Master's Thesis

# An Efficient Numerical Approach for Solving Caputo Fractional Nonlinear Two-Point Boundary Value Problems

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# Abstract

This paper proposed a new type of numerical scheme for solving Multi-term Fractional Nonlinear Differential Equation with Two Point Boundary Value Problems(FBVP).
The proposed methods take lower computational cost than conventional methods. In new techniques, the FBVP transposes to the System of Fractional Nonlinear Initial Value Problems(FIVP). In order to solve the system of FIVP, the Higher-Order Predictor-Corrector Method [1](HOM) is applied. Moreover, we employ shooting method based on Newton's and Halley's methods to approximate the unknown initial values of the system. Several numerical experiments show that the proposed methods give the same rate of convergence in the HOM.

*Keywords :* Capuoto fractional derivative, Fractional differential equation, Fractional order boundary value problem, predictor-corrector methods, Nonlinear shooting methods.

# Contents

# I Introduction

## 1.1 Model Problem

Fractional differential equations recently receive attention. Because, it can more flexibly express many mathematical models than ordinary differential equation. Thus, many physics and engineering parts use that concepts. For problems like biology [2] [3] [4], earthquake modeling [5], Heat flow and diffusion [6], Thermal wave [7] and Neural Network [8] [9] to name but a few.
Thus, various fields need solvers for diverse fractional type of differential equations. This research introduce how to solve FBVP with Caputo operator.

$$\begin{cases} {}^{C}D^{\alpha_2}_{t_0,t}y(t) = f(t, y, {}^{C}D^{\alpha_1}_{t_0,t}) \\ g(y(t), y'(t)) = \gamma_t\big|_{t=a,b} \end{cases} \tag{I.1}$$

where $0 < \alpha_1 < 1,\ 1 < \alpha_2 < 2,\ \alpha_1, \alpha_2 \in \mathbb{R}$, $g$ be a linear function and $\gamma_t$ is for some real number.
In general method construct a matrix for imposing FBVP. But, due to the non-local property, it takes too many computational costs to solve a matrix equation at each time. Thus, this paper suggest new numerical scheme that transform main problem to a system of FIVP. However, that initial value problems doesn't have a whole exact initial value, so if you want to solve that problem, then you must approximate that value which most well solve the system. Thus, this research will use shooting method for renewing an approximation of a initial value problem. That Shooting Methods are Newton's and Halley's Method which have second and third order. The conventional methods basically use predictor-corrector method. So, each system need to find predictor and corrector. But the nonlinear system which comes from original boundary value problem has a special structure that is each equation of transformed system influence other. Therefore, the paper try to use that structure.

## 1.2 Preliminaries

This section briefly present some definitions and theorems of the fractional calculus which is enough to understand contents of this paper. And it will introduce How to solve Fractional differential equation with initial value problem.

### 1.2.1 Fractional Calculus

Some basics of the fractional calculus for the later parts of this paper will be introduced. The origins of fractional calculus begin at 17th century. Leibniz and L'Hospital discussed about "what is $\frac{d^{1/2}y}{dx^{1/2}}$?". After that discussion, various engineers and mathematicians suggest diverse types of fractional differential operator, such as "Riemann-Liouville operator" [10], "Grünwald-Letnikov operator" [10], "Caputo operator" [10], "Caputo-Fabrizio operator" [11] and "Atangana-Baleanu operator" [12]. In this paper we only consider the Caputo operator. Below content will begin with special functions and a brief motivation for the Caputo fractional derivative.

For fractional calculus start from classical result of relation between differential and integration which is called 'Fundamental Theorem of Calculus'.

Now let's define some notation for research's convenient.

**Definition 1.2.1.** *Let's denote $D$ as the operator that maps a differentiable function onto its derivative,*

$$Dy(x) := y'(x) \tag{I.2}$$

*and notation $J$ is the operator that maps a function $f$, assumed to be Riemann integrable on the compact interval $[a, b]$, onto its primitive centered at $a$*

$$J_a y(t) := \int_a^t y(\tau)\, d\tau \tag{I.3}$$

*for $n \in \mathbb{N}$, $D^n$ and $J_a^n$ notation will be used. That means $n-$ford iterates of $D$ and $J_a$ respectively.*

$$D^1 := D,\ J_a^1 := J_a,\ D^n := DD^{n-1},\ \text{and } J_a^n = J_a J_a^{n-1} \tag{I.4}$$

*for $n \geq 2$.*

Following the above definition, let's begin with the integral operator $J_a^n$. In the case $n \in \mathbb{N}$, it can be replace by the following explicite formula.

