method. Section 7 is dedicated to conducting the convergence and error analysis of the proposed methods. In section 8, we construct the fast-quasi TFGW method and presents the numerical examples aimed at evaluating the efficiency, reliability, and accuracy of our numerical methods. The obtained numerical results are thoroughly examined through both tabular and graphical representations. In section 9, we conclude our work.

2. Preliminaries

In this section, we examine several fundamental definitions and characteristics pertaining to tempered variable

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order integrals and derivatives [7, 8].

Definition 2.1 ***Riemann-Liouville tempered integral of order*** $\beta(\tau)$***:***The Riemann-Liouville tempered integral of order $\beta(\tau) \in \mathbb{R}^+$, of a function $u \in L[a,b]$, is defined as

$$\begin{aligned} {}^{\lambda}I_{\tau}^{\beta(\tau)}u(\tau) &= \frac{1}{\Gamma(\beta(\tau))}\int_a^{\tau} e^{-\lambda(\tau-\xi)}\Big(\tau-\xi\Big)^{\beta(\tau)-1}u(\xi)d\xi \\ &= e^{-\lambda\tau}I_{\tau}^{\beta(\tau)}e^{\lambda\tau}u(\tau), \quad a<\tau\le b, \end{aligned}$$

where $I_{\tau}^{\beta(\tau)}$ is the Riemann-Liouville integral of order $\beta(\tau),\ n-1<\beta(\tau)<n,\ n=\lceil\beta(\tau)\rceil,\ n\in\mathbb{N}$ and $\lambda\ge 0$.

Definition 2.2 ***Riemann-Liouville tempered derivative of order*** $\beta(\tau)$***:***The Riemann-Liouville tempered derivative of order $\beta(\tau)$, of a function $u \in AC^n[a,b]$, is defined as

$$\begin{aligned} &{}^{RT}D_{\tau}^{\beta(\tau),\lambda}u(\tau) \\ &= \frac{e^{-\lambda\tau}}{\Gamma(n-\beta(\tau))}\left(\frac{d}{d\tau}\right)^n\int_a^{\tau}\Big(\tau-\xi\Big)^{n-\beta(\tau)-1}e^{\lambda\xi}u(\xi)d\xi \\ &= e^{-\lambda\tau}D_{\tau}^{\beta(\tau)}e^{\lambda\tau}u(\tau), \quad a<\tau\le b, \end{aligned}$$

where $AC^n[a,b]$ is the set of functions with an absolutely continuous $(n-1)$st derivative, and $D_{\tau}^{\beta(\tau)}$ is the Riemann-Liouville derivative of order $\beta(\tau),\ n-1<\beta(\tau)<n, n=\lceil\beta(\tau)\rceil,\ n\in\mathbb{N}$ and $\lambda\ge 0$.

Definition 2.3 ***Caputo tempered derivative of order*** $\beta(\tau)$***:***The Caputo tempered derivative of order $\beta(\tau)$, of a function $u\in AC^n[a,b]$, is defined as

$$\begin{aligned} &{}^{CT}D_{\tau}^{\beta(\tau),\lambda}u(\tau) \\ &= \frac{e^{-\lambda\tau}}{\Gamma(n-\beta(\tau))}\int_a^{\tau}(\tau-\xi)^{n-\beta(\tau)-1}\frac{d^n}{d\xi^n}(e^{\lambda\xi}u(\xi))d\xi \\ &= e^{-\lambda\tau}\,{}^{C}D_{\tau}^{\beta(\tau)}e^{\lambda\tau}u(\tau), \quad a<\tau\le b, \end{aligned}$$

where ${}^{C}D_{\tau}^{\beta(\tau)}$ is the Caputo fractional derivative of order $\beta(\tau)$.

Definition 2.4 ***Mittag-Leffler function:***[33]The one parameter Mittag-Leffler function $E_{\xi}(\tau), \xi>0$ is defined as

$$E_{\xi}(\tau)=\sum_{l=0}^{\infty}\frac{\tau^l}{\Gamma(l\xi+1)}.$$

The two parameter Mittag-Leffler function $E_{\xi,\eta}(\tau), \xi,\eta>0$ is defined as

$$E_{\xi,\eta}(\tau)=\sum_{l=0}^{\infty}\frac{\tau^l}{\Gamma(l\xi+\eta)}.$$

For $\eta=1$ and $\eta=\xi=1$, we have $E_{\xi,1}(\tau)=E_{\xi}(\tau)$ and $E_{1,1}(\tau)=e^{\tau}$, respectively.

Lemma 2.5 Let $u(\tau)=e^{-\lambda\tau}(\tau-a)^{\eta},\ \eta>-1, \lambda\ge 0,\ \beta(\tau)\in\mathbb{R}^+$, then

$${}^{\lambda}I_{\tau}^{\beta(\tau)}u(\tau)=\frac{\Gamma(\eta+1)}{\Gamma(\beta(\tau)+\eta+1)}e^{-\lambda\tau}(\tau-a)^{\beta(\tau)+\eta},$$

$${}^{RT}D_{\tau}^{\beta(\tau),\lambda}u(\tau)=\frac{\Gamma(\eta+1)}{\Gamma(\eta-\beta(\tau)+1)}e^{-\lambda\tau}(\tau-a)^{\eta-\beta(\tau)},$$

$${}^{CT}D_{\tau}^{\beta(\tau),\lambda}u(\tau)=\frac{\Gamma(\eta+1)}{\Gamma(\eta-\beta(\tau)+1)}e^{-\lambda\tau}(\tau-a)^{\eta-\beta(\tau)}.$$

Lemma 2.6 Let $\beta(\tau)>0$ and $u(\tau)\in AC^n[a,b]$, then

$$\begin{aligned} &{}^{\lambda}I_{\tau}^{\beta(\tau)}\,{}^{CT}D_{\tau}^{\beta(\tau),\lambda}u(\tau) \\ &\quad = u(\tau)-\sum_{i=0}^{n-1}\frac{e^{-\lambda\tau}(\tau-a)^i}{i!}\left(\frac{d^i}{d\tau^i}e^{\lambda\tau}u(\tau)\Big|_{\tau=a}\right). \end{aligned}$$

Remark:
Let $\lambda\in\mathbb{R}^+,\ \omega\in\mathbb{R}, \tau\in[0,T],\ T\in\mathbb{R}^+$ and $0<\beta(\tau)<2$, then we have

$$\begin{aligned} &{}^{CT}D_{\tau}^{\beta(\tau),\lambda}e^{-\lambda\tau}\cos(\omega\tau) \\ &\qquad = -\omega^2\tau^{2-\beta(\tau)}e^{-\lambda\tau}E_{2,3-\beta(\tau)}(-\omega^2\tau^2), \end{aligned}$$

and

$$\begin{aligned} &{}^{CT}D_{\tau}^{\beta(\tau),\lambda}e^{-\lambda\tau}\sin(\omega\tau)= \\ &\begin{cases} -\omega^3\tau^{3-\beta(\tau)}e^{-\lambda\tau}E_{2,4-\beta(\tau)}(-\omega^2\tau^2), & 1<\beta(\tau)<2, \\ \omega\tau^{1-\beta(\tau)}e^{-\lambda\tau}E_{2,2-\beta(\tau)}(-\omega^2\tau^2), & 0<\beta(\tau)<1. \end{cases} \end{aligned}$$

We can get the above results by applying the definition 2.3 on $e^{-\lambda\tau}\cos(\omega\tau)$ and $e^{-\lambda\tau}\sin(\omega\tau)$, respectively.

3. Tempered Fractional Gegenbauer Wavelet (TFGW)

We have introduced a generalized version of the fractional Gegenbauer wavelet explicitly designed for applications in tempered calculus, which we named as tempered fractional gegenbauer wavelet (TFGW). This modification enables the use of these wavelets as a basis for wavelet method in solving Caputo-tempered fractional or variable order differential equations.

The shifted Gegenbauer polynomials, $C_m^{\rho}(t)$, of order $m=0,1,2,\cdots$, are defined on the interval $[0,1]$ as

$$\begin{aligned} C_m^{\rho}(t) &= \frac{\Gamma(\rho+0.5)}{\Gamma(2\rho)}\times \\ &\sum_{i=0}^{m}(-1)^{m+i}\frac{\Gamma(2\rho+i+m)}{(m-i)!i!\Gamma(i+\rho+0.5)}t^i, t\in[0,1]. \end{aligned}$$

These polynomials satisfies the following orthogonality condition

$$\int_0^1 C_m^{\rho}(\xi)C_{m'}^{\rho}(\xi)w^{\rho}(\xi)d\xi = h_m^{\rho}\delta_{mm'},$$

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where $w^{\rho}(\xi) = (1-\xi)^{\rho-0.5}\xi^{\rho-0.5}$ and $h_m^{\rho} = \frac{2^{1-4\rho}\Gamma(m+2\rho)\pi}{(m+\rho)m!(\Gamma(\rho))^2}$.

We get the fractional shifted Gegenbauer polynomials by transforming the domain of the shifted Gegenbauer polynomials from $[0,1]$ to $[0,T]$, $T \in \mathbb{R}^+$, by using the given transformation, $t = \frac{s^{\alpha}}{T^{\alpha}}$, as

$$C_m^{\rho,\alpha}(s) = \frac{\Gamma(\rho+0.5)}{\Gamma(2\rho)} \times \sum_{i=0}^{m}(-1)^{m+i}\frac{\Gamma(2\rho+i+m)}{T^{i\alpha}(m-i)!i!\Gamma(i+\rho+0.5)}s^{i\alpha}, s \in [0,T].$$

We define the tempered fractional Gegenbauer wavelet (TFGW) on interval $[0,T]$ as

$$^{\lambda}\psi_{n,m}^{\rho,\alpha}(\tau) = \begin{cases} \frac{2^{\frac{k}{2}}}{\sqrt{L_m^{\rho}}}e^{-\tau\lambda}C_m^{\rho,\alpha}\Big(2^k\tau - nT\Big), & T\frac{n}{2^k} \le \tau < T\frac{n+1}{2^k}, \\ 0, & elsewhere, \end{cases} \quad (1)$$

where $L_m^{\rho} = \frac{2^{1-4\rho}\Gamma(m+2\rho)T^{\alpha}\pi}{\alpha(m+\rho)m!(\Gamma(\rho))^2}$, $n = 0,1,2,\cdots,2^k-1$, is the translation parameter, $k = 0,1,2,\cdots$, is the level of resolution, and $m = 0,1,2,\cdots,M-1$, $M \in \mathbb{Z}^+$, is the degree of the shifted fractional Gegenbauer polynomials. The TFGW have compact support i.e.,

$$Supp(\,^{\lambda}\psi_{n,m}^{\rho,\alpha}(\tau)) = \overline{\{\tau : \,^{\lambda}\psi_{n,m}^{\rho,\alpha}(\tau) \neq 0\}} = \Big[T\frac{n}{2^k}, T\frac{n+1}{2^k}\Big).$$

Corresponding to each $\lambda \ge 0$ and $\rho > -\frac{1}{2}$, we have a different family of fractional wavelets. For $\lambda = 0, \rho = \frac{1}{2}$, we get fractional Legendre wavelet, if $\lambda = 0$, $\rho = 0$, we get fractional Chebyshev wavelet of first kind, when $\lambda = 0$, $\rho = 1$, we get fractional Chebyshev wavelet of second kind and for $\lambda = 0, \rho > -\frac{1}{2}$, we have fractional Gegenbauer wavelet. These wavelets will be of integer order when $\alpha = 1$.

Theorem 3.1 Let $a = T\frac{n}{2^k}$, $b = T\frac{n+1}{2^k}$, $T \in \mathbb{R}^+$, and $0 \le a \le \tau < b < \infty$, then

$$\int_a^b \,^{\lambda}\psi_{n,m}^{\rho,\alpha}(\tau)\,^{\lambda}\psi_{n,m'}^{\rho,\alpha}(\tau)w(\tau)d\tau = \delta_{mm'},$$

where $w(\tau) = \frac{w^{\rho\alpha}(2^k\tau - nT)}{e^{-2\lambda\tau}}$ and $w^{\rho,\alpha}(\xi) = \frac{(T^{\alpha}-\xi^{\alpha})^{\rho-0.5}}{T^{\alpha(2\rho-1)}}\xi^{\alpha\rho+\frac{\alpha}{2}-1}$.

