DE GRUYTER Open Mathematics 2023; 21: 20230160

Research Article

Yoonjin Lee, Jungyun Lee, and Jinjoo Yoo*

Average value of the divisor class numbers of real cubic function fields

https://doi.org/10.1515/math-2023-0160
received June 19, 2023; accepted November 20, 2023

Abstract: We compute an asymptotic formula for the divisor class numbers of *real* cubic function fields $K_m = k(\sqrt[3]{m})$, where $\mathbb{F}_q$ is a finite field with q elements, $q \equiv 1 \pmod 3$, $k \coloneqq \mathbb{F}_q(T)$ is the rational function field, and $m \in \mathbb{F}_q[T]$ is a cube-free polynomial; in this case, the degree of m is divisible by 3. For computation of our asymptotic formula, we find the average value of $|L(s,\chi)|^2$ evaluated at $s = 1$ when χ goes through the primitive cubic *even* Dirichlet characters of $\mathbb{F}_q[T]$, where $L(s,\chi)$ is the associated Dirichlet L-function.

Keywords: L-function, average value of class number, cubic function field, moment over function field

MSC 2020: 11M38, 11R29, 11M06, 11R16, 11R58

1 Introduction

There have been many developments in the study of moments of L-function families since they have many connections to the famous Lindelöf hypothesis for such L-functions [1]. In fact, Gauss [2] made two conjectures on the mean value of the class numbers of quadratic number fields: one is for the imaginary case and the other is for the real case. The conjecture on the imaginary case was proved by Lipschitz [3], Mertens [4], Siegel [5], and Vinogradov [6], and the conjecture on the real case was proved by Siegel [5].

In the function field context as well, there has been active research done on the study of moments of L-function families and class numbers of global function fields (e.g., refer to [7–16]). Let $k \coloneqq \mathbb{F}_q(T)$ be the rational function field and $A \coloneqq \mathbb{F}_q[T]$, where $\mathbb{F}_q$ is a finite field of order q. Let K be a global function field, which is an algebraic extension over k. We say that K is *real* if the infinite place ∞ of k splits completely in K; otherwise, we call K *imaginary*. Inspired by Gauss's conjectures, Hoffstein and Rosen [9] computed the mean value of the (divisor) class numbers h_m of quadratic function fields $k(\sqrt{m})$, where m is a nonsquare polynomial in A. We point out that they computed both cases of imaginary fields and real fields. In detail, for a positive integer n, let A_n^+ be the set of monic polynomials in A of degree n. They obtained the following results: if n is odd (in this case, $k(\sqrt{m})$ is imaginary), then

$$\frac{1}{q^n}\sum_{m\in A_n^+} h_m = \frac{\zeta(2)}{\zeta(3)}q^{\frac{n-1}{2}} - q^{-1},$$

and if n is even (in this case, $k(\sqrt{m})$ is real), then

*** Corresponding author: Jinjoo Yoo,** Department of Mathematical Sciences, Ulsan National Institute of Science and Technology, 50, UNIST-gil, Ulsan 44919, Republic of Korea, e-mail: jinjooyoo@unist.ac.kr
Yoonjin Lee: Department of Mathematics, Ewha Womans University, 52, Ewhayeodae-gil, Seodaemun-gu, Seoul 03760, Republic of Korea, e-mail: yoonjinl@ewha.ac.kr
Jungyun Lee: Department of Mathematics Education, Kangwon National University, 1, Kangwondaehak-gil, Chuncheon-si, Gangwon-do 24341, Republic of Korea, e-mail: lee9311@kangwon.ac.kr

$$\frac{1}{q^n}\sum_{m\in A_n^+} h_m = (q-1)^{-1}\left(\frac{\zeta(2)}{\zeta(3)}q^{\frac{n}{2}} - (2 + (1-q^{-1})(n-1))\right),$$

where $\zeta(s)$ is the zeta function of A. In fact, we note that $h_m = \widetilde{h}_m R_m$ [17, Prop. 14.7], where $\widetilde{h}_m$ is the ideal class number of $k(\sqrt{m})$ (the order of the ideal class group of the maximal order $O_{k(\sqrt{m})}$ of $k(\sqrt{m})$) and R_m is the regulator of $k(\sqrt{m})$.

Now, we discuss the case of *cubic* fields. As a matter of fact, in the number field situation, there has been no work done on the mean value computation for cubic fields yet. On the other hand, in the function field context, Lee et al. [18] obtained an asymptotic formula for the mean value of the divisor class numbers of cubic function fields $K_m = k(\sqrt[3]{m})$, where $q \equiv 1 \pmod 3$, $m \in \mathbb{F}_q[T]$ is a cube-free polynomial, and $\deg(m) \equiv 1 \pmod 3$; in this case, K_m is *imaginary*. Therefore, the goal of this article is to compute an asymptotic formula for the mean value of the divisor class numbers of *real* cubic function fields K_m; in this case, $\deg(m)$ is divisible by 3. We note that the infinite place ∞ of k splits completely in K_m when the degree of m is divisible by 3. To achieve our goal, we compute the mean value of $|L(s,\chi)|^2$ evaluated at $s=1$ when χ average runs through the primitive cubic *even* Dirichlet characters of $\mathbb{F}_q[T]$ as in Theorem 1.1, where $L(s,\chi)$ is the associated Dirichlet L-function.

We state the main results as follows.

Theorem 1.1. *Let h be a polynomial in $A := \mathbb{F}_q[T]$ with $\deg(h) = g+2$. Let S_g be the set of primitive cubic even characters with conductor g. Then, we have the following*:

$$\sum_{\chi\in S_g} |L(1,\chi)|^2 = (C_1 g + C_2)q^{g+2} + O\left(q^{\frac{g}{2}+\varepsilon g+1}\right). \tag{1}$$

Furthermore, the average value of $|L(1,\chi)|^2$ is given as follows:

$$\frac{\sum_{\chi\in S_g}|L(1,\chi)|^2}{|S_g|} = \frac{C_1 g + C_2}{B_1 g + B_2} + O\left(q^{\varepsilon g - \frac{g}{2}}\right),$$

where B_i and C_i $(i=1,2)$ are defined in Notation 1 as follows:

Notation 1.

$$|P| := q^{\deg A} \text{ with } P \in A,$$
ξ_3 is the third root of unity,
$$\mathcal{F}(x,y) := \prod_P (1 - x^{2\deg(P)} - y^{2\deg(P)} - (xy)^{\deg(P)} + (x^2y)^{\deg(P)} + (xy^2)^{\deg(P)}),$$
$$B_1 = \frac{1}{3}\mathcal{F}\left(\frac{1}{q},\frac{1}{q}\right),$$
$$B_2 = \frac{2}{3}\left[\mathcal{F}\left(\frac{1}{q},\frac{1}{q}\right) - \frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{F}(x,x)|_{x=\frac{1}{q}} - \frac{\mathcal{F}\left(\frac{1}{q},\frac{\xi_3}{q}\right)\xi_3^{g+1}}{1-\xi_3} - \frac{\mathcal{F}\left(\frac{\xi_3^2}{q},\frac{1}{q}\right)\xi_3^{2g+2}}{1-\xi_3^2}\right],$$
$$\mathcal{D}(x,y,u,w) := \prod_P (1-x^{\deg(P)})(1-y^{\deg(P)})\left(1 - \frac{(u^3w^3)^{\deg(P)}}{|P|^6} + (x^{\deg(P)} + y^{\deg(P)})(1 - f(u,w))\right),$$
$$C_1 = \frac{\zeta(2)\zeta(3)^2}{3}\mathcal{D}\left(\frac{1}{q},\frac{1}{q},1,1\right),$$
$$C_2 = \frac{\zeta(2)\zeta(3)^2}{3}\left[\mathcal{D}\left(\frac{1}{q},\frac{1}{q},1,1\right) - \frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{D}(x,x,1,1)|_{x=\frac{1}{q}} - \frac{\mathcal{D}\left(\frac{1}{q},\frac{\xi_3}{q},1,1\right)\xi_3^{g+1}}{1-\xi_3} - \frac{\mathcal{D}\left(\frac{\xi_3^2}{q},\frac{1}{q},1,1\right)\xi_3^{2g+2}}{1-\xi_3^2}\right],$$
$$f(u,w) := \frac{u^{3\deg(P)}}{|P|^3} + \frac{(uw)^{\deg(P)}}{|P|^2} + \frac{w^{3\deg(P)}}{|P|^3} - \frac{(u^4w)^{\deg(P)}}{|P|^5} - \frac{(uw^4)^{\deg(P)}}{|P|^5} - \frac{(u^3w^3)^{\deg(P)}}{|P|^6} + \frac{(u^4w^4)^{\deg(P)}}{|P|^8}.$$

As a consequence, we find an asymptotic formula for the average value of the class numbers of cubic real function fields in Theorem 1.2.