**Lemma 1.2.1.** *Let $f$ be Riemann integrable on $[a, b]$. Then, for $a \leq t \leq b$ and $n \in \mathbb{N}$, we have*

$$J_a^n y(t) = \frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} y(\tau) d\tau. \tag{I.5}$$

**Theorem 1.2.1.** *Fundamental Theorem of Calculus Let $f \in C[a, b]$ is real valued function. and let $F : [a, b] \to \mathbb{R}$ is defined by*

$$F(x) := \int_a^x f(t)\, dt. \tag{I.6}$$

*Then, $F$ is differentiable and*

$$F' = f \tag{I.7}$$

In this theorem shows that differential and integration operator have really close relation. When fractional derivative is induced, that relation be a very important point. Moreover, it is an immediate consequence of the fundamental theorem that the following relation holds for the operators $D$ and $J_a$.

**Lemma 1.2.2.** *Let, $m, n \in \mathbb{N}$ such that $m > n$, and let $f$ be a function having a continuous $n$th derivative on the interval $[a, b]$. Then,*

$$D^n y = D^m J_a^{m-n} y. \tag{I.8}$$

Now, let's define a special function. "Gamma function" is really many used in fractional calculus and also in this paper.

**Definition 1.2.2.** *Gamma function $\Gamma : (0, \infty) \longrightarrow \mathbb{R}$, defined by*

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt, \tag{I.9}$$

*is called Euler's Gamma function*

Now, the following definition seems rather natural.

**Definition 1.2.3.** *Riemann-Liouville Integrals Let $\alpha \in \mathbb{R}_+$. The operator $J_a^\alpha$, defined on $L_1[a, b]$ by*

$$J_a^\alpha y(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} y(\tau)\, d\tau \tag{I.10}$$

*for $a \le t \le b$, is called the Riemann-Liouville fractional integral operator of order $\alpha$.*

For $\alpha = 0$, $J_a^0 = I$, the identity operator. And Riemann-Liouville integral operator has following property.

**Theorem 1.2.2.** *Let $\alpha, \beta \in \mathbb{R}_+$, $\alpha, \beta \ge 0$ and $y \in L_1[a, b]$. Then,*

$$J_a^\alpha J_a^\beta y(t) = J_a^{\alpha+\beta} y(t) \tag{I.11}$$

*holds almost everywhere on $[a, b]$. If additionally $y \in C[a, b]$ or $\alpha + \beta \ge 1$, then the identity holds everywhere on $[a, b]$.*

To motive the definition coming up, we recall **Lemma** 1.2.2 that states the identity.

$$D^n y = D^m J_a^{m-n} y.$$

where $m$ and $n$ were integers such that $m > n$. Now assume that $n$ is not an integer. Then $m$ still can be chosen an integer $m$ such that $m > n$. That induced new definition.

**Definition 1.2.4.** *Riemann-Liouville operator let $\alpha \in \mathbb{R}_+$. The operator ${}^{RL}D_a^\alpha$ is defined by*

$${}^{RL}D_a^\alpha y(t) := D^{\lceil \alpha \rceil} J_a^{\lceil \alpha \rceil - \alpha} y(t) = \frac{1}{\Gamma(\lceil \alpha \rceil - \alpha)} \frac{d^{\lceil \alpha \rceil}}{dt^{\lceil \alpha \rceil}} \int_a^t (t-\tau)^{\lfloor \alpha \rfloor - \alpha} y(\tau) d\tau. \tag{I.12}$$

*where $\lceil \quad \rceil$ is ceiling function, and $\lfloor \quad \rfloor$ is a floor function. if $\alpha \in \mathbb{N}$ then, ${}^{RL}D_a^\alpha := D^\alpha$.*

But, it turns out that the Riemann-Liouville derivative have certain disadvantages when trying to model real-world phenomena with fractional differential equations. So there are many alternative operator of the Riemann-Liouville operator. For instance "Caputo operator', "Caputo-Fabrizio operator" and "Atangana-Baleanu operator". And this paper focus on "Caputo operator".