Proof. Since the fractional shifted Gegenbauer polynomials are orthogonal on interval $[0,1]$ as

$$\int_0^1 C_m^{\rho}(\xi)C_{m'}^{\rho}(\xi)w^{\rho}(\xi)d\xi = h_m^{\rho}\delta_{mm'}. \quad (2)$$

Let us consider $\xi = \frac{s^{\alpha}}{T^{\alpha}}$, $d\xi = \frac{\alpha}{T^{\alpha}}s^{\alpha-1}ds$, when $\xi = 0 \Rightarrow s = 0$, and when $\xi = 1 \Rightarrow s = T$. Equation (2) will take the following form

$$\int_0^T C_m^{\rho,\alpha}(s)C_{m'}^{\rho,\alpha}(s)w^{\rho,\alpha}(s)ds = \frac{h_m^{\rho}T^{\alpha}}{\alpha}\delta_{mm'}, \quad (3)$$

where $C_m^{\rho,\alpha}(s) = C_m^{\rho}(\frac{s^{\alpha}}{T^{\alpha}})$ and $w^{\rho,\alpha}(s) = \frac{(T^{\alpha}-s^{\alpha})^{\rho-0.5}}{T^{\alpha(2\rho-1)}}s^{\alpha\rho+\frac{\alpha}{2}-1}$. Multiply and divide by $e^{-2\lambda\tau}$ to equation (3), and substitute $s = 2^k\tau - nT$, $ds = 2^k d\tau$ when $s = 0 \Rightarrow \tau = a$, and when $s = T \Rightarrow \tau = b$ in equation (3) to get

$$\int_{T\frac{n}{2^k}}^{T\frac{n+1}{2^k}} \frac{2^{\frac{k}{2}}}{\sqrt{L_m^{\rho}}}e^{-\lambda\tau}C_m^{\rho,\alpha}(2^k\tau - nT)\frac{2^{\frac{k}{2}}}{\sqrt{L_{m'}^{\rho}}} \times e^{-\lambda\tau}C_{m'}^{\rho,\alpha}(2^k\tau - nT)w(\tau)d\tau = \delta_{mm'}, \quad (4)$$

where $w(\tau) = \frac{w^{\rho\alpha}(2^k\tau-nT)}{e^{-2\lambda\tau}}$ and $L_m^{\rho} = \frac{2^{1-4\rho}\Gamma(m+2\rho)T^{\alpha}\pi}{\alpha(m+\rho)m!(\Gamma(\rho))^2}$. Use (1) in (4) to get the desired result. □

3.1 Function approximations

We can represent any function $u(\tau) \in L_2[0,T)$ in terms of the TFGW series and approximate by truncated the TFGW series as

$$\begin{aligned} u(\tau) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_{n,m}\,^{\lambda}\psi_{n,m}^{\rho,\alpha}(\tau) \quad (5) \\ &\approx \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}\,^{\lambda}\psi_{n,m}^{\rho,\alpha}\tau) \\ &= \mathbf{B}\,^{\lambda}\Psi^{\rho,\alpha}(\tau), \end{aligned}$$

where $\mathbf{B}$ and $^{\lambda}\Psi^{\rho,\alpha}(\tau)$ are $\hat{p} \times 1$, $\hat{p} = 2^kM$, matrices and are given below

$$\mathbf{B} = [b_{0,0}, b_{0,1}, \cdots, b_{0,M-1}, b_{1,0}, b_{1,1}, \cdots, b_{1,M-1}, \cdots, b_{2^k-1,0}, b_{2^k-1,1}, \cdots, b_{2^k-1,M-1}]$$

and

$$^{\lambda}\Psi^{\rho,\alpha}(\tau) = [\,^{\lambda}\psi_{0,0}^{\rho,\alpha}(\tau),\,^{\lambda}\psi_{0,1}^{\rho,\alpha}(\tau), \cdots,\,^{\lambda}\psi_{0,M-1}^{\rho,\alpha}(\tau),\,^{\lambda}\psi_{1,0}^{\rho,\alpha}(\tau),\,^{\lambda}\psi_{1,1}^{\rho,\alpha}(\tau), \cdots,\,^{\lambda}\psi_{1,M-1}^{\rho,\alpha}(\tau), \cdots,\,^{\lambda}\psi_{2^k-1,0}^{\rho,\alpha}(\tau),\,^{\lambda}\psi_{2^k-1,1}^{\rho,\alpha}(\tau), \cdots,\,^{\lambda}\psi_{2^k-1,M-1}^{\rho,\alpha}(\tau)]^T.$$

The collocation points which we have considered for the TFGW are $\tau_j = T\frac{2j-1}{2\hat{p}}$, $j = 1,2,...,\hat{p}$. Expand (5) at the collocation points, we get the TFGW matrix, $^{\lambda}\Psi_{\hat{p}\times\hat{p}}^{\rho,\alpha}$, as

$$^{\lambda}\Psi_{\hat{p}\times\hat{p}}^{\rho,\alpha} = \Big[\,^{\lambda}\Psi^{\rho,\alpha}\Big(T\frac{1}{2\hat{p}}\Big),\,^{\lambda}\Psi^{\rho,\alpha}\Big(T\frac{3}{2\hat{p}}\Big), ...,\,^{\lambda}\Psi^{\rho,\alpha}\Big(T\frac{2\hat{p}-1}{2\hat{p}}\Big)\Big].$$

3.2 The TFGW operational matrix of tempered variable order integration

We have designed the mother wavelet, named as the TFGW, in such a manner that we can easily get the

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$$
\begin{aligned}
{}_a^{\lambda}I_\tau^{\beta(\tau)}\ {}^{\lambda}\psi_{n,m}^{\rho,\alpha}(\tau) &= \frac{2^{\frac{k}{2}}}{\sqrt{L_m^\rho}}\ {}_a^{\lambda}I_\tau^{\beta(\tau)}\left(e^{-\tau\lambda}C_m^{\rho,\alpha}(2^k\tau-nT)\right), \qquad \tau\in[T\frac{n}{2^k},T\frac{n+1}{2^k}),\\
&= \frac{2^{\frac{k}{2}}\Gamma(\rho+0.5)}{\Gamma(2\rho)\sqrt{L_m^\rho}}\sum_{i=0}^{m}(-1)^{m+i}\frac{2^{\alpha ik}\Gamma(2\rho+i+m)}{T^{i\alpha}(m-i)!i!\Gamma(i+\rho+0.5)}\ {}_a^{\lambda}I_\tau^{\beta(\tau)}\left(e^{-\tau\lambda}(\tau-T\frac{n}{2^k})^{i\alpha}\right),\\
\text{or}\quad &= \frac{2^{\frac{k}{2}}\Gamma(\rho+0.5)}{\Gamma(2\rho)\sqrt{L_m^\rho}}\sum_{i=0}^{m}(-1)^{m+i}\frac{2^{\alpha ik}\Gamma(2\rho+i+m)}{T^{i\alpha}(m-i)!i!\Gamma(i+\rho+0.5)}\frac{\Gamma(\alpha i+1)}{\Gamma(\alpha i+\beta(\tau)+1)}e^{-\tau\lambda}(\tau-T\frac{n}{2^k})^{i\alpha+\beta(\tau)}.
\end{aligned}
$$

$$
\begin{aligned}
{}_a^{\lambda}I_\tau^{\beta(\tau)}\ {}^{\lambda}\psi_{n,m}^{\rho,\alpha}(\tau) &= \frac{2^{\frac{k}{2}}\Gamma(\rho+0.5)}{\Gamma(2\rho)\sqrt{L_m^\rho}}\sum_{i=0}^{m}(-1)^{m+i}\frac{2^{\alpha ik}\Gamma(2\rho+i+m)}{T^{i\alpha}(m-i)!i!\Gamma(i+\rho+0.5)\Gamma(\beta(\tau))}\times\\
&\qquad \int_a^\tau e^{-\lambda(\tau-s)}(\tau-s)^{\beta(\tau)-1}e^{-\lambda s}(s-T\frac{n}{2^k})^{i\alpha}ds, \qquad \tau\geq T\frac{n+1}{2^k},\\
&= \frac{2^{\frac{k}{2}}\Gamma(\rho+0.5)}{\Gamma(2\rho)\sqrt{L_m^\rho}}\sum_{i=0}^{m}(-1)^{m+i}\frac{2^{\alpha ik}e^{-\lambda\tau}\Gamma(2\rho+i+m)}{T^{i\alpha}(m-i)!i!\Gamma(i+\rho+0.5)\Gamma(\beta(\tau))}\int_a^\tau(\tau-s)^{\beta(\tau)-1}(s-T\frac{n}{2^k})^{i\alpha}ds,\\
\text{or}\quad &= \frac{2^{\frac{k}{2}}\Gamma(\rho+0.5)}{\Gamma(2\rho)\sqrt{L_m^\rho}}\sum_{i=0}^{m}(-1)^{m+i}\frac{2^{\alpha ik}e^{-\lambda\tau}\Gamma(2\rho+i+m)}{T^{i\alpha}(m-i)!i!\Gamma(i+\rho+0.5)\Gamma(\beta(\tau))}v_i^{n,k}(\tau),
\end{aligned}
$$

where $v_i^{n,k}(\tau)=\int_a^\tau(\tau-s)^{\beta(\tau)-1}(s-T\frac{n}{2^k})^{i\alpha}ds$, $a=T\frac{n}{2^k}$, we can compute it by using the global adaptive technique at different values of the parameters i, n and k in MATLAB. In general, the $\beta(\tau)$ order tempered integral of the TFGW is as follows

$$
\begin{aligned}
{}_a^{\lambda}I_\tau^{\beta(\tau)}\ {}^{\lambda}\psi_{n,m}^{\rho,\alpha}(\tau) &= \frac{2^{\frac{k}{2}}\Gamma(\rho+0.5)}{\Gamma(2\rho)\sqrt{L_m^\rho}}\times \qquad (6)\\
&\sum_{i=0}^{m}\frac{(-1)^{m+i}2^{\alpha ik}\Gamma(2\rho+i+m)}{T^{i\alpha}(m-i)!i!\Gamma(i+\rho+0.5)}\times\\
&\begin{cases}\frac{\Gamma(\alpha i+1)}{\Gamma(\alpha i+\beta(\tau)+1)}e^{-\tau\lambda}(\tau-T\frac{n}{2^k})^{i\alpha+\beta(\tau)}, & T\frac{n}{2^k}\leq\tau<T\frac{n+1}{2^k},\\ \frac{1}{\Gamma(\beta(\tau))}e^{-\lambda\tau}v_i^{n,k}(\tau), & \tau\geq T\frac{n+1}{2^k}.\end{cases}
\end{aligned}
$$

For the construction of operational matrices, let us denote ${}_a^{\lambda}I_\tau^{\beta(\tau)}\ {}^{\lambda}\psi_{n,m}^{\rho,\alpha}(\tau)={}^{\lambda}W_{n,m}^{\beta(\tau)}(\tau)$. We can write (6) in vector notation as

$$
\begin{aligned}
{}^{\lambda}\mathbf{W}^{\beta(\tau)}=[\ &{}^{\lambda}W_{0,0}^{\beta(\tau)}(\tau),\ {}^{\lambda}W_{0,1}^{\beta(\tau)}(\tau),\cdots,\ {}^{\lambda}W_{0,M-1}^{\beta(\tau)}(\tau),\\
&{}^{\lambda}W_{1,0}^{\beta(\tau)}(\tau),\ {}^{\lambda}W_{1,1}^{\beta(\tau)}(\tau),\cdots,\ {}^{\lambda}W_{1,M-1}^{\beta(\tau)}(\tau),\cdots,\\
&{}^{\lambda}W_{2^k-1,0}^{\beta(\tau)}(\tau),\ {}^{\lambda}W_{2^k-1,1}^{\beta(\tau)}(\tau),\cdots,\ {}^{\lambda}W_{2^k-1,M-1}^{\beta(\tau)}(\tau)]^T.
\end{aligned}
$$