Theorem 1.2. *Let M_g be the set of monic cube-free polynomials m in $A \coloneqq \mathbb{F}_q[T]$ such that the degree of m is divisible by 3 and the genus of $K_m = k(\sqrt[3]{m})$ is g, where g is a positive integer. Let h_m be the divisor class number of K_m, which is defined to be the order of the divisor class group of K_m.*

Then, the average value of the class numbers h_m of real cubic function fields K_m is given as follows:

$$\frac{\sum_{m\in M_g} h_m}{|M_g|} = \frac{q^g(C_1 g + C_2)}{B_1 g + B_2} + O(q^{\frac{g}{2}+\varepsilon g}),$$

where B_i and C_i $(i = 1, 2)$ are defined in Notation 1.

We briefly mention the difference between our current work and the previous work [18] as follows. For computation of the mean value of $|L(s,\chi)|^2$ evaluated at $s = 1$, in [18], χ runs through the primitive cubic *odd* Dirichlet characters of $\mathbb{F}_q[T]$; in this article, we deal with the case of the *even* Dirichlet characters of $\mathbb{F}_q[T]$. We emphasize that the computational complexity for $|L(1,\chi)|^2$ with even characters χ increases significantly compared with the case of odd characters χ. In fact, the major difference between the even case and the odd case in terms of complexity comes from the difference between two functional equations of $L(s,\chi)$ for odd and even primitive characters as follows. Let χ be a primitive character of modulus $R \neq 1$. By [1, Theorem 3.9], if χ is odd, then the functional equation is

$$L(s,\chi) = W(\chi) q^{\frac{\deg R - 1}{2}} (q^{-s})^{\deg R - 1} L(1-s, \overline{\chi}),$$

and if χ is even, then the functional equation satisfies the following:

$$(q^{1-s} - 1) L(s,\chi) = W(\chi) q^{\frac{\deg R}{2}} (q^{-s} - 1)(q^{-s})^{\deg R - 1} L(1-s, \overline{\chi}),$$

with $|W(\chi)| = 1$.

For the case where χ is odd, taking the squared modulus of both sides of the functional equation and letting $s = 1$, we obtain [18, Lemma 3.1] the following:

$$|L(1,\chi)|^2 \sum_{n=0}^{\deg R - 1} \left(\sum_{\substack{0 \le i,j < \deg R \\ i+j=n}} L_i(\chi) L_j(\overline{\chi}) \right) q^{-n} + q^{-g} \sum_{n=0}^{\deg R - 2} \left(\sum_{\substack{0 \le i,j < \deg R \\ i+j=n}} L_i(\chi) L_j(\overline{\chi}) \right),$$

where $L_i(\chi) \coloneqq \sum_{\deg a = i, a \in A^+} \chi(a)$.

Unlike the odd case, if χ is even, we need to take derivatives of both sides of the functional equation of $L(s,\chi)$ with respect to s twice. Letting $s = 1$, we obtain

$$|L(1,\chi)|^2 = \frac{1}{2} \left(\sum_{n=1}^{\deg R} \left(\sum_{\substack{0 \le i,j \le \deg R \\ i+j=n}} M_i(\chi) M_j(\overline{\chi}) \right) n^2 q^{-n} + \sum_{n=1}^{\deg R - 1} \left(\sum_{\substack{0 \le i,j \le \deg R \\ i+j=n}} M_i(\chi) M_j(\overline{\chi}) \right) (2g + 4 - n)^2 q^{-\deg R} \right),$$

where $M_i(\chi) \coloneqq qL_{i-1}(\chi) - L_i(\chi)$, $L_{-1}(\chi) \coloneqq 0$, and $L_{g+2}(\chi) = 0$. Due to this difference, we point out that Lemma 3.2 plays a significant role for our main computation.

This article is organized as follows: in Section 2, we recall some basic definitions and necessary lemmas that are useful for our main results; in Section 3, we estimate the value $\sum_\chi |L(1,\chi)|^2$ (Lemmas 3.1–3.5), and for the computation of $\sum_\chi |L(1,\chi)|^2$, we divide the formula of $\sum_\chi |L(1,\chi)|^2$ into three parts; and finally in Section 4, we give the proofs of our main results: Theorems 1.1 and 1.2.

2 Preliminaries

Let $\mathbb{F}_q$ be a finite field of order q, where q is an odd prime power with $q \equiv 1 \pmod 3$. Let $k := \mathbb{F}_q(T)$ be the rational function field and $A := \mathbb{F}_q[T]$ be a polynomial ring. For a nonzero polynomial $f \in A$, the norm of f is defined as $|f| := q^{\deg(f)}$. We denote the set of monic polynomials of A by A^+.

Definition 2.1. The *zeta function* of A, denoted by $\zeta_A(s)$, is defined by the infinite series $\zeta_A(s) := \sum_{f \in A^+} |f|^{-s}$. There are exactly q^d monic polynomials of degree d in A; thus, $\sum_{\deg(f) \le d} \frac{1}{|f|^s} = 1 + \frac{q}{q^s} + \frac{q^2}{q^{2s}} + \cdots + \frac{q^d}{q^{ds}}$, and consequently, $\zeta_A(s) = \frac{1}{1 - q^{1-s}}$ for all complex numbers s with $\mathrm{Re}(s) > 1$. Letting $u := q^{-s}$, we obtain the identity

$$\zeta_A(s) = \frac{1}{1 - q^{1-s}} = \frac{1}{1 - qu} = \sum_{n=0}^{\infty} u^n q^n;$$

we use the fact that $\mathrm{Re}(s) > 1$ is equivalent to $|u| < 1/q$. From now on, for simplicity, we denote $\zeta_A(s)$ by $\zeta(s)$.

Definition 2.2. Let h be a monic polynomial in A. A *Dirichlet character* on A of modulus h is a function $\chi : A \to \mathbb{C}$ that satisfies the following properties: for all $a, b \in A$,

(i) $\chi(ab) = \chi(a)\chi(b)$;
(ii) if $a \equiv b \pmod h$, then $\chi(a) = \chi(b)$;
(iii) $\chi(a) \neq 0$ if and only if $(a, h) = 1$.

The *trivial Dirichlet character* of modulus h is defined by $\chi(a) = \begin{cases} 1 & \text{if}(a,h) = 1, \\ 0 & \text{if}(a,h) \neq 1; \end{cases}$ we denote this by χ_0. The *inverse of a Dirichlet character* χ, denoted by $\overline{\chi}$, is defined by $\overline{\chi}(a) = \overline{\chi(a)}$ for all $a \in A$, where $\overline{\chi(a)}$ is a complex conjugate of $\chi(a)$. We say that a character χ is *even* if $\chi(c) = 1$ for all $c \in \mathbb{F}_q^\times$; otherwise, it is called an *odd* character. A character χ such that $\chi^3 = \chi_0$ and $\chi \neq \chi_0$ is called a *cubic Dirichlet character*.

A Dirichlet character of modulus h induces a homomorphism $(A/hA)^\times \to \mathbb{C}^\times$. Conversely, given such a homomorphism, there is a uniquely corresponding Dirichlet character [17, p. 35]. Abusing the notation, let $\chi : (A/hA)^\times \to \mathbb{C}^\times$ be a Dirichlet character of modulus h. For a Dirichlet character χ of modulus h, we say that we may *define* χ mod f for $f|h$ if there exists $\xi : (A/fA)^\times \to \mathbb{C}^\times$ such that $\xi \circ \varphi_{h,f} = \chi$, where $\varphi_{h,f}$ is a canonical homomorphism from $(A/hA)^\times$ to $(A/fA)^\times$. We note that given a Dirichlet character χ of modulus h, there exists a unique monic polynomial $f \in A$ of minimal degree dividing h such that χ can be defined mod f [19, Theorem 12.6.3].

Definition 2.3. Given a Dirichlet character χ of modulus h, the *conductor of* χ is f if $f \in A$ is a monic polynomial of minimal degree dividing h such that χ can be defined mod f. Let f be the conductor of a Dirichlet character χ. If χ is defined mod f, then we say that χ is *primitive*.

We now introduce the definition of the cubic character χ_p defined by the cubic residue symbol, where $p \in A^+$ is an irreducible polynomial.

Definition 2.4. Let $p \in A^+$ be an irreducible polynomial and a be a polynomial in A. Let Ψ be an isomorphism between the cubic roots of unity in $\mathbb{C}^\times$ and the cubic roots of unity in $\mathbb{F}_q$. We define a *cubic character* χ_p *by means of the cubic residue symbol as follows*: if $p|a$, then $\chi_p(a) = 0$; otherwise, $\chi_p(a) := \alpha$, where α is the unique root of unity such that $a^{\frac{|p|-1}{3}} \equiv \Psi(\alpha) \pmod p$.

This definition can be extended to any monic polynomial $h \in A$. Let $h = \prod_{i=1}^{s} p_i^{e_i}$ be a prime factorization in A, where e_i are positive integers for $1 \le i \le s$. Then, χ_h is defined as follows:

$$\chi_h = \chi_{p_1}^{e_1} \chi_{p_2}^{e_2} \cdots \chi_{p_s}^{e_s}. \tag{2}$$

Then, χ_h is a cubic character of modulus $\prod_{i=1}^{s} p_i$.