**Definition 1.2.5.** *Caputo operator let $\alpha \in \mathbb{R}_+$. The operator ${}^C D_a^\alpha$ is defined by*

$$ {}^C D_a^\alpha y(t) := J_a^{\lceil\alpha\rceil-\alpha} D^{\lceil\alpha\rceil} y(t) = \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)} \int_a^t (t-\tau)^{\lceil\alpha\rceil-\alpha} y^{(\lceil\alpha\rceil)}(\tau) d\tau. \tag{I.13}$$

*where $\lceil \quad \rceil$ is a ceiling function, and $\lfloor \quad \rfloor$ is a floor function. if $\alpha \in \mathbb{N}$ then, ${}^C D_a^\alpha := D^\alpha$.*

Then, there is difference between Riemann-Liouville derivative and Caputo derivative.

**Theorem 1.2.3.** *Let $\alpha \in \mathbb{R}_+$. And assume that $f \in A^{\lceil\alpha\rceil}[a,b]$, ($A^m[a,b], m \in \mathbb{N}$ is the set of functions with an absolutely continuous $(m-1)$st derivative.)*

$$ {}^C D_a^\alpha y(t) = {}^{RL} D_a^\alpha \left[ y - T_{\lfloor\alpha\rfloor}[y;a] \right] \tag{I.14}$$

*almost everywhere. Here $T_m[f;a]$ denote the Taylor polynomial of degree $m$ for the function $y$. centered at $a$; if $m$ is 0, then $T_0[y,a] = 0$.*

And Caputo operator satisfies additive property.

**Theorem 1.2.4.** *Let $f \in C^k[a,b]$ for some $a < b$ and some $k \in \mathbb{N}$. Moreover let $\alpha, \beta > 0$ be such that there exists some $l \in \mathbb{N}$ with $l \leq k$ and $\alpha, \alpha+\beta \in [l-1, l]$. Then,*

$$ D_a^\beta D_a^\alpha y(t) = D_a^{\alpha+\beta} y(t), \tag{I.15}$$

This paper will denote $D_a^\alpha y(t) \equiv {}^C D_a^\alpha y(t)$.

### 1.2.2 Fractional Initial Value Problem

FIVP is consider many branch. For instance, biological branches like tumor model [3], chicken pox disease model [4]. Second, analysis about heat flow [6]. Third, financial branches like Black-Scholes option pricing equation [13]. Fourth, neural networks which is a part of deep learning also have been researched with a fractional differential equation [9]. And there are more branches that use the initial value problem. Thus, two things will be introduced. First, definition of fractional initial value problem. Second, some theorems for it. They are about existence, equivalent form of solution and uniqueness of that form. Every below theorems or lemmas come from [10].

$$ \begin{cases} D_a^\alpha y(t) = f(t, y(t)), & t \in [a,b] \\ y^{(k)}(a) = y_k, & k = 0, 1, \cdots, \lfloor\alpha\rfloor \end{cases} \tag{I.16}$$

Then, the existence of solution for FIVP (I.16) is guaranteed by below theorem.

**Theorem 1.2.5.** *Let* $0 < \alpha$ *and* $y_a^{(0)}, \cdots, y^{([\alpha])} \in \mathbb{R},\ K > 0$ *and* $h^* > 0$.
*Define* $G := \left\{(x, y)\middle| x \in [a, b^*], |b^* - a| = h^*, \left|y - \sum_{k=0}^{[\alpha]}(x-a)^k y_a^{(k)}/k!\right| \le K\right\}$.
*and let the function* $f : G \to \mathbb{R}$ *be continuous. Beside, define* $M := \sup_{(x,z)\in G} |f(x, z)|$ *and*

$$h := \begin{cases} h^* & \text{if } M = 0 \\ \min\left\{h^*, (K\Gamma(\alpha+1)/M)^{1/\alpha}\right\} & \text{else} \end{cases} \tag{I.17}$$

*Then, there exists a number* $b$ *such that* $|a - b| = h$ *and also exists function* $y \in C[a, b]$ *solving the initial value problem* (I.16).

Existence of solution of (I.16) is guaranteed by Theorem (1.2.5). Then, next Lemma is going to show that what equation is equivalent to solution of (I.16) and that following lemma will be adapted to numerical method for.