Expand (7) at the collocation points to get

$$
{}^{\lambda}\mathbf{W}_{\hat{p}\times\hat{p}}^{\beta(\tau)} = \left[{}^{\lambda}\mathbf{W}^{\beta(\tau)}\left(T\frac{1}{2\hat{p}}\right),\ {}^{\lambda}\mathbf{W}^{\beta(\tau)}\left(T\frac{3}{2\hat{p}}\right),\cdots,\ {}^{\lambda}\mathbf{W}^{\beta(\tau)}\left(T\frac{2\hat{p}-1}{2\hat{p}}\right)\right].
$$

3.3 The TFGW operational matrix of tempered variable order integration for Boundary Value Problems

When tackling boundary value problems with tempered variable orders derivatives, it's crucial to utilize an additional significant operational matrix for tempered variable order integration. Consider any function $g\in L_2[0,T)$ and apply the tempered $\beta(\tau)$ order integral on (1) from 0 to T at $\tau=T$, we have

$$
\begin{aligned}
&g(\tau)\ {}_a^{\lambda}I_T^{\beta(T)}\ {}^{\lambda}\psi_{n,m}^{\rho,\alpha}(T) \qquad (7)\\
&= g(\tau)\frac{2^{\frac{k}{2}}}{\sqrt{L_m^\rho}}\ {}_a^{\lambda}I_T^{\beta(T)}e^{-\tau\lambda}C_m^{\rho,\alpha}\left(2^k\tau-nT\right),\\
&= g(\tau)\frac{2^{\frac{k}{2}}\Gamma(\rho+0.5)}{\Gamma(2\rho)\sqrt{L_m^\rho}}\times\\
&\sum_{i=0}^{m}(-1)^{m+i}\frac{2^{\alpha ik}\Gamma(2\rho+i+m)}{T^{i\alpha}(m-i)!i!\Gamma(i+\rho+0.5)\Gamma(\beta(T))}\times\\
&\int_{T\frac{n}{2^k}}^{T\frac{n+1}{2^k}}e^{-\lambda(T-s)}(T-s)^{\beta(T)-1}e^{-\lambda s}(s-T\frac{n}{2^k})^{i\alpha}ds,\\
&= g(\tau)\ {}^{\lambda,\rho}R_{n,m}^{\beta(T),\alpha},
\end{aligned}
$$

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where

$$ {}^{\lambda,\rho}R^{\beta(T),\alpha}_{n,m} = \frac{2^{\frac{k}{2}}\Gamma(\rho+0.5)}{\Gamma(2\rho)\sqrt{L^{\rho}_m}} \times \sum_{i=0}^{m}(-1)^{m+i}\frac{2^{\alpha i k}\Gamma(2\rho+i+m)}{T^{i\alpha}(m-i)!i!\Gamma(i+\rho+0.5)\Gamma(\beta(T))} \times \int_{T\frac{n}{2^k}}^{T\frac{n+1}{2^k}} e^{-\lambda T}(T-s)^{\beta(T)-1}\left(s-T\frac{n}{2^k}\right)^{i\alpha}ds, $$

and it can be computed with the help of global adaptive technique at different values of the parameters n, m, i and k in MATLAB. Expand (7) at the collocation points to get

$$ {}^{\lambda}\mathbf{P}^{g(\tau),\beta(T)}_{\hat{p}\times\hat{p}} = {}^{\lambda,\rho}\mathbf{R}^{\beta(T),\alpha}_{\hat{p}\times 1}\mathbf{G}_{1\times\hat{p}}, $$

where

$$ \begin{aligned} {}^{\lambda,\rho}\mathbf{R}^{\beta(T),\alpha} = \Big[& {}^{\lambda,\rho}R^{\beta(T),\alpha}_{0,0}, {}^{\lambda,\rho}R^{\beta(T),\alpha}_{0,1}, \cdots, \\ & {}^{\lambda,\rho}R^{\beta(T),\alpha}_{0,M-1}, {}^{\lambda,\rho}R^{\beta(T),\alpha}_{1,0}, {}^{\lambda,\rho}R^{\beta(T),\alpha}_{1,1}, \\ & \cdots, {}^{\lambda,\rho}R^{\beta(T),\alpha}_{1,M-1}, \cdots, {}^{\lambda,\rho}R^{\beta(T),\alpha}_{2^k-1,0}, \\ & {}^{\lambda,\rho}R^{\beta(T),\alpha}_{2^k-1,1}, \cdots, {}^{\lambda,\rho}R^{\beta(T),\alpha}_{2^k-1,M-1}\Big]^T \end{aligned} $$

and $\mathbf{G}_{1\times\hat{p}} = \left[g(T\frac{1}{2\hat{p}}), g(T\frac{3}{2\hat{p}}), \cdots, g(T\frac{2\hat{p}-1}{2\hat{p}})\right]$.

4. Methodology of the TFGW Method for the Tempered Variable Order Ordinary Differential Equations

In this subsection, we will describe the procedure of implementation of TFGW operational matrices technique for the solution of tempered variable order ordinary boundary value problems.

Consider the following tempered variable order ordinary boundary value problem

$$ \begin{aligned} & {}^{CT}D^{\zeta(x),\lambda}_x u(x) + a(x)\,{}^{CT}D^{\eta(x),\lambda}_x u(x) + d(x)u(x) = f(x), \\ & 0 \le x \le T,\ 1 < \zeta(x) \le 2,\ 0 < \eta(x) \le 1, \\ & u(0) = c_0,\ u(T) = c_1,\ T > 0. \end{aligned} \tag{8} $$

Apply the proposed TFGW operational matrices method on the above equation (8) by using the indirect approach. According to the approach, we first approximate the tempered second order derivative by the TFGW series (5). The next step is to apply the Riemann-Liouville tempered integral of order 2 and then compute the Caputo tempered derivative of order $\zeta(x)$ and $\eta(x)$ along the utilization of boundary conditions to have

$$ {}^{CT}D^{\zeta(x),\lambda}_x u(x) \approx \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}\, {}^{\lambda}_{a}I^{2-\zeta(x)}_x\, {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(x), \tag{9} $$

$$ \begin{aligned} {}^{CT}D^{\eta(x),\lambda}_x u(x) \approx & \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}\, {}^{\lambda}_{a}I^{2-\eta(x)}_x\, {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(x) \\ & - \frac{x^{1-\eta(x)}e^{\lambda(T-x)}}{T\Gamma(2-\eta(x))}\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}\, {}^{\lambda}_{a}I^{2}_T\, {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(T) \\ & + \frac{x^{1-\eta(x)}e^{\lambda(T-x)}}{T\Gamma(2-\eta(x))}c_1 - \frac{x^{1-\eta(x)}e^{-\lambda x}}{T\Gamma(2-\eta(x))}c_0, \end{aligned} \tag{10} $$

$$ \begin{aligned} u(x) \approx & \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}\, {}^{\lambda}_{a}I^{2}_x\, {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(x) \\ & - \frac{xe^{\lambda(T-x)}}{T}\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}\, {}^{\lambda}_{a}I^{2}_T\, {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(T) \\ & + \frac{xe^{\lambda(T-x)}}{T}c_1 + (1-\frac{x}{T})e^{-\lambda x}c_0. \end{aligned} \tag{11} $$

Substitute the equations (9), (10) and (11) in (8) to get

$$ \begin{aligned} \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}\Big(& {}^{\lambda}_{a}I^{2-\zeta(x)}_x\, {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(x) + \\ & a(x)\lambda_a I^{2-\eta(x)}_x\, {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(x) + d(x)\, {}^{\lambda}_{a}I^{2}_x\, {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(x) \\ & + r(x)\, {}^{\lambda}_{a}I^{2}_T\, {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(T)\Big) \approx h(x), \end{aligned} \tag{12} $$

where $r(x) = -e^{\lambda(T-x)}\frac{x}{T}\left(a(x)\frac{x^{-\eta(x)}}{\Gamma(2-\eta(x))} + d(x)\right)$ and

$$ \begin{aligned} h(x) = & f(x) - d(x)\left(\frac{xe^{\lambda(T-x)}}{T}c_1 + (1-\frac{x}{T})e^{-\lambda x}c_0\right) \\ & -a(x)\left(\frac{x^{1-\eta(x)}e^{\lambda(T-x)}}{T\Gamma(2-\eta(x))}c_1 - \frac{x^{1-\eta(x)}e^{-\lambda x}}{T\Gamma(2-\xi(x))}c_0\right). \end{aligned} $$

Expand (12) at the collocation points, we have

$$ \begin{aligned} \mathbf{B}\Big(& {}^{\lambda}\mathbf{W}^{2-\zeta(\tau)}_{\hat{p}\times\hat{p}} + {}^{\lambda}\mathbf{W}^{2-\eta(\tau)}_{\hat{p}\times\hat{p}}\mathbf{A}_{\hat{p}\times\hat{p}} \\ & + {}^{\lambda}\mathbf{W}^{2}_{\hat{p}\times\hat{p}}\mathbf{D}_{\hat{p}\times\hat{p}} + {}^{\lambda}\mathbf{P}^{r(x),2}_{\hat{p}\times\hat{p}}\Big) \approx \mathbf{H}_{1\times\hat{p}}, \end{aligned} \tag{13} $$

where $\mathbf{H}_{1\times\hat{p}} = [h(\frac{1}{2\hat{p}}), h(\frac{3}{2\hat{p}}), \cdots, h(\frac{2\hat{p}-1}{2\hat{p}})]^T$ and the matrices ${}^{\lambda}\mathbf{W}^{2-\beta(x)}_{\hat{p}\times\hat{p}}$, ${}^{\lambda}\mathbf{W}^{2}_{\hat{p}\times\hat{p}}$, ${}^{\lambda}\mathbf{W}^{2-\xi(x)}_{\hat{p}\times\hat{p}}$ and ${}^{\lambda}\mathbf{P}^{r(x),2}_{\hat{p}\times\hat{p}}$ are given in Section 3. The matrices $\mathbf{A}_{\hat{p}\times\hat{p}}$ and $\mathbf{D}_{\hat{p}\times\hat{p}}$ are diagonal, and are given below

$$ \mathbf{A}_{\hat{p}\times\hat{p}} = \begin{bmatrix} a(x_1) & 0 & \cdots & 0 \\ 0 & a(x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a(x_{\hat{p}}) \end{bmatrix} $$

and

$$ \mathbf{D}_{\hat{p}\times\hat{p}} = \begin{bmatrix} d(x_1) & 0 & \cdots & 0 \\ 0 & d(x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d(x_{\hat{p}}) \end{bmatrix}. $$

We solve (13) by matrix inversion method to get $\mathbf{B}$ and substitute it in (11) to get the solution $u(x)$ at the collocation points $x = x_i,\ i = 0, 1, \cdots, \hat{p}$.

4.1 Application

Consider the above problem (8) along with the following value of the parameters:

$$c_0 = 1,\ c_1 = 2e^{-\lambda},\ T = 1,$$

$$\begin{aligned} g(x) &= e^{-\lambda x}\Big(\frac{\Gamma(l+4)}{\Gamma(l+4-\zeta(x))}x^{l+3-\zeta(x)} \\ &-\pi^3 t^{3-\zeta(x)}E_{2,4-\zeta(x)}(-\pi^2x^2) \\ &+a(x)\big(\frac{\Gamma(l+4)}{\Gamma(l+4-\eta(x))}x^{l+3-\eta(x)} \\ &+\pi t^{1-\eta(x)}E_{2,2-\eta(x)}(-\pi^2x^2)\big) \\ &+b(x)\big(x^{l+3}+\sin(\pi x)+1\big)\Big), \end{aligned}$$

$\zeta(x) = \frac{5}{2} - \cos(x), \eta(x) = \frac{1}{10} + \sin(x),\ a(x) = \cos(x)$ and $d(x) = e^x$. The exact solution is $u(x) = e^{-\lambda x}x^{l+3} + e^{-\lambda x}\sin(\pi x) + e^{-\lambda x},\ l \in \mathbb{R}^+$.

For the implementation of the method, we consider the following value of the parameters: $\alpha = 0.5$, $\lambda = 0.56$, $\rho = 0.6$, $l = 3.125$, and apply the proposed TFGW method for the solution of (8) at different values of k and M.

We get more accurate result while increasing the level of resolution k and degree M as shown in Figure 1. Figure 2 is used to plot the maximum absolute error by the TFGW method against the parameters M and k. It shows that error decreases while increasing the value of the parameters k and M, as shown in the convergence analysis of the method.