We now define an important set M_g as follows. Let M_g be the set of monic cube-free polynomials m in $A := \mathbb{F}_q[T]$ such that the degree of m is divisible by 3 and the genus of $K_m = k(\sqrt[3]{m})$ is g, where g is a positive integer. Since m is a monic cube-free polynomial, there are monic square-free polynomials m_1 and m_2 in A with $(m_1, m_2) = 1$ such that $m = m_1 m_2^2$. By [13, Lemma 3.2], we obtain $\frac{(3-1)(\deg(m_1)+\deg(m_2)-2)}{2} = g$, i.e., $\deg(m_1) + \deg(m_2) = g + 2$. Therefore, the set M_g can be written as follows:

$$M_g = \{m = m_1 m_2^2 \in A^+ \mid m_1, m_2 \text{ are square-free polynomials with } (m_1, m_2) = 1, \deg(m_1 m_2^2) \equiv 0 \pmod 3, \deg(m_1) + \deg(m_2) = g + 2\}, \tag{3}$$

Let χ_m be the cubic character associated with K_m, where $m \in M_g$. From the condition that 3 divides the degree of m, the character χ_m is even [8, p. 1273]. In addition, we note that the conductor of χ_m is $m_1 m_2$.

Definition 2.5. Let χ be a Dirichlet character. The associated *Dirichlet L-function* is defined for $\mathrm{Re}(s) > 1$ by

$$L(s, \chi) = \sum_{f \in A^+} \frac{\chi(f)}{|f|^s}.$$

For the proof of our main results, we introduce a crucial lemma, i.e., known as *Perron's formula*. For convenience, we let A_n^+ ($A_{\leq n}^+$) be the set of monic polynomials of A of degree n (degree $\leq n$), respectively.

Lemma 2.6. [8, Lemma 2.1] (Perron's formula) *If the generating series $\mathcal{A}(u) = \sum_{f \in A^+} a(f) u^{\deg f}$ is absolutely convergent in $|u| \leq r < 1$, then*

$$\sum_{f \in A_n^+} a(f) = \frac{1}{2\pi i} \int_{|u|=r} \frac{\mathcal{A}(u)}{u^n} \frac{du}{u}$$

and

$$\sum_{f \in A_{\leq n}^+} a(f) = \frac{1}{2\pi i} \int_{|u|=r} \frac{\mathcal{A}(u)}{u^n(1-u)} \frac{du}{u}.$$

3 Necessary lemmas for main computations

In this section, we prove five important lemmas for finding the second moment of the class numbers of real cubic function fields with $q \equiv 1 \pmod 3$.

We define a set S_g to be

$$S_g := \{\chi_m | m \in M_g\}; \tag{4}$$

we note that $\mathcal{S}_g$ is a set of primitive cubic even characters with conductor whose degree is g.

For our computation, we need the following lemma.

Lemma 3.1. *Let q be an odd prime power such that $q \equiv 1 \pmod 3$ and χ_a be a cubic character defined in equation* (2). *Let $\mathcal{S}_g$ be the set that is defined in equation* (4). *Then, we have the following*:

$$\chi^{\mathrm{av}}(a) := \sum_{\chi \in \mathcal{S}_g} \chi(a) = \sum_{\substack{d_1 + d_2 = g+2 \\ d_1 + 2d_2 \equiv 0 \pmod 3}} \sum_{\substack{m_1 \in \mathfrak{A}_{d_1} \\ m_2 \in \mathfrak{A}_{d_2} \\ (m_1, m_2) = 1}} \chi_a(m_1 m_2^2),$$

where $d_i = \deg(m_i)$ and $\mathfrak{A}_{d_i}$ is the set of monic square-free polynomials of A of degree d_i for $i = 1, 2$.

Proof. By equation (4), we obtain $\chi = \chi_m$ for $m \in M_g$, where $\chi \in \mathcal{S}_g$. According to the description of M_g in equation (3), we have $d_1 + d_2 = g + 2$ and $d_1 + 2d_2 \equiv 0 \pmod 3$, where $d_i = \deg(m_i)$ for $i = 1, 2$. Then, we obtain

$$\sum_{\chi \in \mathcal{S}_g} \chi(a) = \sum_{m \in M_g} \chi_m(a) = \sum_{m \in M_g} \chi_a(m);$$

the second equality, which follows from reciprocity law [17, Theorem 3.5] under the assumption that q is an odd prime power and $q \equiv 1 \pmod 3$. The rest of the proof follows immediately from [8, Lemma 2.9]. □

We recall that A^+ refers to the set of monic polynomials of $A = \mathbb{F}_q[T]$ and $\mathfrak{A}_d$ is the set of monic square-free polynomials of A of degree d. From now on, we denote by $\mathfrak{A}$ the set of monic square-free polynomials of A. In addition, for some $f \in A^+$, we denote by $ab^2 = \square$ if ab^2 can be written as f^3. If not, we put $ab^2 \neq \square$.

Notation 2.

$$\mathcal{A}_{\leq g,\square} := \sum_{\substack{a,b\in A^+\\ \deg(ab)\leq g\\ ab^2=\square}} \frac{\chi^{\mathrm{av}}(ab^2)}{|ab|} \qquad \widetilde{\mathcal{A}}_{\leq g} := \sum_{\substack{a,b\in A^+\\ \deg(ab)\leq g\\ ab^2\neq\square}} \frac{\chi^{\mathrm{av}}(ab^2)}{|ab|}$$

$$\mathcal{B}_{=v,\square} := \sum_{\substack{a,b\in A^+\\ \deg(ab)=v\\ ab^2=\square}} \chi^{\mathrm{av}}(ab^2) \qquad \widetilde{\mathcal{B}}_{=v} := \sum_{\substack{a,b\in A^+\\ \deg(ab)=v\\ ab^2\neq\square}} \chi^{\mathrm{av}}(ab^2)$$

$$\mathcal{B}_{\leq v,\square} := \sum_{\substack{a,b\in A^+\\ \deg(ab)\leq v\\ ab^2=\square}} \chi^{\mathrm{av}}(ab^2) \qquad \widetilde{\mathcal{B}}_{\leq v} := \sum_{\substack{a,b\in A^+\\ \deg(ab)\leq v\\ ab^2\neq\square}} \chi^{\mathrm{av}}(ab^2)$$

$$\mathcal{G}_v := \sum_{\substack{a,b\in A^+\\ \deg(ab)=v}} \frac{\chi^{\mathrm{av}}(ab^2)}{|ab|} \qquad \mathcal{H}_i(g-1) := \sum_{\substack{a,b\in A^+\\ \deg(ab)\leq g-1}} (\deg(ab)^i \chi^{\mathrm{av}}(ab^2))$$

Lemma 3.2. *Let $\mathcal{S}_g$ be a set of all primitive cubic even characters with conductor g as defined in equation* (4). *Then, we obtain the following*:

$$\begin{aligned}\sum_{\chi\in\mathcal{S}_g} |L(1,\chi)|^2 &= \frac{(g+2)^2}{2}\mathcal{G}_{g+2} - \frac{(g^2+6g+7)}{2}\mathcal{G}_{g+1} + \mathcal{A}_{\leq g,\square} + \widetilde{\mathcal{A}}_{\leq g}\\ &+ q^{-g-2}\Bigg(\frac{(q-1)^2}{2}\mathcal{H}_2(g-1) - 2(q-1)(q+gq-g-2)\mathcal{H}_1(g-1)\\ &+ (2(q-1)(g+2)(qg-g-2) + q(2q-1)\Bigg)(\mathcal{B}_{\leq g-1,\square} + \widetilde{\mathcal{B}}_{\leq g-1})\\ &- \left(q(g+3)^2 - \frac{(g+4)^2}{2}\right)(\mathcal{B}_{=g,\square} + \widetilde{\mathcal{B}}_{=g}) + \frac{(g+3)^2}{2}(\mathcal{B}_{=g+1,\square} + \widetilde{\mathcal{B}}_{=g+1}) - (q-1)^2(g+2)^2\Bigg).\end{aligned}$$

Proof. We first claim that

$$\begin{aligned}|L(1,\chi)|^2 &= \frac{(g+2)^2}{2}\sum_{\substack{a,b\in A^+\\ \deg(ab)=g+2}} \frac{\chi(a)\overline{\chi}(b)}{|ab|} - \frac{(g^2+6g+7)}{2}\sum_{\substack{a,b\in A^+\\ \deg(ab)=g+1}} \frac{\chi(a)\overline{\chi}(b)}{|ab|} + \sum_{\substack{a,b\in A^+\\ \deg(ab)\leq g}} \frac{\chi(a)\overline{\chi}(b)}{|ab|}\\ &+ q^{-g-2}\Bigg(\frac{(q-1)^2}{2}\sum_{\substack{a,b\in A^+\\ \deg(ab)\leq g-1}} (\deg(ab))^2\chi(a)\overline{\chi}(b)\\ &- 2(q-1)(q+gq-g-2)\sum_{\substack{a,b\in A^+\\ \deg(ab)\leq g-1}} \deg(ab)\chi(a)\overline{\chi}(b)\\ &+ (2(q-1)(g+2)(gq-g-2) + q(2q-1))\sum_{\substack{a,b\in A^+\\ \deg(ab)\leq g-1}} \chi(a)\overline{\chi}(b)\\ &- \left(q(g+3)^2 - \frac{(g+4)^2}{2}\right)\sum_{\substack{a,b\in A^+\\ \deg(ab)=g}} \chi(a)\overline{\chi}(b)\\ &+ \frac{(g+3)^2}{2}\sum_{\substack{a,b\in A^+\\ \deg(ab)=g+1}} \chi(a)\overline{\chi}(b) - (q-1)^2(g+2)^2\Bigg),\end{aligned} \tag{5}$$

where $\overline{\chi}(b) = \chi(b^2)$ is the inverse of $\chi(b)$. Using this claim and Lemma 3.1, the result follows immediately as desired.