**Lemma 1.2.3.** *Assume the hypotheses of Theorem* (1.2.5). *The function* $y \in C[a, b]$ *is a solution of the initial value problem* (I.16) *if and only if it is a solution of the nonlinear Volterra integral equation of the second kind.*

$$y(t) = \sum_{k=0}^{[\alpha]} \frac{(x-a)^k}{k!} y_k + \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau, \tag{I.18}$$

Next, following theorem guarantee uniqueness of solution.

**Theorem 1.2.6.** *Let* $0 < \alpha$ *and* $y_a^{(0)}, \cdots, y^{([\alpha])} \in \mathbb{R},\ K > 0$ *and* $h^* > 0$. *Define the set* $G$ *as in Theorem* (1.2.5) *and let the function* $f : G \to \mathbb{R}$ *continuous and fulfill a Lipschitz condition with respect to the second variable, i.e*

$$|f(x, y_1) - f(x, y_2)| \le L|y_1 - y_2| \tag{I.19}$$

*with some* $L > 0$ *independent of* $x, y_1$ *and* $y_2$. *Then, denoting* $h$ *as in Theorem* (1.2.5). *there exists a uniquely defined function* $y \in C[a, b]$ *solving the initial value problem* (I.16).

# II Fractional Two-Point Boundary Value Problem

The main part of this research suggest a new numerical scheme for solving the following FBVP:

$$\begin{cases} D_a^{\alpha_2} y(t) = f(t, y, D_a^{\alpha_1} y(t)) \\ g(y(t), y'(t)) = \gamma_t \big|_{t=a,b}, \end{cases} \tag{II.1}$$

where $1 < \alpha_2 < 2,\ 0 < \alpha_1 < 1,\ \alpha_2, \alpha_1 \in \mathbb{R}$ and g be linear function.

If conventional method is used for solving above problem then it need many computational cost to solve a dense matrix and multi-dimensional solver. By the way, there are some theorems for FBVP to transpose system of FIVP. Then, the numerical method for it need less computational cost than conventional one for fractional boundary value problem. However, original FBVP doesn't provide a initial value $y(a)$ or $y'(a)$ data directly. Thus, to approximate that data, shooting method will be used. For examples are Newton's or Halley's methods. And that shooting methods are iterated when the error between result given data is reduced. "Figure 1" will be show outline of above explanation.

$$\begin{cases} D_a^{\alpha_2} y(t) = f(t, y, D_a^{\alpha_1} y(t)) \\ g(y(t), y'(t)) = \gamma_t|_{t=a,b} \end{cases}$$
$$1 < \alpha_2 < 2,\ 0 < \alpha_1 < 1, \alpha_1, \alpha_2 \in \mathbb{R}$$

$$\Downarrow$$

$$\begin{cases} D_a^{\alpha_1} y(t) = w(t) & y(t_0) = y_a \\ D_a^{1-\alpha_1} w(t) = z(t) & w(t_0) = 0 \\ D_a^{\alpha_2 - 1} z(t) = f(t, y(t), w(t)), & z(t_0) = s \end{cases}$$

$$\Downarrow$$

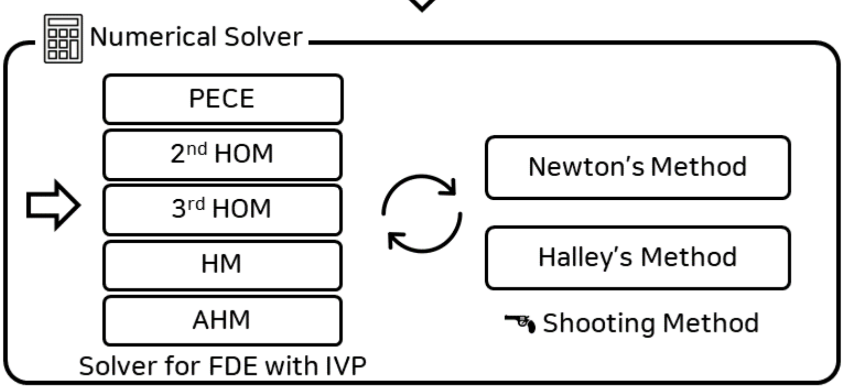



Figure 1: Diagram about Solving Process of Fractional Two-Point Boundary Value Problem

## 2.1 Description about Fractional Boundary Value Problem

FBVP can be transformed to system of FIVP by following theorem from [10].