5. The Fast TFGW Method

Since the traditional numerical methods take much more memory of the system and computational time for solving fractional or variable order partial differential equations. In this section, we will work to save the memory and computational time by introducing an efficient method, named as the fast TFGW method. According to the fast TFGW method, we first discretize the tempered fractional time derivative by fast algorithm, as given in Subsection 5.1, and then utilize the TFGW method, given in Section 4.

5.1 Fast Evaluation of the Tempered FractionalDerivative

In this subsection, we will work to save the memory and computational time by using an efficient algorithm for the tempered time fractional derivative, ${}^{CT}D_t^{\gamma,\lambda}u(t),\ 0 \le \gamma < 1$. This algorithm will be based on the efficient sum-of-exponentials (SOE) approximation [20, 21]. Let us discretize the time domain $[0, T],\ T \in \mathbb{R}^+$ as $t_r = r\Delta t_r,\ r = 0, 1, 2, \cdots, R \in \mathbb{N}$, and $\Delta t_r = t_r - t_{r-1}$. Consider the tempered fractional derivative at $t = t_r$ and split it in terms of local and history part as

$$\begin{aligned} {}^{CT}D_t^{\gamma,\lambda}u(t_r) &= \frac{e^{-\lambda t_r}}{\Gamma(1-\gamma)}\times \\ &\int_{t_0}^{t_r}(t_r-\tau)^{-\gamma}\frac{d}{d\tau}(e^{\lambda\tau}u(\tau))d\tau = H_r + L_r, \end{aligned} \tag{14}$$

where $H_r = \frac{e^{-\lambda t_r}}{\Gamma(1-\gamma)}\int_{t_0}^{t_{r-1}}(t_r-\tau)^{-\gamma}\frac{d}{d\tau}(e^{\lambda\tau}u(\tau))d\tau$ and $L_r = \frac{e^{-\lambda t_r}}{\Gamma(1-\gamma)}\int_{t_{r-1}}^{t_r}(t_r-\tau)^{-\gamma}\frac{d}{d\tau}(e^{\lambda\tau}u(\tau))d\tau$ are called the history and local part, respectively. For the evaluation of local part, we used the forward difference formula to approximate $\frac{d}{d\tau}e^{\lambda\tau}u(\tau)$ on $[t_{r-1}, t_r]$, and integrate the remaining integrand to get

$$L_r \approx \frac{u(t_r) - e^{-\lambda\Delta t_r}u(t_{r-1})}{\Delta t_r^{\gamma}\Gamma(2-\gamma)}. \tag{15}$$

For the evaluation of history part, H_r, we apply the integration by parts to obtain

$$\begin{aligned} H_r &= \frac{e^{-\lambda t_r}}{\Gamma(1-\gamma)}\Big[\Delta t_r^{-\gamma}e^{\lambda t_{r-1}}u(t_{r-1}) - t_r^{-\gamma}e^{\lambda t_0}u(t_0) \\ &\qquad -\gamma\int_{t_0}^{t_{r-1}}\frac{e^{\lambda\tau}u(\tau)}{(t_r-\tau)^{\gamma+1}}d\tau\Big]. \end{aligned} \tag{16}$$

After utilizing the sum of exponential (SOE) approximation for the integrand term, $\frac{1}{(t_r-\tau)^{\gamma+1}}$, in equation (16), we get

$$\begin{aligned} H_r &\approx \frac{e^{-\lambda t_r}}{\Gamma(1-\gamma)}\Big[\frac{e^{\lambda t_{r-1}}}{\Delta t_r^{\gamma}}u(t_{r-1}) - \frac{e^{\lambda t_0}}{t_r^{\gamma}}u(t_0) \\ &\qquad -\gamma\sum_{j=1}^{J}w_jQ_H^j(t_r)\Big], \end{aligned} \tag{17}$$

where $Q_H^j(t_r) = \int_{t_0}^{t_{r-1}}e^{-t_rs_j+(s_j+\lambda)\tau}u(\tau)d\tau$, and s_j, w_j, J are given in [20].

By following the procedure in [34], we extract the following discretized expression of $Q_H^j(t_r)$ for the tempered fractional derivative as

$$\begin{aligned} Q_H^j(t_r) &\approx e^{-s_j\Delta t_r}Q_H^j(t_{r-1}) + \\ &\frac{e^{-s_j\Delta t_r}}{\Delta t_{r-1}s_j^2}\Big[e^{\lambda t_{r-2}}u(t_{r-2})\big(1-e^{-s_j\Delta t_{r-1}}(s_j\Delta t_{r-1}+1)\big) \\ &\qquad +e^{\lambda t_{r-1}}u(t_{r-1})\big(e^{-s_j\Delta t_{r-1}}-1+s_j\Delta t_{r-1}\big)\Big]. \end{aligned}$$

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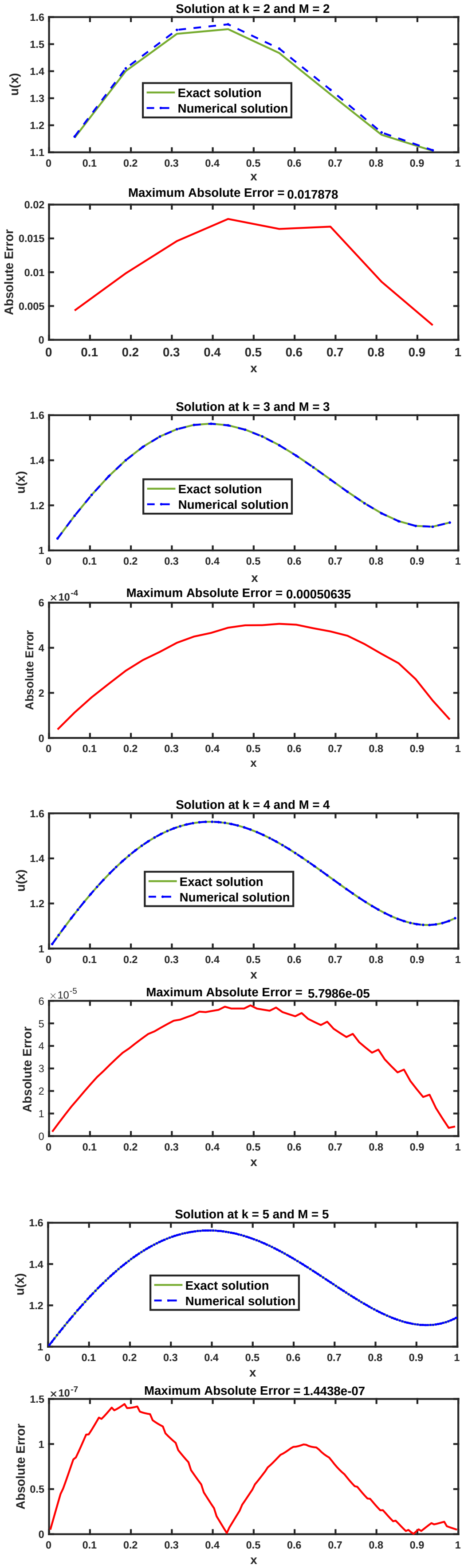

Figure 1. Solutions of (8) by the TFGW method at different values of k and M.

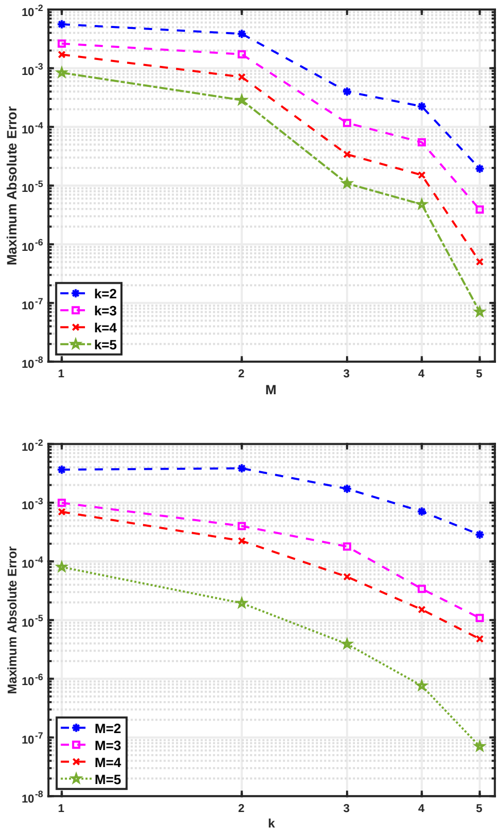

Figure 2. The maximum absolute errors are calculated for (8) by using the TFGW method at different values of M and k.

Substitute equation (17) and (15) in equation (14) to get

$$\begin{aligned} {}^{CT}D_t^{\gamma,\lambda}u(t_r) &= \frac{1}{\Delta t_r^{\gamma}\Gamma(2-\gamma)}u(t_r) + \\ &\frac{1}{\Delta t_r^{\gamma}\Gamma(2-\gamma)}\Big((1-\gamma)e^{-\lambda\Delta t_{r-1}} - e^{-\lambda\Delta t_r}\Big)u(t_{r-1}) - \\ &\frac{e^{-\lambda(t_r-t_0)}}{\Gamma(1-\gamma)t_r^{\gamma}}u(t_0) - \gamma\frac{e^{-\lambda t_r}}{\Gamma(1-\gamma)}\sum_{j=1}^{J} w_j Q_H^j(t_r) + {}^{CT}\mathbb{R}^r. \end{aligned} \tag{18}$$

In accordance with equation (18), the fast algorithm for the tempered fractional operator is applicable for an order $0 < \gamma < 1$ over the interval $[0, T]$ and can be expressed as:

$$\begin{aligned} {}^{CT}D_t^{\gamma,\lambda}u(t_r) &= \frac{1}{\Delta t_r^{\gamma}\Gamma(2-\gamma)}u(t_r) + \\ &\frac{1}{\Delta t_r^{\gamma}\Gamma(2-\gamma)}\Big((1-\gamma)e^{-\lambda\Delta t_{r-1}} - e^{-\lambda\Delta t_r}\Big)u(t_{r-1}) - \\ &\frac{e^{-\lambda(t_r-t_0)}}{\Gamma(1-\gamma)t_r^{\gamma}}u(t_0) - \gamma\frac{e^{-\lambda t_r}}{\Gamma(1-\gamma)}\sum_{j=1}^{J} w_j Q_H^j(t_r) + {}^{CT}\mathbb{R}^r, \\ &= {}^{CT}\mathbb{D}_t^{\gamma,\lambda}u(t_r) + {}^{CT}\mathbb{R}^r, \qquad r > 2. \end{aligned} \tag{19}$$

When $r = 1$, then $Q_H^j(t_1) = 0$ and the above expression will becomes as

$$
\begin{aligned}
{}^{CT}\mathbb{D}_t^{\gamma,\lambda}u(t_1) &= \frac{1}{\Delta t_1^{\gamma}\Gamma(2-\gamma)}u(t_1) + \\
&\frac{1}{\Delta t_1^{\gamma}\Gamma(2-\gamma)}\Big(1-\gamma-e^{-\lambda\Delta t_1}\Big)u(t_0) \\
&-\frac{e^{-\lambda\Delta t_1}}{\Gamma(1-\gamma)t_1^{\gamma}}u(t_0).
\end{aligned}
$$

Since we know the L_1 approximation of tempered fractional derivative is given in [35] as

$$
\begin{aligned}
{}_0^{CT}D_t^{\gamma,\lambda}u(t_r) &= \frac{\Delta t_r^{-\gamma}}{\Gamma(2-\gamma)}\Big[a_0^{(\gamma)}u(t_r) - \\
&\sum_{l=1}^{r-1}\Big(a_{r-l-1}^{(\gamma)}-a_{r-l}^{(\gamma)}\Big)e^{-\lambda(t_l-t_r)}u(t_l) \\
&-a_{r-1}^{(\gamma)}e^{-\lambda(t_0-t_r)}u(t_0)\Big] + \mathbb{R}^r, \\
&= {}^{CT}\mathbb{D}_t^{\gamma,\lambda}u(t_r) + \mathbb{R}^r,
\end{aligned}
$$

where $a_j^{(\gamma)} = (j+1)^{\gamma} - j^{1-\gamma}$, $0 \le j \le r-1$. They also provided the error bound for the above discretized technique, which enables us to determine the error bound for the technique outlined in equation (19).