Now, it is sufficient to prove our claim (5). By [1, p. 250, proof of Lemma 3.11], we have the following:

$$\begin{aligned}(q^{1-s}-1)^2|L(s,\chi)|^2 &= \sum_{n=0}^{g+2}\left(\sum_{\substack{0\le i,j\le g+2\\ i+j=n}} M_i(\chi)M_j(\overline{\chi})\right)q^{-ns}\\ &\quad + q^{-g-2}\sum_{n=0}^{g+1}\left(\sum_{\substack{0\le i,j\le g+2\\ i+j=n}} M_i(\chi)M_j(\overline{\chi})\right)q^{(1-s)(2g+4-n)},\end{aligned} \tag{6}$$

where $M_i(\chi) := qL_{i-1}(\chi) - L_i(\chi)$, $L_i(\chi) := \sum_{\deg a=i, a\in A^+}\chi(a)$, $L_{-1}(\chi) := 0$, and $L_{g+2}(\chi) = 0$. We note that $L_0(\chi) = \sum_{\deg a=0, a\in A^+}\chi(a) = q-1$ since χ is an even character.

Taking the derivatives of both sides of equation (6) twice with respect to s and letting $s = 1$, we have the following:

$$\begin{aligned}|L(1,\chi)|^2 &= \frac{1}{2}\left(\sum_{n=1}^{g+2}\left(\sum_{\substack{0\le i,j\le g+2\\ i+j=n}} M_i(\chi)M_j(\overline{\chi})\right)n^2q^{-n}\right.\\ &\quad\left. + \sum_{n=1}^{g+1}\left(\sum_{\substack{0\le i,j\le g+2\\ i+j=n}} M_i(\chi)M_j(\overline{\chi})\right)(2g+4-n)^2q^{(-g-2)}\right)\\ &:= \frac{1}{2}(\mathcal{L}_1 + q^{-g-2}\mathcal{L}_2).\end{aligned} \tag{7}$$

We first compute the first term of equation (7), i.e., $\mathcal{L}_1 = \sum_{n=1}^{g+2}\left(\sum_{\substack{0\le i,j\le g+2\\ i+j=n}} M_i(\chi)M_j(\overline{\chi})\right)n^2q^{-n}$.

$$\begin{aligned}\mathcal{L}_1 &= \sum_{n=0}^{g+2}\left(\sum_{\substack{0\le i,j\le g+2\\ i+j=n}} M_i(\chi)M_j(\overline{\chi})\right)n^2q^{-n}\\ &= \sum_{n=0}^{g+2}q^2\left(\sum_{\substack{0\le i,j\le g+2\\ i+j=n}} L_{i-1}(\chi)L_{j-1}(\overline{\chi})\right)n^2q^{-n} - \sum_{n=0}^{g+2}q\left(\sum_{\substack{0\le i,j\le g+2\\ i+j=n}} L_{i-1}(\chi)L_j(\overline{\chi})\right)n^2q^{-n}\\ &\quad - q\sum_{n=0}^{g+2}\left(\sum_{\substack{0\le i,j\le g+2\\ i+j=n}} L_i(\chi)L_{j-1}(\overline{\chi})\right)n^2q^{-n} + \sum_{n=0}^{g+2}\left(\sum_{\substack{0\le i,j\le g+2\\ i+j=n}} L_i(\chi)L_j(\overline{\chi})\right)n^2q^{-n}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{g}\left(\sum_{\substack{0\le i,j\le g+1\\ i+j=n}} L_i(\chi)L_j(\overline{\chi})\right)(n+2)^2q^{-n}\\
&\quad - 2\sum_{n=0}^{g+1}\left(\sum_{\substack{0\le i,j\le g+1\\ i+j=n}} L_i(\chi)L_j(\overline{\chi})\right)(n+1)^2q^{-n} + \sum_{n=0}^{g+2}\left(\sum_{\substack{0\le i,j\le g+2\\ i+j=n}} L_i(\chi)L_j(\overline{\chi})\right)n^2q^{-n}\\
&= \sum_{n=0}^{g}\left(\sum_{\substack{0\le i,j\le g+1\\ i+j=n}} L_i(\chi)L_j(\overline{\chi})\right)(n+2)^2q^{-n} - 2\sum_{n=0}^{g+1}\left(\sum_{\substack{0\le i,j\le g+1\\ i+j=n}} L_i(\chi)L_j(\overline{\chi})\right)(n+1)^2q^{-n}\\
&\quad + \sum_{n=0}^{g+2}\left(\sum_{\substack{0\le i,j\le g+1\\ i+j=n}} L_i(\chi)L_j(\overline{\chi})\right)n^2q^{-n} + (L_0(\chi)L_{g+2}(\overline{\chi}) + L_{g+2}(\chi)L_0(\overline{\chi}))(g+2)^2q^{-(g+2)}\\
&= -\left(\sum_{\substack{0\le i,j\le g+1\\ i+j=g+1}} L_i(\chi)L_j(\overline{\chi})\right)(g+1)^2q^{-(g+1)} + \left(\sum_{\substack{0\le i,j\le g+1\\ i+j=g+2}} L_i(\chi)L_j(\overline{\chi})\right)(g+2)^2q^{-(g+2)}\\
&\quad - 4\left(\sum_{\substack{0\le i,j\le g+1\\ i+j=g+1}} L_i(\chi)L_j(\overline{\chi})\right)(g+1)q^{-(g+1)} - 2\left(\sum_{\substack{0\le i,j\le g+1\\ i+j=g+1}} L_i(\chi)L_j(\overline{\chi})\right)q^{-(g+1)}\\
&\quad + 2\sum_{n=0}^{g}\left(\sum_{\substack{0\le i,j\le g+1\\ i+j=n}} L_i(\chi)L_j(\overline{\chi})\right)q^{-n} + (L_0(\chi)L_{g+2}(\overline{\chi}) + L_{g+2}(\chi)L_0(\overline{\chi}))(g+2)^2q^{-(g+2)}\\
&= -(g+1)^2\sum_{\substack{a,b\in A^+\\ \deg(ab)=g+1}}\frac{\chi(a)\overline{\chi}(b)}{|ab|} + (g+2)^2\sum_{\substack{a,b\in A^+\\ \deg(ab)=g+2}}\frac{\chi(a)\overline{\chi}(b)}{|ab|}\\
&\quad - 4(g+1)\sum_{\substack{a,b\in A^+\\ \deg(ab)=g+1}}\frac{\chi(a)\overline{\chi}(b)}{|ab|} - 2\sum_{\substack{a,b\in A^+\\ \deg(ab)=g+1}}\frac{\chi(a)\overline{\chi}(b)}{|ab|} + 2\sum_{\substack{a,b\in A^+\\ \deg(ab)\le g}}\frac{\chi(a)\overline{\chi}(b)}{|ab|}\\
&= (g+2)^2\sum_{\substack{a,b\in A^+\\ \deg(ab)=g+2}}\frac{\chi(a)\overline{\chi}(b)}{|ab|} - (g^2+6g+7)\sum_{\substack{a,b\in A^+\\ \deg(ab)=g+1}}\frac{\chi(a)\overline{\chi}(b)}{|ab|}\\
&\quad + 2\sum_{\substack{a,b\in A^+\\ \deg(ab)\le g}}\frac{\chi(a)\overline{\chi}(b)}{|ab|}.
\end{aligned}
$$

Now, we compute $\mathcal{L}_2$, i.e., the second term of equation (7). We note that

$$
\begin{aligned}
\mathcal{L}_2 &= \sum_{n=1}^{g+1}\left(\sum_{\substack{0\le i,j\le g+2\\ i+j=n}} M_i(\chi)M_j(\overline{\chi})\right)(2g+4-n)^2\\
&= \sum_{n=0}^{g+1}\left(\sum_{\substack{0\le i,j\le g+2\\ i+j=n}} M_i(\chi)M_j(\overline{\chi})\right)(2g+4-n)^2 - (q-1)^2(2g+4)^2;
\end{aligned}
$$

we use the fact that $M_0(\chi) = q\mathcal{L}_{-1}(\chi) - \mathcal{L}_0(\chi) = -(q-1)$.

$$\mathcal{L}_2 + (q-1)^2(2g+4)^2 = q^2\sum_{n=0}^{g-1}\left(\sum_{\substack{0\le i,j\le g+1\\ i+j=n}} L_i(\chi)L_j(\overline{\chi})\right)(2g+2-n)^2 - 2q\sum_{n=0}^{g}\left(\sum_{\substack{0\le i,j\le g+1\\ i+j=n}} L_i(\chi)L_j(\overline{\chi})\right)(2g+3-n)^2$$
$$+\sum_{n=0}^{g+1}\left(\sum_{\substack{0\le i,j\le g+2\\ i+j=n}} L_i(\chi)L_j(\overline{\chi})\right)(2g+4-n)^2.$$