**Theorem 2.1.1.** *Multi-term Fractional Differential Equations with integer order term*

$$\begin{cases} D_a^{\alpha_n} y(t) = f(t, y(t), D_a^{\alpha_1} y(t), D_a^{\alpha_2} y(t), \cdots, D_a^{\alpha_{n-1}} y(t)), \\ y^{(j)}(a) = y_a^{(j)},\ j = 0, 1, \cdots, [\alpha_n], \end{cases} \tag{II.2}$$

*where* $\alpha_n > \alpha_{n-1} > \cdots \alpha_1 > 0,\ \alpha_j - \alpha_{j-1} \leq 1$ *for all* $j = 2, 3, \cdots, n$ *and* $0 < \alpha_1 < 1$. *Then, we can define* $\beta_j,\ j = 1, 2, \cdots, n,$

$$\begin{cases} \beta_1 := \alpha_1, \\ \beta_j := \alpha_j - \alpha_{j-1}, \quad j = 2, 3, \cdots, n, \end{cases} \tag{II.3}$$

*Then, this initial value problem* (II.1) *is equivalent to the system of equations*

$$\begin{aligned} D_a^{\beta_1} y_0(t) &= y_1(t), \\ D_a^{\beta_2} y_1(t) &= y_2(t), \\ &\vdots \\ D_a^{\beta_{n-1}} y_{n-1} &= y_n(t), \\ D_a^{\beta_n} y_n &= f(t, y_1, y_2, \cdots, y_{n-2}, y_{n-1}), \end{aligned} \tag{II.4a}$$

*together with the initial conditions.*

$$y_j(t_0) = \begin{cases} y^{(0)} & \text{if } j = 1, \\ y^{(l)} & \text{if } \alpha_{j-1} = l \in \mathbb{N}, \\ 0 & \text{else}, \end{cases} \tag{II.4b}$$

*in the following sense.*

*1. Whenever the function* $y \in C^{\lceil \alpha_n} [a, b]$ *is a solution of the multi-term equation* II.2 *with initial conditions, the vector-valued function* $Y := (y_1, \cdots, y_n)^T$ *with*

$$y_j(t) := \begin{cases} y(t) & \text{if } j = 1, \\ D_a^{\alpha_{j-1}} y(t) & \text{if } j = 2, \end{cases} \tag{II.5}$$

*is a solution of the multi-order fractional differential system* (II.4a) *with initial conditions* (II.4b)

*2. Whenever the vector-valued function* $Y := (y_1, \cdots, y_n)^T$ *is a solution of the multi-order fractional differential systema* (II.4a) *with initial condition* (II.4b), *the function* $y := y_1$ *is a solution of the multi-term equation* II.2 *with initial conditions.*

There are many two-point boundary conditions. But in this research just consider Dirichlet and Robin boundary conditions.

Robin boundary condition is considered, similarly FBVP can be written as,

$$\begin{cases} D_a^{\alpha_2} y(t) = f(t, y(t), D_a^{\alpha_1} y(t))), \\ a_1 y(t_0) + b_1 y'(t_0) = \gamma_1,\ a_2 y(t_N) + b_2 y'(t_N) = \gamma_2 \end{cases} \tag{II.6}$$

Then,

$$\begin{cases} D_a^{\alpha_1} y(t) = w(t), & y(t_0) = y_a = (\gamma_1 - b_1 s)/a_1 \\ D_a^{1-\alpha_1} w(t) = z(t), & w(t_0) = 0 \\ D_a^{\alpha_2 - 1} z(t) = f(t, y(t), w(t)), & z(t_0) = s. \end{cases} \tag{II.7}$$

if $b_1$ and $b_2$ are zeros, that case is called "Dirichlet Boundary Condition".

## 2.2 Numerical Methods

### 2.2.1 Fractional Two-Point Boundary Value Problem

Each equations of system (II.7) are FIVP. Then, solution is

$$\begin{cases} y(t) = y_0 + \frac{1}{\Gamma(\alpha_1)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha_1 - 1} w(\tau) d\tau, & y_0 = (\gamma_1 - b_1 s)/a_1 \\ w(t) = w_0 + \frac{1}{\Gamma(1-\alpha_1)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{-\alpha_1} z(\tau) d\tau, & w_0 = 0 \\ z(t) = z_0 + \frac{1}{\Gamma(\alpha_2 - 1)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha_2 - 2} f(\tau, y(\tau), w(\tau)) d\tau, & z_0 = s. \end{cases} \tag{II.8}$$

Thus, numerical schemes for FIVP are needed. And in order to find unknown value $z_0 = s$ which induce boundary condition at the right end point $a_2 y(b) + b_2 y'(b) = \gamma_2$, shooting methods are used.