5.2 Methodology of the fast TFGW method for the tempered variable order partial differential equations

This section is devoted to the development of methodology based on the fast algorithm for time fractional tempered derivative and the TFGW operational matrices technique for variable order differential equations for the solution of Caputo-tempered variable order partial differential equations.

Consider the Caputo-tempered variable order partial differential equation of the following form

$$
\begin{aligned}
&{}^{CT}D_t^{\gamma,\lambda}u(x,t) = {}^{CT}D_x^{\beta(x),\lambda}u(x,t) \\
&\quad + {}^{CT}D_x^{\xi(x),\lambda}u(x,t) + u(x,t) + f(x,t), \\
&\quad 1<\beta(x)\le 2,\ 0<\xi(x)\le 1,\ 0<\gamma\le 1, \\
&\quad u(x,0) = u_0(x),\ 0\le x\le T, \\
&\quad u(0,t) = A_0(t),\ u(T,t) = C_0(t),\ 0\le t\le T.
\end{aligned} \tag{20}
$$

Discretize the equation (20) at $t = t_r$ and then apply the fast algorithm for the tempered time-fractional derivative of order γ, given in equation (18), to get

$$
\begin{aligned}
&\left(\frac{1}{\Delta t_r^{\gamma}\Gamma(2-\gamma)}-1\right)u(x,t_r) - {}^{CT}D_x^{\beta(x),\lambda}u(x,t_r) \\
&\quad - {}^{CT}D_t^{\xi(x),\lambda}u(x,t_r) = f(x,t_r) \\
&\quad -\frac{1}{\Delta t_r^{\gamma}\Gamma(2-\gamma)}\Big((1-\gamma)e^{-\lambda\Delta t_{r-1}} \\
&\quad -e^{-\lambda\Delta t_r}\Big)u(x,t_{r-1}) + \frac{e^{-\lambda(t_r-t_0)}}{\Gamma(1-\gamma)t_r^{\gamma}}u(x,t_0) \\
&\quad +\gamma\frac{e^{-\lambda\Delta t_r}}{\Gamma(1-\gamma)}\sum_{j=1}^{J}w_jQ_H^j(x,t_r), \\
&\quad u(x,t_0) = u_0(x), \\
&\quad u(0,t_r) = A_0(t_r), \\
&\quad u(T,t_r) = C_0(t_r),
\end{aligned} \tag{21}
$$

where

$$
\begin{aligned}
Q_H^j(x,t_r) &= e^{-s_j\Delta t_r}Q_H^j(x,t_{r-1}) + \frac{e^{-s_j\Delta t_r}}{\Delta t_{r-1}s_j^2}\times \\
&\Big[e^{\lambda t_{r-2}}u(x,t_{r-2})\big(1-e^{-s_j\Delta t_{r-1}}(s_j\Delta t_{r-1}+1)\big) \\
&+e^{\lambda t_{r-1}}u(x,t_{r-1})\big(e^{-s_j\Delta t_{r-1}}-1+s_j\Delta t_{r-1}\big)\Big].
\end{aligned}
$$

Apply the TFGW matrices technique on the discretized equation (21), and utilize the boundary conditions to have

$$
{}^{CT}D_x^{\beta(x),\lambda}u(x,t_r) \approx \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}^r\, {}_a^{\lambda}I_x^{2-\beta(x)}\, {}^{\lambda}\psi_{n,m}^{\rho,\alpha}(x), \tag{22}
$$

$$
\begin{aligned}
{}^{CT}D_x^{\xi(x),\lambda}u(x,t_r) &\approx \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}^r\, {}_a^{\lambda}I_x^{2-\xi(x)}\, {}^{\lambda}\psi_{n,m}^{\rho,\alpha}(x) \\
&-\frac{x^{1-\xi(x)}e^{\lambda(T-x)}}{T\Gamma(2-\xi(x))}\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}^r\, {}_a^{\lambda}I_T^{2}\, {}^{\lambda}\psi_{n,m}^{\rho,\alpha}(T) \\
&+\frac{x^{1-\xi(x)}e^{\lambda(T-x)}}{T\Gamma(2-\xi(x))}C_0(t_r) - \frac{x^{1-\xi(x)}e^{-\lambda x}}{T\Gamma(2-\xi(x))}A_0(t_r),
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
u(x,t_r) &\approx \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}^r\lambda_a I_x^2\, {}^{\lambda}\psi_{n,m}^{\rho,\alpha}(x) \\
&-\frac{xe^{\lambda(T-x)}}{T}\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}^r\lambda_a I_T^2\, {}^{\lambda}\psi_{n,m}^{\rho,\alpha}(T) \\
&+\frac{xe^{\lambda(T-x)}}{T}C_0(t_r) + (1-\frac{x}{T})e^{-\lambda x}A_0(t_r).
\end{aligned} \tag{24}
$$

Substitute the equations (22), (23) and (24) in (21) to get

$$
\begin{aligned}
&\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}^r\Big(e_r^{\gamma}\, {}_a^{\lambda}I_x^{2}\, {}^{\lambda}\psi_{n,m}^{\rho,\alpha}(x) - \\
&{}_a^{\lambda}I_x^{2-\beta(x)}\, {}^{\lambda}\psi_{n,m}^{\rho,\alpha}(x) - {}_a^{\lambda}I_x^{2-\xi(x)}\, {}^{\lambda}\psi_{n,m}^{\rho,\alpha}(x) \\
&+q(x)\, {}_a^{\lambda}I_T^{2}\, {}^{\lambda}\psi_{n,m}^{\rho,\alpha}(T)\Big) \approx g(x,t_r),
\end{aligned}
$$

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where $e_r^\gamma = \left(\frac{1}{\Delta t_r^\gamma \Gamma(2-\gamma)} - 1\right)$, $q(x) = e^{\lambda(T-x)} \frac{x}{T}\left(\frac{x^{-\xi(x)}}{\Gamma(2-\xi(x))} - e_r^\gamma\right)$ and

$$\begin{aligned} g(x,t_r) &= f(x,t_r) - \frac{1}{\Delta t_r^\gamma \Gamma(2-\gamma)} \times \\ &\left((1-\gamma)e^{-\lambda\Delta t_{r-1}} - e^{-\lambda \Delta t_r}\right) u(x,t_{r-1}) \\ &+ \frac{e^{-\lambda(t_r - t_0)}}{\Gamma(1-\gamma) t_r^\gamma} u(x,t_0) + \gamma \frac{e^{-\lambda \Delta t_r}}{\Gamma(1-\gamma)} \sum_{j=1}^{J} w_j Q_H^j(x,t_r) \\ &- e_r^\gamma \frac{x e^{\lambda(T-x)}}{T} C_0(t_r) - e_r^\gamma (1 - \frac{x}{T}) e^{-\lambda x} A_0(t_r) \\ &+ \frac{x^{1-\xi(x)} e^{\lambda(T-x)}}{T\Gamma(2-\xi(x))} C_0(t_r) - \frac{x^{1-\xi(x)} e^{-\lambda x}}{T\Gamma(2-\xi(x))} A_0(t_r). \end{aligned}$$

Expand (25) at the collocation points, we have

$$\mathbf{B}^r \Big(e_r^\gamma \, {}^\lambda\mathbf{W}^2_{\hat{p}\times\hat{p}} - \lambda \mathbf{W}^{2-\beta(\tau)}_{\hat{p}\times\hat{p}} - \lambda \mathbf{W}^{2-\xi(\tau)}_{\hat{p}\times\hat{p}} + {}^\lambda\mathbf{P}^{q(x),2}_{\hat{p}\times\hat{p}} \Big) \approx \mathbf{G}^r_{1\times\hat{p}}, \tag{25}$$

where $\mathbf{G}^r_{1\times\hat{p}} = [g(\frac{1}{2\hat{p}}, t_r), g(\frac{3}{2\hat{p}}, t_r), \cdots, g(\frac{2\hat{p}-1}{2\hat{p}}, t_r)]^T$, and the matrices ${}^\lambda\mathbf{W}^{2-\beta(x)}_{\hat{p}\times\hat{p}}$, ${}^\lambda\mathbf{W}^2_{\hat{p}\times\hat{p}}$, ${}^\lambda\mathbf{W}^{2-\xi(x)}_{\hat{p}\times\hat{p}}$ and ${}^\lambda\mathbf{P}^{q(x),2}_{\hat{p}\times\hat{p}}$ are given in Section 3. For each value of t_r, we solve (25) by matrix inversion method to get $\mathbf{B}^r$ and substitute it in (24) to get the solution $u(x,t_r)$ at $t = t_r,\ r = 0, 1, \cdots, R$.

6. The L1- TFGW Method

In this section, we will develop another method for the solution of Caputo-tempered partial differential equations featuring both time fractional and spatial variable order derivatives. The present method is proposed by merging the L1 approximation with the TFGW method. According to the L1- TFGW method, we first discretize the tempered fractional time derivative by the L1 approximations, and then utilize the TFGW method on the obtained spatial variable order equation.

6.1 Methodology of the L1-TFGW method for the tempered variable order partial differential equations

This section is devoted to the development of methodology based on the L1 approximations for time fractional tempered derivative and the TFGW operational matrices technique for variable order differential equations for the solution of Caputo-tempered variable order partial differential equations.

Discretize the equation (20) at $t = t_r$ and then apply the L1 approximations to the tempered time-fractional derivative of order γ, given in [35], to get

$$\begin{aligned} &\left(\frac{d_0^\gamma}{\Delta t_r^\gamma \Gamma(2-\gamma)} - 1\right) u(x,t_r) - {}^{CT}D_x^{\beta(x),\lambda} u(x,t_r) \\ &- D_t^{\xi(x),\lambda} u(x,t_r) = f(x,t_r) + \\ &\frac{\Delta t_r^{-\gamma}}{\Gamma(2-\gamma)} \sum_{l=1}^{r-1} \left(d_{r-l-1}^\gamma - d_{r-l}^\gamma\right) e^{\lambda(t_l - t_r)} u(x,t_l) \\ &- d_{r-1}^\gamma e^{\lambda(t_0 - t_r)} u(x,t_0), \end{aligned} \tag{26}$$

where $d_j^\gamma = (j+1)^{1-\gamma} - (j)^{1-\gamma}, 0 \le j \le r-1$. Let $h_r^\gamma = \left(\frac{d_0^\gamma}{\Delta t_r^\gamma \Gamma(2-\gamma)} - 1\right)$, $p(x) = e^{\lambda(T-x)} \frac{x}{T}\left(\frac{x^{-\xi(x)}}{\Gamma(2-\xi(x))} - h_r^\gamma\right)$ and

$$\begin{aligned} d(x,t_r) &= f(x,t_r) + \\ &\frac{\Delta t_r^{-\gamma}}{\Gamma(2-\gamma)} \sum_{l=1}^{r-1} \left(d_{r-l-1}^\gamma - d_{r-l}^\gamma\right) e^{\lambda(t_l - t_r)} u(x,t_l) \\ &- d_{r-1}^\gamma e^{\lambda(t_0 - t_r)} u(x,t_0) - h_r^\gamma \frac{x e^{\lambda(T-x)}}{T} C_0(t_r) \\ &- h_r^\gamma (1 - \frac{x}{T}) e^{-\lambda x} A_0(t_r) \\ &+ \frac{x^{1-\xi(x)} e^{\lambda(T-x)}}{T\Gamma(2-\xi(x))} C_0(t_r) - \frac{x^{1-\xi(x)} e^{-\lambda x}}{T\Gamma(2-\xi(x))} A_0(t_r). \end{aligned}$$

Substitute the equations (22), (23) and (24) in (26) to get

$$\begin{aligned} \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} b_{n,m}^r \Big(& h_r^\gamma \, {}^\lambda_a I_x^2 \, {}^\lambda\psi_{n,m}^{\rho,\alpha}(x) \\ & - {}^\lambda_a I_x^{2-\beta(x)} \, {}^\lambda\psi_{n,m}^{\rho,\alpha}(x) - {}^\lambda_a I_x^{2-\xi(x)} \, {}^\lambda\psi_{n,m}^{\rho,\alpha}(x) \\ & + q(x) \, {}^\lambda_a I_T^2 \, {}^\lambda\psi_{n,m}^{\rho,\alpha}(T) \Big) \approx d(x,t_r), \end{aligned} \tag{27}$$

Expand (27) at the collocation points, we have

$$\mathbf{B}^r \Big(h_r^\gamma \, {}^\lambda\mathbf{W}^2_{\hat{p}\times\hat{p}} - {}^\lambda\mathbf{W}^{2-\beta(\tau)}_{\hat{p}\times\hat{p}} - {}^\lambda\mathbf{W}^{2-\xi(\tau)}_{\hat{p}\times\hat{p}} + {}^\lambda\mathbf{P}^{q(x),2}_{\hat{p}\times\hat{p}} \Big) \approx \mathbf{D}^r_{1\times\hat{p}}, \tag{28}$$

where $\mathbf{D}^r_{1\times\hat{p}} = [d(\frac{1}{2\hat{p}}, t_r), d(\frac{3}{2\hat{p}}, t_r), \cdots, d(\frac{2\hat{p}-1}{2\hat{p}}, t_r)]^T$. For each value of t_r, we solve (28) by matrix inversion method to get $\mathbf{B}^r$ and substitute it in (24) to get the solution $u(x,t_r)$ at $t = t_r,\ r = 0, 1, \cdots, R$.