For simplicity, let $\mathcal{M}_n := \left(\sum_{\substack{0\le i,j\le g+1\\ i+j=n}} L_i(\chi)L_j(\overline{\chi})\right)$ and $r := g+2$. Then, we obtain

$$\begin{aligned}\mathcal{L}_2 + (q-1)^2(2g+4)^2 &= q^2\sum_{n=0}^{r-3}\mathcal{M}_n(2r-n-2)^2 - 2q\sum_{n=0}^{r-2}\mathcal{M}_n(2r-n-1)^2 + \sum_{n=0}^{r-1}\mathcal{M}_n(2r-n)^2\\ &= \underbrace{\sum_{n=0}^{r-3}(q^2(2r-n-2)^2 - 2q(2r-n-1)^2 + (2r-n)^2)\mathcal{M}_n}_{\mathcal{N}_1}\\ &\quad \underbrace{-(2q(r+1)^2 - (r+2)^2)\mathcal{M}_{r-2} + (r+1)^2\mathcal{M}_{r-1}}_{\mathcal{N}_2}.\end{aligned}$$

Using $\mathcal{M}_n = \sum_{\substack{a,b\in A^+\\ \deg(ab)=n}}\chi(a)\overline{\chi}(b)$, $\mathcal{N}_1$ and $\mathcal{N}_2$ can be computed as follows:

$$\begin{aligned}\mathcal{N}_1 &= (q-1)^2\sum_{n=0}^{g-1}n^2\mathcal{M}_n - 4(q-1)(rq-q-r)\sum_{n=0}^{g-1}n\mathcal{M}_n + (4r^2(q-1)^2 - 8rq(q-1) + 4q^2 - 2q)\sum_{n=0}^{g-1}\mathcal{M}_n\\ &= (q-1)^2\sum_{\substack{a,b\in A^+\\ \deg(ab)\le g-1}}(\deg(ab))^2\chi(a)\overline{\chi}(b) - 4(q-1)(q+gq-g-2)\sum_{\substack{a,b\in A^+\\ \deg(ab)\le g-1}}\deg(ab)\chi(a)\overline{\chi}(b)\\ &\quad + (4(q-1)(g+2)(gq-g-2) + 2q(2q-1))\sum_{\substack{a,b\in A^+\\ \deg(ab)\le g-1}}\chi(a)\overline{\chi}(b)\end{aligned}$$

and

$$\mathcal{N}_2 = (-2q(g+3)^2 + (g+4)^2)\sum_{\substack{a,b\in A^+\\ \deg(ab)=g}}\chi(a)\overline{\chi}(b) + (g+3)^2\sum_{\substack{a,b\in A^+\\ \deg(ab)=g+1}}\chi(a)\overline{\chi}(b);$$

thus, we obtain the desired result. □

From now on, we compute the asymptotic values of $\mathcal{A}_{\le g,\square}$, $\mathcal{B}_{=v,\square}$, and $\mathcal{B}_{\le v,\square}$ in Lemma 3.3, $\widetilde{\mathcal{A}}_{\le g}$, $\widetilde{\mathcal{B}}_{=v}$, and $\widetilde{\mathcal{B}}_{\le v}$ in Lemma 3.4, and finally, $\mathcal{G}_v$ and $\mathcal{H}_i(g-1)$ $(i=1,2)$ in Lemma 3.5.

Lemma 3.3. *Let $\chi \in \mathcal{S}_g$, where the set $\mathcal{S}_g$ is as defined in equation* (4). *Let v be a nonnegative integer, and let all the notations be the same as in Notations* 1 *and* 2. *Then, we have the following*:

(*i*) $\mathcal{A}_{\le g,\square} = C_1 gq^{g+2} + C_2 q^{g+2} + O(q^{\frac{g}{2}+\varepsilon g})$;

(*ii*) $\mathcal{B}_{=v,\square} = O_g(q^{\frac{v}{6}+g+(2v+g)\varepsilon})$;

(*iii*) $\mathcal{B}_{\le v,\square} = O_g(q^{\frac{v}{6}+g+(2v+g)\varepsilon})$.

Proof. We first compute $\mathcal{A}_{\le g,\square}$. By Lemma 3.1, we have

$$\mathcal{A}_{\le g,\square} = \sum_{\substack{d_1+d_2=g+2\\ d_1+2d_2\equiv 0 \pmod 3}} \sum_{\substack{m_1\in\mathfrak{A}_{d_1}\\ m_2\in\mathfrak{A}_{d_2}\\ (m_1,m_2)=1}} \sum_{\substack{a,b\in A^+\\ \deg(ab)\le g\\ (ab,m_1m_2)=1\\ ab^2=\square}} \frac{1}{|ab|},$$

where $d_i = \deg(m_i)$ for $i = 1, 2$. We consider the following generating series, which is defined in [18, Section 3.1.1]:

$$C(x,y,u,w) := \sum_{\substack{m_1,m_2\in\mathfrak{A}\\ (m_1,m_2)=1}} \sum_{\substack{a,b\in A^+\\ (ab,m_1m_2)=1\\ ab^2=\square}} x^{\deg(m_1)}y^{\deg(m_2)}\frac{u^{\deg(a)}w^{\deg(b)}}{|ab|}. \tag{8}$$

As given in [18, Section 3.1.1], we can obtain the following:

$$\begin{aligned}\mathcal{A}_{\le g,\square} &= \sum_{\substack{d_1+d_2=g+2\\ d_1+2d_2\equiv 0 \pmod 3}} \sum_{e=0}^{g} \frac{1}{(2\pi i)^4} \int_{|x|=\frac{1}{q^{\varepsilon+1}}}\int_{|y|=\frac{1}{q^{\varepsilon+1}}}\int_{|w|=\frac{1}{q^{\varepsilon}}}\int_{|u|=\frac{1}{q^{2\varepsilon}}} \frac{C(x,y,u,w)}{x^{d_1}y^{d_2}w^e u^{g-e}(1-u)}\frac{\mathrm{d}u}{u}\frac{\mathrm{d}w}{w}\frac{\mathrm{d}y}{y}\frac{\mathrm{d}x}{x}\\ &= \frac{\zeta(2)\zeta(3)^2}{(2\pi i)^2} \sum_{\substack{d_1+d_2=g+2\\ d_1+2d_2\equiv 0 \pmod 3}} \int_{|x|=\frac{1}{q^{\varepsilon+1}}}\int_{|y|=\frac{1}{q^{\varepsilon+1}}} \frac{\mathcal{D}(x,y,1,1)}{(1-qx)(1-qy)x^{d_1}y^{d_2}}\frac{\mathrm{d}y}{y}\frac{\mathrm{d}x}{x} + O(q^{\frac{g}{2}+\varepsilon g}),\end{aligned} \tag{9}$$

where $e := \deg(b)$ and $\mathcal{D}(x,y,u,w)$ is defined in Notation 1.

Now, we compute the integrals over x and y of equation (9). Let $2g+1 \equiv \alpha \pmod 3$ and $g \equiv \beta \pmod 3$, where $0 \le \alpha, \beta \le 2$. Since $d_1 + d_2 = g + 2$ and $d_1 + 2d_2 \equiv 0 \pmod 3$, we have

$$d_1 \equiv 2d_1 + 2d_2 = 2(g+2) \equiv 2g+1 \equiv \alpha \pmod 3.$$

Thus, as in [18, Section 3.1.1], equation (9) is

$$\frac{\zeta(2)\zeta(3)^2}{(2\pi i)^2}\int_{|x|=\frac{1}{q^3}}\int_{|y|=\frac{1}{q^2}} \frac{\mathcal{D}(x,y,1,1)}{(1-qx)(1-qy)(y^3-x^3)}\left[\frac{y^{2+\alpha-\beta}}{x^{g+1+\alpha-\beta}} - \frac{x^{3-\alpha}}{y^{g+2-\alpha}}\right]\frac{\mathrm{d}y}{y}\frac{\mathrm{d}x}{x}. \tag{10}$$

Following the computation method in [18, Section 3.1.1], we obtain $\mathcal{A}_{\le g,\square} = C_1 g q^{g+2} + C_2 q^{g+2} + O(q^{\frac{g}{2}+\varepsilon g})$ as desired.