### 2.2.2 Numerical Methods for Fractional Initial Value Problem

Conventionally, "Predictor-Corrector Method" is adopted for numerical scheme. That is used to solve ordinary differential equation. In detail, It find an unknown function that satisfies a given differential equation. That process have two part, first, "prediction" part, it predicts points which fitted to the function value and derivative values from preceding set. Second, "corrector" part, it refines the previous approximated value which gained from "prediction" part, to obtain same point. Fractional differential equation can use "Predictor-Corrector Method" also, and it will be introduced bellow.

Before start, for convenience let's denote $y_j$ as approximated value of $y(t_j)$ and $f_j \equiv f(t_j, y_j), \quad j = 1, \cdots, N$. However, if $j = 0$ then, $f_0 = f(t_0, y(t_0))$. Because initial value $y(t_0)$ is given exactly. First, let the domain $\Omega$ to be

$$\Phi_N := \{\, t_j \,|\, a = t_0 < \cdots < t_j < \cdots < t_n < t_{n+1} < \cdots < t_N = b\}. \tag{II.9}$$

for simplicity, step size is uniform, that means $t_{j+1} - t_j = h,\ j = 0, 1 \cdots, N-1$. Thus (I.18) can be rewritten at time $t_{n+1}$ as follows

$$y(t_{n+1}) = g(t_{n+1}) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha - 1} f(\tau, y(\tau)) d\tau. \tag{II.10}$$

where $g(t_{n+1}) = \sum_{k=0}^{[\alpha]} \frac{(t_{n+1} - a)^k}{k!} y_k$. For numerical approximations of $y(t)$, it need to interpolate the term $f(\tau, y(\tau))$, given sufficient continuity, over each interval $I_j = [t_j, t_{j+1}],\ j =$

$0, 1, \cdots, N-1$, by some types of interpolating polynomials. For this purpose, the Linear and Quadratic order Lagrange polynomials are often taken into consideration.

Thus, let's consider linear order Lagrange polynomials. On each interval $I_j = [t_j, t_{j+1}]$, Second order Lagrange interpolation of $f(\tau, y(\tau))$ is

$$f(\tau, y(\tau)) = L_j^1 f(\tau) + R_j(f(\tau)), \quad \tau \in [t_j, t_{j+1}] \tag{II.11}$$

where

$$L_j^1 f(\tau) = \frac{t_{j+1}-\tau}{h} f(t_j, y(t_j)) - \frac{t_j - \tau}{h} f(t_{j+1}, y(t_{j+1}))), \; j = 0, 1, \cdots, N,$$

and

$$R_j^1(f(\tau)) = \frac{f''(\xi_j)}{2}(\tau - t_j)(\tau - t_{j+1}), \; \xi_j \in (t_j, t_{j+1}).$$

$$\begin{aligned} y(t_{n+1}) &= g(t_{n+1}) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} \left( L_j^1 f(\tau) + R_j^1(f(\tau)) \right) d\tau, \\ &= g(t_{n+1}) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \left( B_{n+1}^{1,j} f(t_j, y(t_j)) + B_{n+1}^{2,j} f(t_{j+1}, y(t_{j+1})) \right) + T_{n+1}. \end{aligned} \tag{II.12}$$

where

$$B_{n+1}^{1,j} = \frac{1}{h} \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} (t_{j+1} - \tau) d\tau,$$

$$B_{n+1}^{2,j} = -\frac{1}{h} \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} (t_j - \tau) d\tau,$$

$$T_{n+1} = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} R_j^1(f(\tau)) \tau.$$

When $j = n$, it need $y(t_{n+1})$. But that term doesn't known. Thus, the $f(t_{n+1}, y(t_{n+1}))$ is approximated to $f_{n+1}^P$ is needed. And that term must be 'predict', thus, it is called "predictor"

$$y_{n+1} = g(t_{n+1}) + \frac{1}{\Gamma(\alpha)} \left[ \sum_{j=0}^{n-1} \left( B_{n+1}^{1,j} f_j + B_{n+1}^{2,j} f_{j+1} \right) + B_{n+1}^{1,n} f_n + B_{n+1}^{2,n} f_{n+1}^P \right], \tag{II.13}$$

There are two types of 'predictor' for Second-order corrector. The initial is "Predictor-Evaluate Corrector-Evaluate Method(PECE)" type [14], and another is "Second-order High-Order Predictor Corrector Method(SHOM)"type [1].