7. Analysis

This section is dedicated to conducting a comprehensive error and convergence analysis encountered during the implementation and application of the proposed methods.

7.1 Bessel's Inequality for TFGW sequence

Lemma 7.1 Let

$$\begin{aligned} {}^\lambda\Psi^{\rho,\alpha}(\tau) = [\, & {}^\lambda\psi_{0,0}^{\rho,\alpha}(\tau),\ {}^\lambda\psi_{0,1}^{\rho,\alpha}(\tau), \cdots, \\ & {}^\lambda\psi_{0,M-1}^{\rho,\alpha}(\tau),\ {}^\lambda\psi_{1,0}^{\rho,\alpha}(\tau),\ {}^\lambda\psi_{1,1}^{\rho,\alpha}(\tau), \cdots, \\ & {}^\lambda\psi_{1,M-1}^{\rho,\alpha}(\tau), \cdots, \lambda\psi_{2^k-1,0}^{\rho,\alpha}(\tau), \\ & {}^\lambda\psi_{2^k-1,1}^{\rho,\alpha}(\tau), \cdots, {}^\lambda\psi_{2^k-1,M-1}^{\rho,\alpha}(\tau)]^T \end{aligned}$$

be the orthonormal sequence in $L_2[0,T)$, then from (5) we have

$$\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1}|b_{n,m}|^2 \le \|u\|^2.$$

Proof. Since $u(\tau) \approx \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}\ {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(\tau)$ and $b_{n,m} = \langle u(\tau),\ {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(\tau)\rangle = \int_0^T u(\tau)\ {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(\tau)w(\tau)d\tau.$

Let $S(\tau) = u(\tau) - \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}\ {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(\tau)$, then we have

$$\begin{aligned}\|S\|^2 &= \langle u(\tau) - \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}\ {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(\tau)\\ &\quad u(\tau) - \sum_{n'=0}^{2^k-1}\sum_{m'=0}^{M-1} b_{n',m'}\ {}^{\lambda}\psi^{\rho,\alpha}_{n',m'}(\tau)\rangle,\\ &= \|u\|^2 - \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}\langle\ {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(\tau), u(\tau)\rangle\\ &\quad - \sum_{n'=0}^{2^k-1}\sum_{m'=0}^{M-1} \overline{b_{n',m'}}\langle u(\tau),\ {}^{\lambda}\psi^{\rho,\alpha}_{n',m'}(\tau)\rangle\\ &+ \sum_{n'=0}^{2^k-1}\sum_{m'=0}^{M-1}\overline{b_{n',m'}}\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}\langle\ {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(\tau),\ {}^{\lambda}\psi^{\rho,\alpha}_{n',m'}(\tau)\rangle.\end{aligned}$$

If ${}^{\lambda}\Psi^{\rho,\alpha}(\tau)$ is an orthonormal sequence in $L_2[0,T)$, then we have

$$\|S\|^2 = \|u\|^2 - \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1}|\langle u(\tau),\ {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(\tau)\rangle|^2 \ge 0,$$

or

$$\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1}|\langle u(\tau),\ {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(\tau)\rangle|^2 \le \|u\|^2.$$

From Theorem 3.1, the $TFGW$ sequence, ${}^{\lambda}\Psi^{\rho,\alpha}(\tau)$, forms the orthonormal basis of $L_2[0,T)$, this implies that

$$\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1}|\langle u(\tau),\ {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(\tau)\rangle|^2 = \|u\|^2.\square$$

7.2 Convergence Analysis

Theorem 7.2 Let k and M approaches to ∞, then the series $\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}\ {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(\tau)$ converges to $u(\tau)$.

Proof. Let $V_{k,M}$ and $Z_{k',M'}$ be the sequences of partial sums of $b_{n,m}\ {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(\tau)$, such that

$$V_{k,M} = \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}\ {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(\tau)$$

and

$$Z_{k',M'} = \sum_{n=0}^{2^{k'}-1}\sum_{m=0}^{M'-1} b_{n,m}\ {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(\tau),$$

where $k > k'$ and $M > M'$. Firstly, we will show that the sequence of partial sums of $b_{n,m}\ {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(\tau)$ is a Cauchy sequence. For this purpose, we consider

$$\begin{aligned}&\|V_{k,M} - Z_{k',M'}\|^2\\ &= \|\sum_{n=2^{k'}}^{2^k-1}\sum_{m=M'}^{M-1} b_{n,m}\ {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(\tau)\|^2,\\ &= \langle\sum_{n=2^{k'}}^{2^k-1}\sum_{m=M'}^{M-1} b_{n,m}\ {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(\tau), \sum_{p=2^{k'}}^{2^k-1}\sum_{q=M'}^{M-1} b_{p,q}\ {}^{\lambda}\psi^{\rho,\alpha}_{p,q}(\tau)\rangle,\\ &= \sum_{n=2^{k'}}^{2^k-1}\sum_{m=M'}^{M-1} b_{n,m}\sum_{p=2^{k'}}^{2^k-1}\sum_{q=M'}^{M-1}\overline{b_{p,q}}\langle\ {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(\tau),\ {}^{\lambda}\psi^{\rho,\alpha}_{p,q}(\tau)\rangle,\end{aligned}$$

From Theorem 3.1, the sequence $\lambda\Psi^{\rho,\alpha}(\tau)$ forms an orthonormal basis of $L^2[0,T]$, then

$$\|V_{k,M} - Z_{k',M'}\|^2 = \sum_{n=2^{k'}}^{2^k-1}\sum_{m=M'}^{M-1}|b_{n,m}|^2.$$

From the Lemma 7.1 (Bessel inequality), we have $\sum_{n=2^{k'}}^{2^k-1}\sum_{m=M'}^{M-1}|b_{n,m}|^2$ is convergent and therefore $\|V_{k,M} - Z_{k',M'}\|^2$ approaches to zero with the increase of k, k', M, M'. This implies that $V_{k,M}$ is a Cauchy sequence and it converges to, say, $f(\tau) \in L^2[0,T]$, we will show that $f(\tau) = u(\tau)$. For this purpose, consider the following inner product

$$\begin{aligned}&\langle f(\tau) - u(\tau),\ {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(\tau)\rangle\\ &= \langle f(\tau), \lambda\psi^{\rho,\alpha}_{n,m}(\tau)\rangle - \langle u(\tau),\ {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(\tau)\rangle,\\ &= \lim_{k,M\to\infty}\langle V_{k,M}(\tau),\ {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(\tau)\rangle - b_{n,m} = 0.\end{aligned}$$

This implies that $\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}\ {}^{\lambda}\psi^{\rho,\alpha}_{n,m}(\tau)$ converges to $u(\tau)$ when $k, M \to \infty$. $\square$

7.3 Error Bound

Theorem 7.3 Let the TFGW series (5) be the best approximations of any function $u(\tau) \in L^2[0,T)$, and the M^{th} order derivative of $u(\tau)$ is bounded by some constant $H \in \mathbb{R}^+$ such that $\left|u^M(\tau)\right| \le H$, then

$$\|u(\tau) - u_{kM}(\tau)\|_{L^2[0,T)} \le \frac{C_3}{2^{k(M+\frac{1}{2})}\sqrt{2M+1}M!}.$$

Proof. Let $a = T\frac{n}{2^k}$, $b = T\frac{n+1}{2^k}$, $T \in \mathbb{R}^+$, expand $u(\tau)$ by using the Taylor's polynomials of degree $(M-1)^{th}$ about $\tau = a$, we get

$$\left|u(\tau) - \overline{u}(\tau)\right| = \tfrac{(\tau-a)^M}{M!}\left|u^M(l(\tau))\right| \le H\tfrac{(\tau-a)^M}{M!},$$

where $\overline{u}(\tau) = u(a) + (\tau-a)u'(a) + \frac{(\tau-a)^2}{2!}u''(a) + \cdots + \frac{(\tau-a)^{M-1}}{(M-1)!}u^{M-1}(a)$, $l(\tau) \in (0,T)$. Since $u_{kM}(\tau) =$

$\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}\ {}^{\lambda}\psi_{n,m}^{\rho,\alpha}(\tau)$ is the best approximation of $u(\tau)$, this implies that

$$\begin{aligned}
&\|u(\tau)-u_{kM}(\tau)\|^2_{L^2[0,T)}\\
&=\int_a^b (u(\tau)-\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b_{n,m}\ {}^{\lambda}\psi_{n,m}^{\rho,\alpha}(\tau))^2 d\tau,\\
&\le \int_a^b \left(\left|u(\tau)-\bar{u}(\tau)\right|\right)^2 d\tau,\\
&\le \frac{H^2}{M!^2}\int_a^b (\tau-a)^{2M} d\tau,
\end{aligned}$$

or

$$\|u(\tau)-u_{kM}(\tau)\|_{L^2[0,T)} \le \frac{C_3}{2^{k(M+\frac{1}{2})}\sqrt{2M+1}M!},$$

where $C_3 = HT^{M+\frac{1}{2}}$. This implies that error approaches to zero with the increase of M or k or both.□

For the convergence of quasilinearization technique, we have the following theorem

Theorem 7.4 Consider the second order boundary value problem in [36]. Let $u(\tau)$, $u'(\tau)$ and $u''(\tau)$ are bounded by some constants such that $\max_\tau |u(\tau)| \le w$, $\max_\tau |u'(\tau)| \le w$ and $\max_\tau |u''(\tau)| \le p$, then

$$\max_\tau |u_{q+1}(\tau)-u_q(\tau)| \le \frac{T^2\frac{p}{8}}{1-\frac{T^2w}{4}}(\max_\tau(|u_q(\tau)-u_{q-1}(\tau)|)^2,$$

where $w,\ p \in \mathbb{R}^+$.

Proof. For proof, we refer the readers to [36]. □

Theorem 7.4 shows that quasilinearization technique has quadratic convergence and converges to the exact solution when $q \to \infty$.

Lemma 7.5 [35] Suppose $u(t) \in C^2[t_0,t_r]$, and define $\mathbb{R}^r = {}_0^{CT}D_t^{\gamma,\lambda}u(t_r) - {}^{CT}\mathbb{D}_t^{\gamma,\lambda}u(t_r)$, then we can conclude that

$$\begin{aligned}
|\mathbb{R}^r| \le\ & \frac{1}{2\Gamma(1-\gamma)}\left[\frac{1}{4}+\frac{\gamma}{(1-\gamma)(2-\gamma)}\right]\times\\
&\left[\lambda^2\max_{t_0\le t\le t_r}|u(t)|+2\lambda\max_{t_0\le t\le t_r}|u'(t)|+\right.\\
&\left.\max_{t_0\le t\le t_r}|u''(t)|\right]\Delta t^{2-\gamma}.
\end{aligned}$$

Incorporating the Lemma 7.5 mentioned above, we proceed to calculate the error bound for the operator which is given in equation (19).