For computation of $\mathcal{B}_{=v,\square}$, we consider the following generating series:

$$\tilde{C}(x,y,u,w) := \sum_{\substack{m_1,m_2\in\mathfrak{A}\\ (m_1,m_2)=1}} \sum_{\substack{a,b\in A^+\\ (ab,m_1m_2)=1\\ \deg(ab)=v\\ ab^2=\square}} x^{\deg(m_1)}y^{\deg(m_2)}u^{\deg(a)}w^{\deg(b)}. \tag{11}$$

As in [18, Section 3.2.1], the generating series $\tilde{C}(x,y,u,w)$ can be expressed as follows:

$$\tilde{C}(x,y,u,w) = \frac{1}{1-quw}\frac{1}{1-qu^3}\frac{1}{1-qw^3}\frac{1}{1-qx}\frac{1}{1-qy}\tilde{\mathcal{D}}(x,y,u,w), \tag{12}$$

where

$$\begin{aligned}\tilde{\mathcal{D}}(x,y,u,w) := \prod_P &(1 - x^{2\deg(P)} - y^{2\deg(P)} - (xy)^{\deg(P)} + (x^2y)^{\deg(P)} + (xy^2)^{\deg(P)} - (u^3w^3)^{\deg(P)}\\ &+ (xu^3w^3)^{\deg(P)} + (yu^3w^3)^{\deg(P)} - (xyu^3w^3)^{\deg(P)} - \tilde{f}(u,w)(x^{\deg(P)} + y^{\deg(P)} - x^{2\deg(P)} - 2(xy)^{\deg(P)}\\ &- y^{2\deg(P)} + (x^2y)^{\deg(P)} + (xy^2)^{\deg(P)})) \text{ and}\end{aligned}$$

$$\tilde{f}(u,w) := u^{3\deg(P)} + (uw)^{\deg(P)} + w^{3\deg(P)} - (u^4w)^{\deg(P)} - (uw^4)^{\deg(P)} - (u^3w^3)^{\deg(P)} + (u^4w^4)^{\deg(P)}.$$

We note that $\tilde{\mathcal{D}}(x,y,u,w)$ converges absolutely when $|x| < \frac{1}{\sqrt{q}}$, $|y| < \frac{1}{\sqrt{q}}$, $|u| < \frac{1}{q^6}$, and $|w| < \frac{1}{q^6}$. Applying Lemma 2.6 to equation (12) four times and $e := \deg(b)$, we obtain

$$\sum_{\substack{d_1+d_2=g+2\\ d_1+2d_2\equiv 0 \pmod 3}} \sum_{e=0}^{v} \frac{1}{(2\pi i)^4} \int_{|x|=\frac{1}{q^{\varepsilon+1}}} \int_{|y|=\frac{1}{q^{\varepsilon+1}}} \int_{|w|=\frac{1}{q^{\varepsilon+\frac{1}{6}}}} \int_{|u|=\frac{1}{q^{2\varepsilon+\frac{1}{6}}}} \\ \times \frac{\tilde{\mathcal{D}}(x,y,u,w)}{(1-quw)(1-qu^3)(1-qw^3)(1-qx)(1-qy)x^{d_1}y^{d_2}w^e u^{v-e}} \frac{\mathrm{d}u}{u}\frac{\mathrm{d}w}{w}\frac{\mathrm{d}y}{y}\frac{\mathrm{d}x}{x}. \tag{13}$$

Using the same computation as in $\mathcal{A}_{\le g,\square}$, the integrals over u and w of equation (13) can be evaluated as follows:

$$\begin{aligned} &\frac{1}{(2\pi i)^2} \int_{|w|=\frac{1}{q^{\varepsilon+\frac{1}{6}}}} \int_{|u|=\frac{1}{q^{2\varepsilon+\frac{1}{6}}}} \sum_{e=0}^{v} \frac{\tilde{\mathcal{D}}(x,y,u,w)}{(1-quw)(1-qu^3)(1-qw^3)w^e u^{v-e}} \frac{\mathrm{d}u}{u}\frac{\mathrm{d}w}{w} \\ &= \frac{1}{(2\pi i)^2} \int_{|w|=\frac{1}{q^{\varepsilon+\frac{1}{6}}}} \frac{\mathrm{d}w}{(1-qw^3)w^{v+1}} \int_{|u|=\frac{1}{q^{2\varepsilon+\frac{1}{6}}}} \frac{\tilde{\mathcal{D}}(x,y,u,w)(w^{v+1}-u^{v+1})\mathrm{d}u}{(1-quw)(1-qu^3)(w-u)u^{v+1}} \\ &= \frac{1}{(2\pi i)^2} \int_{|w|=\frac{1}{q^{\varepsilon+\frac{1}{6}}}} \frac{\mathrm{d}w}{(1-qw^3)} \int_{|u|=\frac{1}{q^{2\varepsilon+\frac{1}{6}}}} \frac{\tilde{\mathcal{D}}(x,y,u,w)\mathrm{d}u}{(1-quw)(1-qu^3)(w-u)u^{v+1}} \\ &\quad - \frac{1}{(2\pi i)^2} \int_{|w|=\frac{1}{q^{\varepsilon+\frac{1}{6}}}} \frac{\mathrm{d}w}{(1-qw^3)w^{v+1}} \int_{|u|=\frac{1}{q^{2\varepsilon+\frac{1}{6}}}} \frac{\tilde{\mathcal{D}}(x,y,u,w)\mathrm{d}u}{(1-quw)(1-qu^3)(w-u)}. \end{aligned} \tag{14}$$

Noting that the second double integral of equation (14) vanishes since the integrand has no pole inside the regions, we have the following:

$$\begin{aligned} \mathcal{B}_{=v,\square} &= \frac{1}{(2\pi i)^2} \sum_{\substack{d_1+d_2=g+2\\ d_1+2d_2\equiv 0 \pmod 3}} \int_{|x|=\frac{1}{q^{1+\varepsilon}}} \int_{|y|=\frac{1}{q^{1+\varepsilon}}} \frac{1}{(1-qx)(1-qy)y^{d_2}x^{d_1}} \\ &\quad \times \int_{|w|=\frac{1}{q^{\varepsilon+\frac{1}{6}}}} \frac{\mathrm{d}w}{(1-qw^3)} \int_{|u|=\frac{1}{q^{2\varepsilon+\frac{1}{6}}}} \frac{\tilde{\mathcal{D}}(x,y,u,w)\mathrm{d}u}{(1-quw)(1-qu^3)(w-u)u^{v+1}} \frac{\mathrm{d}y}{y}\frac{\mathrm{d}x}{x} \\ &= O_g(q^{\frac{v}{6}+g+(2v+g+4)\varepsilon}). \end{aligned} \tag{15}$$

Finally, we compute $\mathcal{B}_{\le v,\square}$. We consider the generating series (equation (11)). Applying Lemma 2.6 to equation (12) four times with $e \coloneqq \deg(b)$, we obtain

$$\sum_{\substack{d_1+d_2=g+2\\ d_1+2d_2\equiv 0 \pmod 3}} \sum_{e=0}^{v} \frac{1}{(2\pi i)^4} \int_{|x|=\frac{1}{q^{\varepsilon+1}}} \int_{|y|=\frac{1}{q^{\varepsilon+1}}} \int_{|w|=\frac{1}{q^{\varepsilon+\frac{1}{6}}}} \int_{|u|=\frac{1}{q^{2\varepsilon+\frac{1}{6}}}} \\ \times \frac{\tilde{\mathcal{D}}(x,y,u,w)}{(1-quw)(1-qu^3)(1-qw^3)(1-qx)(1-qy)x^{d_1}y^{d_2}w^e u^{v-e}(1-u)} \frac{\mathrm{d}u}{u}\frac{\mathrm{d}w}{w}\frac{\mathrm{d}y}{y}\frac{\mathrm{d}x}{x}. \tag{16}$$

As in [18, Section 3.2.1], we have the following:

$$\begin{aligned} \mathcal{B}_{\le v,\square} &= \frac{\zeta(2)^3}{(2\pi i)^2} \sum_{\substack{d_1+d_2=g+2\\ d_1+2d_2\equiv 0 \pmod 3}} \int_{|x|=\frac{1}{q^{1+\varepsilon}}} \int_{|y|=\frac{1}{q^{1+\varepsilon}}} \frac{1}{(1-qx)(1-qy)y^{d_2}x^{d_1}} \\ &\quad \times \int_{|w|=\frac{1}{q^{\varepsilon+\frac{1}{6}}}} \frac{\mathrm{d}w}{(1-qw^3)} \int_{|u|=\frac{1}{q^{2\varepsilon+\frac{1}{6}}}} \frac{\tilde{\mathcal{D}}(x,y,u,w)\mathrm{d}u}{(1-quw)(1-qu^3)(1-u)(w-u)u^{v+1}} \frac{\mathrm{d}y}{y}\frac{\mathrm{d}x}{x} \\ &= O_g(q^{\frac{v}{6}+g+(2v+g+4)\varepsilon}). \end{aligned} \tag{17}$$

□

Lemma 3.4. *For nonnegative integers g and ν, let $\widetilde{\mathcal{A}}_{\leq g}$, $\widetilde{\mathcal{B}}_{=\nu}$, and $\widetilde{\mathcal{B}}_{=\nu}$ be the same as in Notation 2. Then, we have the following*:

(i) $\widetilde{\mathcal{A}}_{\leq g} = O_g(q^{\frac{g}{2}+\varepsilon g+1}(g+1))$;

(ii) $\widetilde{\mathcal{B}}_{=\nu} = O_g(q^{\frac{g}{2}+\varepsilon g+\nu})$;

(iii) $\widetilde{\mathcal{B}}_{\leq \nu} = O_g(q^{\frac{g}{2}+\varepsilon g+\nu})$.