Theorem 7.6 If we assume that $u(t) \in C^2[t_0,t_r]$, and let ${}^{CT}\mathbb{R}^r = {}^{CT}D_t^{\gamma,\lambda}u(t_r) - {}^{CT}\mathbb{D}_t^{\gamma,\lambda}u(t_r)$ for $0<\gamma<1$, then

$$\begin{aligned}
\left|{}^{CT}\mathbb{R}^r\right| \le\ & \frac{1}{2\Gamma(1-\gamma)}\left[\frac{1}{4}+\frac{\gamma}{(1-\gamma)(2-\gamma)}\right]\times\\
&\left[\lambda^2\max_{t_0\le t\le t_r}|u(t)|+2\lambda\max_{t_0\le t\le t_r}|u'(t)|\right.\\
&\left.+\max_{t_0\le t\le t_r}|u''(t)|\right]\Delta t^{2-\gamma}\\
&+\frac{\epsilon\gamma e^{-\lambda\Delta t}}{\Gamma(1-\gamma)}\max_{t_0\le t\le t_{r-1}}|u(t)|.
\end{aligned}$$

proof. Clearly, the only difference between the fast algorithm designed for the tempered fractional operator and the discretization using L_1 is the addition of the history component which is specified in equation (17). The truncation error arises from the history part can be calculated by using the absolute error bound ϵ for the sum of exponential (SOE) approximations [34] as:

$$\begin{aligned}
&|{}^{CT}D_t^{\gamma,\lambda}u(t_r) - {}^{CT}\mathbb{D}_t^{\gamma,\lambda}u(t_r)| = |{}^{CT}\mathbb{R}^r|\\
&\le \left|\mathbb{R}^r + \epsilon\gamma\frac{e^{-\lambda t_r}}{\Gamma(1-\gamma)}\sum_{l=1}^{r-1}\int_{t_l-1}^{t_l}\prod_{1,l}P_\lambda(s)ds\right|,
\end{aligned}$$

where $\prod_{1,l}P_\lambda(t) = \frac{t_l-t}{\Delta t}P_\lambda(t_{l-1}) + \frac{t-t_{l-1}}{\Delta t}P_\lambda(t_l)$ and $P_\lambda(t) = e^{\lambda t}u(t)$. Using the triangular inequality, we get

$$|{}^{CT}\mathbb{R}^r| \le |\mathbb{R}^r| + \epsilon\gamma\frac{e^{-\lambda t_r}}{\Gamma(1-\gamma)}\sum_{l=1}^{r-1}\int_{t_l-1}^{t_l}|\prod_{1,l}P_\lambda(s)|ds,$$

where

$$\sum_{l=1}^{r-1}\int_{t_l-1}^{t_l}|\prod_{1,l}P_\lambda(s)|ds \le \max_{t_0\le t\le t_{r-1}}|u(t)|e^{t_{r-1}}t_{r-1}. \quad (29)$$

Combining Lemma 7.5 and equation (29), we get the required result. □

8. Applications

In this section, we implement the fast TFGW method, the L1- TFGW method and the fast-quasi TFGW method on linear and nonlinear Caputo-tempered variable order partial differential equations, respectively.

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8.1 Linear Caputo-tempered variable order partial differential equation

Consider the following tempered variable order boundary value problem

$$\begin{aligned} {}^{CT}D_t^{\gamma,\lambda}u(x,t) &= a(x)\,{}^{CT}D_t^{\beta(x),\lambda}u(x,t) \\ &+b(x)u(x,t)+f(x,t), \\ &0\le t\le 1,\ 0\le x\le T,\ T>0, \\ u(x,0) &= e^{-\lambda x}x^{w}, \\ u(0,t) &= e^{-\lambda t}t^{v}, \\ u(T,t) &= e^{-\lambda(T+t)}\left(T^{w}+t^{v}\right), \end{aligned} \tag{30}$$

where $\beta(x) = 2.3 - e^{-x}$, $0 < \gamma < 1$, and $f(x,t) = e^{-\lambda(x+t)}\left(\frac{\Gamma(v+1)}{\Gamma(v+1-\gamma)}t^{v-\gamma} - a(x)\frac{\Gamma(w+1)}{\Gamma(w+1-\beta(x))}x^{w-\beta(x)} - b(x)(x^{w}+t^{v})\right)$. The exact solution of (30) is $u(x,t) = e^{-\lambda(x+t)}\left(x^{w}+t^{v}\right),\ w,v\in\mathbb{R}^{+}$.

For the solution of equation (30), we firstly consider the following values of the parameters: $w = 4.125$, $v = 3.754$, $\lambda = 5$, $\rho = 0.5$, $\alpha = 0.95$, $\gamma = 0.75$, $\beta(x) = 2.3 - e^{-x}$, $T = 1$, $b(x) = 1$, and $a(x) = \sqrt{x}$. Figure 3 is used to plot the exact solution, $u_{ex}(x,t)$, solution by the fast TFGW method, $u_{app}(x,t)$, and their corresponding absolute error. It shows that solution by the fast TFGW method is in full agreement with the exact solution.

Figure 4 is used to plot the maximum absolute error by the fast TFGW method ($\alpha = 0.8$, $\gamma = 0.7$, $\rho = 0.5$, $\lambda = 8$, $R = 800$) against the parameters M and k. Figure 4 shows that maximum absolute error decreases while increasing the value of the parameters k and M, as shown in Theorem 7.3 of Section 7.

Based on Table 1, it is clear that the fast TFGW method, demonstrates a convergence rate that is outlined in the Theorem 7.6 of Section 7. Moreover, it indicate that with an increase in time steps, the error diminishes.

The comparison between the TFGW method, The L1-TFGW method and the fast TFGW method is given in Table 2. Table 2 shows that all the methods, the TFGW, the L1- TFGW and the fast TFGW method, are almost equally accurate but the fast TFGW method takes very less computational time as compared to the TFGW method and the L1- TFGW method, which is the main purpose of introducing the fast TFGW method. The results produced by the fast TFGW method and the L1-TFGW method are almost similar.

A comparison of computational time among these three methods, the TFGW method, the fast TFGW method, and the L1-TFGW method, is provided in Figure 5. For Figure 5, we have used the following values of the parameters $k = 3$, $M = 3$, $\lambda = 5$, $\rho = 0.5$, $\alpha = 0.95$,

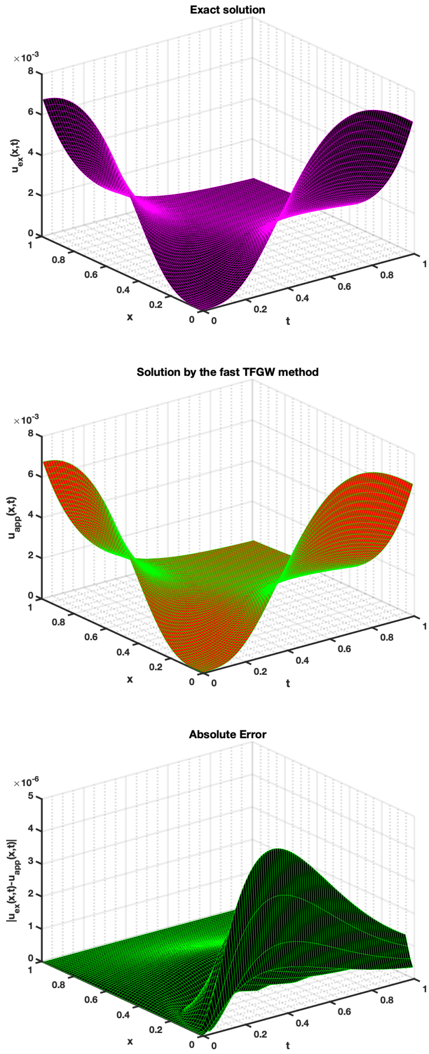

Figure 3. Exact solution, solutions by the fast TFGW method ($k = 4$, $M = 4$ and $R = 100$) and their corresponding absolute error.

$\beta(x) = 2.3 - e^{-x}$ and $\gamma = 0.75$. Figure 5 shows that the fast TFGW method takes very less computational time as compared to the TFGW method and the L1- TFGW method. The computational costs is considered by measuring the CPU time taken by the TFGW method, the L1-TFGW method and the fast TFGW method against the total number of time steps R, by fixing $k = 2$, $M = 2$. When $R = [1024,\ 2048,\ 4096,\ 8192]$, the CPU time (in second) taken by the fast TFGW method are $[0.3308,\ 0.5035,\ 0.8708,\ 1.5681]$, this means that CPU consumption rate of the fast TFGW method is almost $O(R)$, whereas the time taken by the L1- TFGW method are $[0.3076,\ 1.1292,\ 4.2754,\ 15.7772]$ which means that the CPU consumption rate of the L1 TFGW method is almost $O(R^2)$.

It illustrates that the fast TFGW method exhibits greater efficiency compared to both the TFGW method and the L1-TFGW method.

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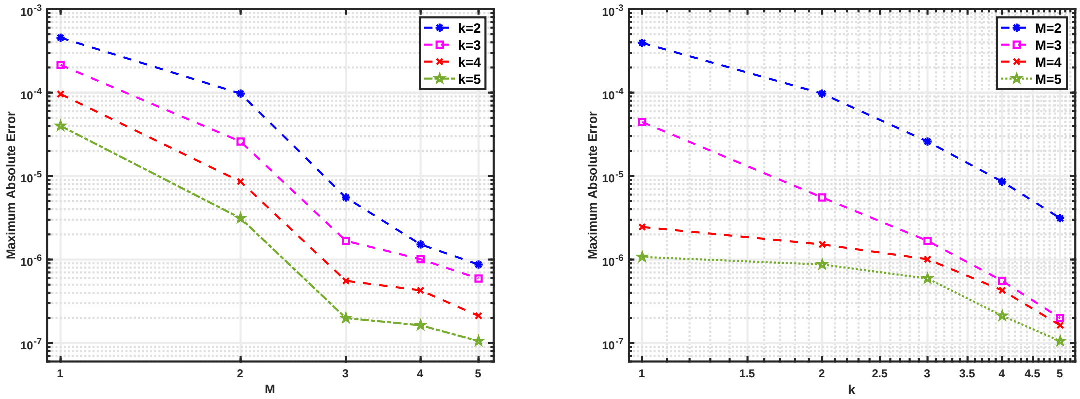

Figure 4. The maximum absolute errors by the fast TFGW method for $R = 800$, and different values of M and k.

Table 1. The maximum error and order of convergence at $w = 2,\ v = 2$, and for different values of time steps, R, and space steps, $\hat{p}$.

Time steps R	Space steps $\hat{p} = 2^k M$	Error $\gamma = 0.75$	Order	Error $\gamma = 0.90$	Order
32	$M = 1,\ k = 1$	2.0663×10^{-4}	–	3.6183×10^{-4}	–
64		9.0180×10^{-5}	1.1962	1.7184×10^{-4}	1.0743
128		3.8661×10^{-5}	1.2219	8.1002×10^{-5}	1.0850
256		1.6448×10^{-5}	1.2330	3.7991×10^{-5}	1.0923
512		6.9627×10^{-6}	1.2402	1.7775×10^{-5}	1.0958
1024		2.9391×10^{-6}	1.2443	8.3057×10^{-6}	1.0977
2048		1.2386×10^{-6}	1.2466	3.8782×10^{-6}	1.0987
4096		5.2149×10^{-7}	1.2480	1.8101×10^{-6}	1.0993
32	$M = 4,\ k = 4$	1.0463×10^{-4}	-	2.1369×10^{-4}	–
64		4.8657×10^{-5}	1.1046	1.0819×10^{-4}	0.9820
128		2.1458×10^{-5}	1.1812	5.2876×10^{-5}	1.0329
256		9.2722×10^{-6}	1.2105	2.5264×10^{-5}	1.0655
512		3.9521×10^{-6}	1.2303	1.1936×10^{-5}	1.0818
1024		1.6734×10^{-6}	1.2399	5.6017×10^{-6}	1.0913
2048		7.0562×10^{-7}	1.2458	2.6205×10^{-6}	1.0960
4096		2.9658×10^{-7}	1.2505	1.2237×10^{-6}	1.0986

Table 2. Comparison of the TFGW method, the L1- TFGW method and the fast TFGW method when $\gamma = 0.75,\ \beta(x) = 2.3 - e^{-x},\ w = 4.125,\ v = 3.754$, and for different values of time and space steps.