Proof. We note that

$$\chi^{\mathrm{av}}(ab^2) = \chi^{\mathrm{av}}(f) = O_g(q^{\frac{g}{2}+\varepsilon g+1}); \tag{18}$$

this follows by using a similar computation method as [18, Section 3.1.2]. By equation (18), we obtain

$$\widetilde{\mathcal{A}}_{\leq g} = O_g\left(q^{\frac{g}{2}+\varepsilon g+1} \sum_{\substack{a,b\in A^+\\ \deg(ab)\leq g\\ ab^2\neq\square}} \frac{1}{|ab|}\right) = O_g(q^{\frac{g}{2}+\varepsilon g+1}(g+1)).$$

The desired results for $\widetilde{\mathcal{B}}_{=\nu} = O_g(q^{\frac{g}{2}+\varepsilon g+\nu+1})$ and $\widetilde{\mathcal{B}}_{\leq \nu} = O_g(q^{\frac{g}{2}+\varepsilon g+\nu+1})$ follow immediately from equation (18). □

Lemma 3.5. *For nonnegative integers g and ν, we have the following*:

(i) $\mathcal{G}_\nu = O_g(q^{-\frac{5\nu}{6}+g+(2\nu+g)\varepsilon}) + O_g(q^{\frac{g}{2}+\varepsilon g+1})$;

(ii) $\mathcal{H}_i(g-1) = (g-1)^i O_g(q^{\frac{3}{2}g+\varepsilon g})$ *for* $i = 1, 2$,

where $\mathcal{G}_\nu$ and $\mathcal{H}_i(g-1)$ are defined in Notation 2.

Proof. (i) Using Lemmas 3.3 and 3.4, we obtain the following:

$$\begin{aligned}\sum_{\substack{a,b\in A^+\\ \deg(ab)=\nu}} \frac{\chi^{\mathrm{av}}(ab^2)}{|ab|} &= \frac{1}{q^\nu} \sum_{\substack{a,b\in A^+\\ \deg(ab)=\nu}} \chi^{\mathrm{av}}(ab^2)\\ &= \frac{1}{q^\nu}\left(\sum_{\substack{a,b\in A^+\\ \deg(ab)=\nu\\ ab^2=\square}} \chi^{\mathrm{av}}(ab^2) + \sum_{\substack{a,b\in A^+\\ \deg(ab)=\nu\\ ab^2\neq\square}} \chi^{\mathrm{av}}(ab^2)\right) = \frac{1}{q^\nu}(\mathcal{B}_{=\nu,\square} + \widetilde{\mathcal{B}}_{=\nu})\\ &= \frac{1}{q^\nu}(O_g(q^{\frac{\nu}{6}+g+(2\nu+g+4)\varepsilon}) + O_g(q^{\frac{g}{2}+\varepsilon g+\nu+1})).\end{aligned}$$

(ii) For $i = 1, 2$, the desired results hold by Lemmas 3.3 and 3.4 as follows:

$$\begin{aligned}\sum_{\substack{a,b\in A^+\\ \deg(ab)\leq g-1}} \deg(ab)^i \chi^{\mathrm{av}}(ab^2) \leq (g-1)^i \sum_{\substack{a,b\in A^+\\ \deg(ab)\leq g-1}} \chi^{\mathrm{av}}(ab^2) &= (g-1)^i(\mathcal{B}_{\leq g-1,\square} + \widetilde{\mathcal{B}}_{\leq g-1})\\ &= (g-1)^i(O_g(q^{\frac{g-1}{6}+g+(3g+2)\varepsilon}) + O_g(q^{\frac{3}{2}g+\varepsilon g}))\\ &= (g-1)^i O_g(q^{\frac{3}{2}g+\varepsilon g}).\end{aligned}$$

□

4 Average value of the divisor of class numbers

In this section, we prove our main results: computing the average value of $\sum_\chi |L(1,\chi)|^2$ (Theorem 1.1) and finding an asymptotic formula for the average value of the divisor class numbers of cubic real function fields (Theorem 1.2), where χ runs through the primitive cubic even Dirichlet characters of A. For estimating the average value of $\sum_\chi |L(1,\chi)|^2$, the following lemma plays an important role.

Lemma 4.1. *Let q be an odd prime power such that $q \equiv 1 \pmod 3$ and M_g be the set that is defined in equation* (3). *Then, we have the following*:

$$|M_g| = B_1 g q^{g+2} + B_2 q^{g+2} + O(q^{\frac{g}{2}+\varepsilon g}),$$

where $\mathcal{F}(x,y) \coloneqq \prod_P (1 - x^{2\deg(P)} - y^{2\deg(P)} - (xy)^{\deg(P)} + (x^2y)^{\deg(P)} + (xy^2)^{\deg(P)})$, $B_1 = \frac{1}{3}\mathcal{F}\left(\frac{1}{q}, \frac{1}{q}\right)$, and

$$B_2 = \frac{2}{3}\left[\mathcal{F}\left(\frac{1}{q},\frac{1}{q}\right) - \frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{F}(x,x)|_{x=\frac{1}{q}} - \frac{\mathcal{F}\left(\frac{1}{q},\frac{\xi_3}{q}\right)\xi_3^{g+1}}{1-\xi_3} - \frac{\mathcal{F}\left(\frac{\xi_3^2}{q},\frac{1}{q}\right)\xi_3^{2g+2}}{1-\xi_3^2}\right].$$

Proof. For $i = 1, 2$, let $d_i \coloneqq \deg m_i$. Using equation (3), the cardinality of M_g can be represented as follows:

$$|M_g| = \sum_{\substack{d_1+d_2=g+2 \\ d_1+2d_2\equiv 0 \pmod 3}} \sum_{\substack{m_1,m_2\in\mathfrak{A} \\ (m_1,m_2)=1}} 1.$$

For the computation of $|M_g|$, we consider the generating series

$$C(x,y) = \sum_{\substack{m_1,m_2\in\mathfrak{A} \\ (m_1,m_2)=1}} x^{\deg(m_1)} y^{\deg(m_2)}.$$

As in the proof of [18, Lemma 4.1], we obtain

$$C(x,y) = \frac{1}{1-qx}\frac{1}{1-qy}\mathcal{F}(x,y), \tag{19}$$

where $\mathcal{F}(x,y) = \prod_P (1 - x^{2\deg(P)} - y^{2\deg(P)} - (xy)^{\deg(P)} + (x^2y)^{\deg(P)} + (xy^2)^{\deg(P)})$. We note that $\mathcal{F}(x,y)$ converges absolutely when $|x| < \frac{1}{\sqrt{q}}$ and $|y| < \frac{1}{\sqrt{q}}$. Applying Lemma 2.6 to equation (19) twice, we obtain

$$|M_g| = \sum_{\substack{d_1+d_2=g+2 \\ d_1+2d_2\equiv 0 \pmod 3}} \frac{1}{(2\pi i)^2} \int_{|x|=\frac{1}{q^{\varepsilon+1}}} \int_{|y|=\frac{1}{q^{\varepsilon+1}}} \frac{\mathcal{F}(x,y)}{(1-qx)(1-qy)x^{d_1}y^{d_2}} \frac{\mathrm{d}y}{y}\frac{\mathrm{d}x}{x}.$$

By a similar computation method used in equation (9), we have the desired result. □

Proof of Theorem 1.1. By Lemma 3.2, we have the following:

$$\begin{aligned}
\sum_{\chi\in\mathcal{S}_g} |L(1,\chi)|^2 = {} & \frac{(g+2)^2}{2}\mathcal{G}_{g+2} - \frac{(g^2+6g+7)}{2}\mathcal{G}_{g+1} + \mathcal{A}_{\le g,\square} + \widetilde{\mathcal{A}}_{\le g} \\
& + q^{-g-2}\left(\frac{(q-1)^2}{2}\mathcal{H}_2(g-1) - 2(q-1)(q+gq-g-2)\mathcal{H}_1(g-1)\right) \\
& + q^{-g-2}(2(q-1)(g+2)(qg-g-2) + q(2q-1))(\mathcal{B}_{\le g-1,\square} + \widetilde{\mathcal{B}}_{\le g-1}) \\
& - q^{-g-2}\left(\left(q(g+3)^2 - \frac{(g+4)^2}{2}\right)(\mathcal{B}_{=g,\square} + \widetilde{\mathcal{B}}_{=g}) + \frac{(g+3)^2}{2}(\mathcal{B}_{=g+1,\square} + \widetilde{\mathcal{B}}_{=g+1})\right) \\
& - q^{-g-2}((q-1)^2(g+2)^2);
\end{aligned} \tag{20}$$

all the notations follow from Lemmas 3.3–3.5. We now compute each summand of equation (20) as follows: the values of $\mathcal{A}_{\leq g,\square}$ and $\widetilde{\mathcal{A}}_{\leq g}$ follow from Lemmas 3.3 and 3.4, respectively. By Lemma 3.5, we obtain $\mathcal{G}_{g+1} = \mathcal{G}_{g+2} = O(q^{\frac{g}{2}+\varepsilon g+1})$. Thus, the first summand of equation (20) can be computed as follows:

$$\begin{aligned}\frac{(g+2)^2}{2}\mathcal{G}_{g+2} - \frac{(g^2+6g+7)}{2}\mathcal{G}_{g+1} + \mathcal{A}_{\leq g,\square} + \widetilde{\mathcal{A}}_{\leq g} \\ = \frac{(g+2)^2}{2}O_g(q^{\frac{g}{2}+\varepsilon g+1}) - \frac{(g^2+6g+7)}{2}O_g(q^{\frac{g}{2}+\varepsilon g+1}) + C_1 g q^{g+2} + C_2 q^{g+2} + O_g(q^{\frac{g}{2}+\varepsilon g}) + O_g(q^{\frac{g}{2}+\varepsilon g+1}(g+1)) \\ = C_1 g q^{g+2} + C_2 q^{g+2} + O_g(q^{\frac{g}{2}+\varepsilon g+1}).\end{aligned} \tag{21}$$

Using Lemma 3.5, the second summand of equation (20) is

$$\begin{aligned} & q^{-g-2}\left(\frac{(q-1)^2}{2}\mathcal{H}_2(g-1) - 2(q-1)(q+gq-g-2)\mathcal{H}_1(g-1)\right) \\ &= q^{-g-2}\left(\frac{(q-1)^2}{2}(g-1)^2 O_g(q^{\frac{3}{2}g+\varepsilon g}) - 2(q-1)(q+gq-g-2)(g-1)O_g(q^{\frac{3}{2}g+\varepsilon g})\right) \\ &= \frac{(g-1)^2}{2}O_g(q^{\frac{1}{2}g+\varepsilon g}) - 2(g-1)O_g(q^{\frac{1}{2}g+\varepsilon g}) = O_g(q^{\frac{1}{2}g+\varepsilon g}).\end{aligned} \tag{22}$$

By Lemmas 3.3 and 3.4, we obtain the following:

$$\begin{aligned}\mathcal{B}_{\leq g-1,\square} + \widetilde{\mathcal{B}}_{\leq g-1} &= O_g(q^{\frac{g-1}{6}+g+(3g-2)\varepsilon}) + O_g(q^{\frac{3}{2}g+\varepsilon g}) = O_g(q^{\frac{3}{2}g+\varepsilon g}) \\ \mathcal{B}_{=g,\square} + \widetilde{\mathcal{B}}_{=g} &= O_g(q^{\frac{7g}{6}+3\varepsilon g}) + O_g(q^{\frac{3}{2}g+\varepsilon g}) = O_g(q^{\frac{3}{2}g+\varepsilon g}) \\ \mathcal{B}_{=g+1,\square} + \widetilde{\mathcal{B}}_{=g+1} &= O_g(q^{\frac{g+1}{6}+g+(3g+2)\varepsilon}) + O_g(q^{\frac{3}{2}g+\varepsilon g+1}) = O_g(q^{\frac{3}{2}g+\varepsilon g+1}).\end{aligned}$$

Therefore, the sum of the third and fourth summands of equation (20) is

$$q^{-g-2}O_g(q^{\frac{3}{2}g+\varepsilon g+1}) = O_g(q^{\frac{1}{2}g+\varepsilon g}). \tag{23}$$

Combining equations (21), (22), and (23) altogether, we obtain

$$\sum_{\chi\in\mathcal{S}_g}|L(1,\chi)|^2 = (21) + (22) + (23) - q^{-g-2}(q-1)^2(g+2)^2 = C_1 g q^{g+2} + C_2 q^{g+2} + O\left(q^{\frac{g}{2}+\varepsilon g+1}\right).$$

Therefore, the average value of $|L(1,\chi)|^2$ is obtained directly from Lemma 4.1. □

Proof of Theorem 1.2. Since 3 divides the degree of m, we can see that the infinite place ∞ of k splits completely in $K_m = k(\sqrt[3]{m})$; therefore, K_m is real.

By [16, Theorem 1.5], we obtain

$$|L(1,\chi_m)|^2 = (q-1)^2\frac{\widetilde{h}_m R_m}{\sqrt{|d_m|}}, \tag{24}$$

where $\widetilde{h}_m$ is the order of the ideal class group of the maximal order O_{K_m} of K_m, R_m is the regulator of K_m, and d_m is the discriminant of K_m. In fact, we note that $h_m = \widetilde{h}_m R_m$. In addition, the discriminant of K_m is $(m_1m_2)^2$, which follows from [16, Theorem 1.2]. Therefore, the denominator of equation (24) $\sqrt{|d_m|}$ is equal to $=\sqrt{|(m_1m_2)^2|} = q^{g+2}$, where we use the fact that the degree of m_1m_2 is $g+2$.

Consequently, using equation (24), Lemma 4.1, and Theorem 1.1 all together, for $m \in M_g$, the average value of h_m is given as follows:

$$\frac{\sum_{m\in M_g}h_m}{|M_g|} = \frac{q^{g+2}}{(q-1)^2}\frac{\sum_{m\in M_g}|L(1,\chi_m)|^2}{|M_g|} = \frac{q^{g+2}}{(q-1)^2}\left(\frac{C_1g+C_2}{B_1g+B_2} + O\left(q^{\varepsilon g-\frac{g}{2}}\right)\right);$$

thus, we obtain the result. □

Acknowledgement: The authors would like to thank the reviewers for their very helpful suggestions, which improved the clarity of this article.

Funding information: Y. Lee is supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (Grant No. 2019R1A6A1A11051177) and the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (NRF-2022R1A2C1003203). J. Lee is supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2020R1I1A3057692). J. Yoo is supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (Nos. 2020R1A4A1016649 and 2022R1A2C1009297), and the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2021R1I1A1A01047765).

Conflict of interest: The authors state that there is no conflict of interest.

References

[1] J. C. Andrade and M. Yiasemides, *The fourth power mean of Dirichlet L-functions in* $\mathbb{F}_q[T]$, Rev. Mat. Complut. **34** (2021), no. 1, 239–296, DOI: https://doi.org/10.1007/s13163-020-00350-2.
[2] C. F. Gauss, *Disquisitiones Arithmeticae*, English ed., Springer-Verlag, New York, 1986.
[3] R. Lipschitz, *Ueber die asymptotischen Gesetze von gewisser Gattungen zahlentheoretischer Funktionen*, Monatsber. Preuss. Akad. Wiss. (1865), 174–185.
[4] F. Mertens, *Ueber einige asymptotische Gesetze der Zahlentheorie*, J. Reine Angew. Math. **77** (1874), 289–338.
[5] C. L. Siegel, *The average measure of quadratic forms with given determinant and signature*, Ann. Math. **45** (1944), no. 4, 667–685, DOI: https://doi.org/10.2307/1969296.
[6] I. M. Vinogradov, *On the number of integer points in a sphere*, Izv. Ross. Akad. Nauk Ser. Mat. **27** (1963), 957–968.
[7] S. Baier and M. P. Young, *Mean values with cubic characters*, J. Number Theory **130** (2010), no. 4, 879–903, DOI: https://doi.org/10.1016/j.jnt.2009.11.007.
[8] C. David, A. Florea, and M. Lalín, *The mean values of cubic L-functions over function fields*, Algebra Number Theory **16** (2022), no. 5, 1259–1326, DOI: https://doi.org/10.2140/ant.2022.16.1259.
[9] J. Hoffstein and M. Rosen, *Average values of L-series in function fields*, J. Reine Angew. Math. **426** (1992), 117–150, DOI: https://doi.org/10.1515/crll.1992.426.117.
[10] H. Jung, *Note on average of class numbers of cubic function field*, Korean J. Math. **22** (2014), no. 3, 419–427, DOI: https://doi.org/10.11568/kjm.2014.22.3.419.
[11] J. Lee and Y. Lee, *Non-vanishing of L-functions for cyclotomic characters in function fields*, Proc. Amer. Math. Soc. **150** (2022), no. 2, 455–468, DOI: https://doi.org/10.1090/proc/15144.
[12] J. Lee and Y. Lee, *Mean values of derivatives of L-functions for cyclotomic characters in function fields*, Taiwanese J. Math. **26** (2022), no. 6, 1113–1131, DOI: https://doi.org/10.11650/tjm/220701.
[13] Y. Lee and J. Yoo, *Indivisibility of divisor class numbers of Kummer extensions over the rational function field*, J. Number Theory **192** (2018), 270–292, DOI: https://doi.org/10.1016/j.jnt.2018.04.016.
[14] W. Luo, *On Hecke L-series associated with cubic characters*, Compos. Math. **140** (2004), no. 5, 1191–1196, DOI: https://doi.org/10.1112/S0010437X0400051X.
[15] J. Yoo and Y. Lee, *Infinite families of cyclotomic function fields with any prescribed class group rank*, J. Pure Appl. Algebra **225** (2021), no. 9, 106658, DOI: https://doi.org/10.1016/j.jpaa.2020.106658.
[16] M. Rosen, *Average value of class numbers in cyclic extensions of the rational function field*, CMS Conference Proceedings, Vol. 15, American Mathematical Society, Providence, 1994, pp. 307–323.
[17] M. I. Rosen, *Number Theory in Function Fields*, Springer-Verlag, New York, 2002.
[18] J. Lee, Y. Lee, and J. Yoo, *The mean value of the class numbers of cubic function fields*, J. Math. Anal. Appl. **517** (2023), no. 1, 126582, DOI: https://doi.org/10.1016/j.jmaa.2022.126582.
[19] G. D. V. Salvador, *Topics in the Theory of Algebraic Function Fields*, Birkhäuser, Boston, 2006.