Time steps R	Space steps $\hat{p}$	The fast TFGW method E_{FTFGW} CPU_{FTFGW}	CPU time(sec) E_{L1}	The L1- TFGW method CPU_{L1}	CPU time(sec) E_{TFGW}	The TFGW method CPU_{TFGW}	CPU time(sec)
160	$k = 1, M = 1$	1.9084×10^{-3}	0.0812	1.9084×10^{-3}	0.0396	1.9391×10^{-3}	0.7001
320	$k = 2, M = 2$	1.5186×10^{-4}	0.1488	1.5186×10^{-4}	0.06107	1.5292×10^{-4}	2.3715
640	$k = 3,\ M = 3$	9.2022×10^{-7}	0.1972	9.2022×10^{-7}	0.1223	7.8065×10^{-7}	8.8944
1280	$k = 4,\ M = 4$	3.0521×10^{-7}	0.4726	3.0521×10^{-7}	0.6565	3.2632×10^{-7}	65.8420
2560	$k = 5,\ M = 5$	1.2065×10^{-7}	2.8243	1.2065×10^{-7}	4.4486	6.8298×10^{-8}	193.0953

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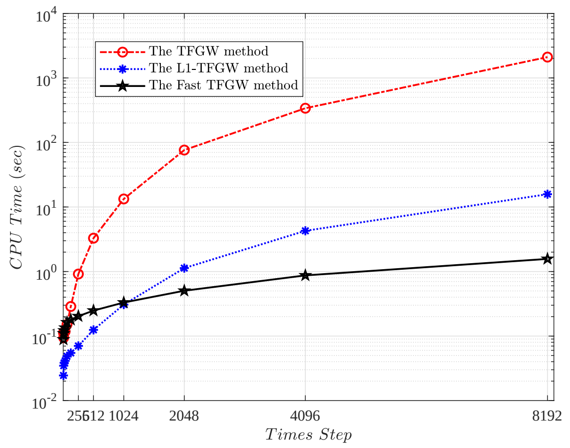

Figure 5. Comparison of the fast TFGW, the L1- TFGW and the TFGW method are presented for $M = 2$, $k = 2$ and at different values of time steps R.

8.2 The fast-quasi TFGW method: Nonlinear Caputo-tempered variable order partial differential equation

We have also extended the fast TFGW method for the solution of nonlinear tempered variable order partial differential equations. For that purpose, we propose a method, which we named as the fast-quasi TFGW method, by merging the quasilinearization technique with the fast TFGW method. According to the proposed method, we first descritize the nonlinear tempered variable order partial differential equation by quasilinearization technique and then utilize the fast TFGW operational matrices method, given in Section (5), at each iteration of quasilinearization technique.

Consider the nonlinear tempered variable order Burger's type equation of the form

$$\begin{aligned} &{}^{CT}D_t^{\gamma,\lambda}u(x,t) + u(x,t)\,{}^{CT}D_x^{\eta(x),\lambda}u(x,t) \\ &\quad = D_x^{\beta(x),\lambda}u(x,t) + f(x,t), \\ &\quad 0 \le \eta(x) \le 1,\ 1 \le \beta(x) \le 2, \\ &\quad u(x,0) = 0,\ 0 \le x \le 1, \\ &\quad u(0,t) = 0,\ u(1,t) = e^{-\lambda(1+t)}t^{2.9},\ 0 \le t \le 1, \end{aligned} \tag{31}$$

where $\beta(x) = 2.5 - \cos(x)$, $\eta(x) = \frac{1+\sin(x)}{2}$, $0 < \gamma < 1$, and $f(x,t) = e^{-\lambda(x+t)}\left(\frac{x^{2.9}\Gamma(3.9)}{\Gamma(3.9-\gamma)}t^{2.9-\gamma} + t^{2.9}e^{-\lambda(x+t)}\frac{\Gamma(3.9)}{\Gamma(3.9-\eta(x))}x^{5.8-\eta(x)} - \frac{t^{2.9}\Gamma(3.9)}{\Gamma(3.9-\beta(x))}x^{2.9-\beta(x)}\right)$.

The exact solution of (31) is $u(x,t) = e^{-\lambda(x+t)}\left(xt\right)^{2.9}$.

The firs step is to apply the quasilinearization technique to the above problem (31), we get

$$\begin{aligned} &{}^{CT}D_t^{\gamma,\lambda}u_{q+1}(x,t) + u_q(x,t)\,{}^{CT}D_x^{\eta(x),\lambda}u_{q+1}(x,t) \\ &\quad + {}^{CT}D_x^{\eta(x),\lambda}u_q(x,t)u_{q+1}(x,t) = \\ &\quad D_x^{\beta(x),\lambda}u_{q+1}(x,t) + f(x,t) + \\ &\quad u_q(x,t)\,{}^{CT}D_x^{\eta(x),\lambda}u_q(x,t), \\ &\quad u_{q+1}(x,0) = 0, \quad 0 \le x \le 1, \\ &\quad u_{q+1}(0,t) = 0, \\ &\quad u_{q+1}(1,t) = e^{-\lambda(1+t)}t^{2.9}, \quad 0 \le t \le 1. \end{aligned} \tag{32}$$

The second step is to implement the fast TFGW method on (32) for each $q = 0, 1, 2, \cdots$. For the solution of equation (32), we consider the following value of the parameters: $\lambda = 0.05$, $\rho = 0.75$, $\alpha = 0.59$, $\gamma = 0.75$, $\eta(x) = \frac{1+\sin(x)}{2}$ and $\beta(x) = 2.5 - \cos(x)$. We have considered $k = 3$, $M = 3$, $q = 5$ and $R = 200$ for the construction of Figure 6. Figure 6 is used to plot the exact solution, $u_{ex}(x,t)$, solution by the fast-quasi TFGW method $u_{app}(x,t)$ and their corresponding absolute error. It shows that solution by the fast-quasi TFGW method is in full agreement with the exact solution.

We want to show the effect of quasilinearization technique when it is merge with the fast TFGW method for the solution of nonlinear problems. For this purpose, we consider the following value of the parameters $\lambda = 0.05$, $\rho = 0.75$, $\alpha = 0.6$, $\gamma = 0.51, \eta(x) = \frac{1+\sin(x)}{2}$, $\beta(x) = 2.5 - \cos(x)$, and the results are shown in Figure 7. It shows that the maximum absolute errors decreases with the increase of q, k, M and R.

9. Conclusion

In this paper, four main objectives were pursued. Firstly, the TFGW method were introduced for solving Caputo-tempered variable order differential equations. To achieve this, we constructed the TFGW matrix, ${}^{\lambda}\mathbf{\Psi}^{\rho,\alpha}_{\hat{p}\times\hat{p}}$, and its operational matrices of tempered variable order integration, ${}^{\lambda}\mathbf{W}^{\beta(\tau)}_{\hat{p}\times\hat{p}}$ and ${}^{\lambda}\mathbf{P}^{g(\tau),\beta(T)}_{\hat{p}\times\hat{p}}$. These operational matrices helped to reduce computational costs due to their numerous zero entries. These matrices were used to convert the Caputo-tempered variable order differential equations into a system of algebraic equations. This process streamlined the computational process and making it more efficient. Results produced by the TFGW method are shown in Figure 1. It indicates that our results are in good agreement with the exact solution and we can get better result by increasing the value of the parameters k and M, as given in the analysis of the method. According to the Theorems 7.2 and 7.3, we conclude that the solution is achieved using the TFGW method, $\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} b^{\lambda}_{n,m}\psi^{\rho,\alpha}_{n,m}(\tau)$, converges to the exact solution when k or M or both approaches to ∞.

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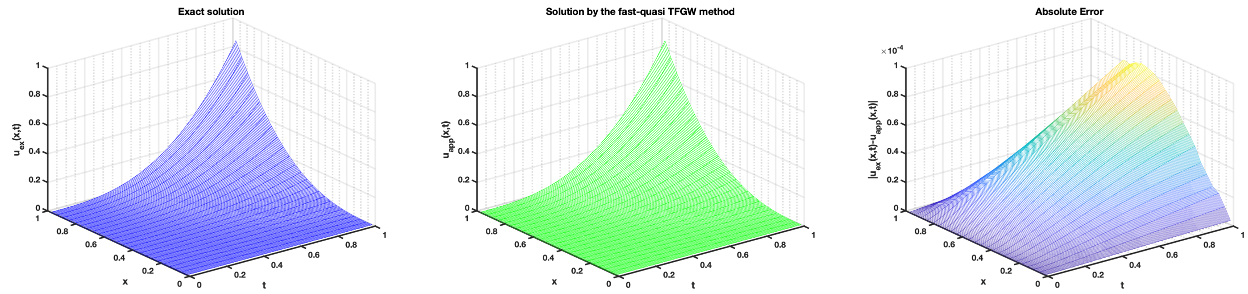

Figure 6. Solutions by the fast-quasi TFGW method and their corresponding absolute error.

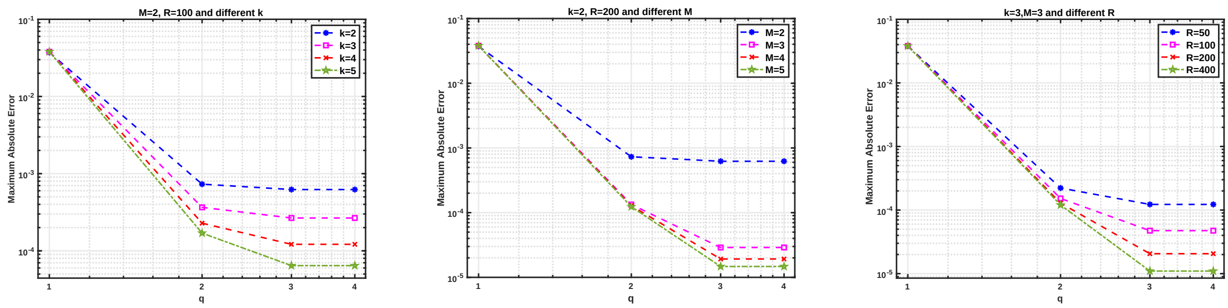

Figure 7. The maximum absolute errors against q for the fast-quasi TFGW method at different values of M, k and R.

Secondly, the fast TFGW method was introduced for solving Caputo-tempered partial differential equations with both time fractional and spatial variable order derivatives. This method aimed to reduce computational costs and enhance efficiency. The result produced by the fast TFGW method are shown in Figure 3 and Table 2. According to Figure 3, our results are in full agreement with the exact solution. Table 2 is used to enlist the absolute error and computational time by the TFGW method, the L1- TFGW method and the fast TFGW method. According to the Table 2, the fast TFGW method takes very less computational time as compared to both the TFGW method and the L1-TFGW method, which is the main advantage of the fast TFGW method over the TFGW method and the L1- TFGW method. According to the Theorems 7.2, 7.3 and 7.6, we conclude that the solution by the fast TFGW method converges to the exact solution when k or M or R or all approaches to ∞.

The third objective was to introduced the L1- TFGW method for solving Caputo-tempered partial differential equations with both time fractional and spatial variable order derivatives. The aim of introducing this method is to show that the L1- TFGW method is more efficient as compared to the TFGW method, but less efficient as compared to the fast TFGW method. According to the Figure 5, the fast TFGW method exhibits greater efficiency as compared the TFGW method and the L1- TFGW method.

Lastly, the fast-quasi TFGW method was introduced for solving nonlinear Caputo-tempered partial differential equations with time fractional and space variable order derivatives. The results are shown in Figure 6 and Figure 7. They indicates that the results produced by the fast-quasi TFGW method are in good agreement with the exact solution and are getting more accurate while increasing the value of the parameters k, M, R and q, as given in the analysis of the method. According to the Theorems 7.2, 7.3, 7.6, 7.4, and Figure 7, the solution by the fast-quasi TFGW method converges to the exact solution when k, M, R and q approaches to ∞.

Convergence analysis, error analysis and methodology of the methods were also conducted. These methods were aimed at advancing the field of applied mathematics by providing efficient and accurate methods for solving linear and nonlinear Caputo-tempered variable order differential equations.

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Authors contributions
All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.
Availability of data and materials
No data were analyzed during this study.
Conflict of interests
Authors declared that there is no competing interests for this paper.

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