"PECE"is the most used conventional method. It use rectangle rule to replace the right-hand side of (II.10).

$$y_{n+1}^P = g(t_{n+1}) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} p_{n+1}^j f_j, \tag{II.14}$$

where

$$p_{n+1}^j = \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} d\tau,$$

The convergence analysis shows that the error is expected to behave as

$$\max_{j=1,\cdots,n} |y(t_j) - y_j| = O(h^q), \tag{II.15}$$

where $q = \min(2, 1+\alpha)$.

SHOM is another conventional one. This method have better convergence rate than PECE. Convergence rate of PECE depend on derivative order. But SHOM have fixed second-order convergence rate which does not depend on it.

$$y_{n+1}^P = g(t_{n+1}) + G(t_{n+1}) + b_{n+1}^1 f_{n-1} + b_{n+1}^2 f_n, \tag{II.16}$$

where

$$G(t_{n+1}) = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \left( B_{n+1}^{1,j} f_j + B_{n+1}^{2,j+1} f_{j+1} \right),$$

$$b_{n+1}^1 = \frac{1}{h} \int_{t_n}^{t_{n+1}} (t_{n+1}-\tau)^{\alpha-1}(t_n - \tau)d\tau,$$

$$b_{n+1}^2 = -\frac{1}{h} \int_{t_n}^{t_{n+1}} (t_{n+1}-\tau)^{\alpha-1}(t_{n-1} - \tau)d\tau.$$

The Global error $E_{n+1}$ of LHOM is

$$\max_{j=1,\cdots,n} |y(t_j) - y_j| = O(h^2), \tag{II.17}$$

(II.17) is guaranteed by the below theorem from [1].

Below global error theorem show convergence rate of SHOM.

**Theorem 2.2.1.** *Global Error of Second-order HOM Let define $E_{n+1}$ is global error. Suppose $f(\cdot, y(\cdot)) \in C^2[a,b]$ and furthermore is Lipschitz continuous in the second argument in Theorem* (1.2.6) *then we have*

$$E_{n+1} = |y(t_{n+1}) - y_{n+1}| \leq O(h^2). \tag{II.18}$$

Therefore, SHOM have better convergence rate than PECE.
Similarly, Quadratic Lagrange interpolation of $f(\tau, y(\tau))$ is defined as,

$$f(\tau, y(\tau)) = L_j^2 f(\tau) + R_j^2(f(\tau)), \quad t \in [t_j, t_{j+1}], \tag{II.19}$$

where

$$L_j^2 f(\tau) = f(t_{j-1}, y(t_{j-1}))Q_j^1(\tau) + f(t_j, y(t_j))Q_j^2(\tau) + f(t_{j+1}, y(t_{j+1}))Q_j^3(\tau) \tag{II.20}$$

$$Q_j^1(\tau) = \frac{(t_j - \tau)(t_{j+1}-\tau)}{(t_j - t_{j-1})(t_{j+1} - t_{j-1})},\ Q_j^2(\tau) = \frac{(t_{j-1} - \tau)(t_{j+1}-\tau)}{(t_{j-1} - t_j)(t_{j+1} - t_j)},\ Q_j^3(\tau) = \frac{(t_{j-1} - \tau)(t_j-\tau)}{(t_{j-1} - t_{j+1})(t_j - t_{j+1})},$$

$$R_j^2(f(\tau)) = \frac{f'''(\xi_j)}{6}(\tau - t_{j-1})(\tau - t_j)(\tau - t_{j+1}),\ \xi_j \in (t_j, t_{j+1}),\ j = 1, \cdots, N.$$

But, $I_0 = [t_0, t_1]$ interval can not interpolate by Quadratic interpolation. So it will use a $t_{1/2}$ point. If exact value is known, just use it, or doesn't know it, approximate $y_{1/2} \equiv y_0$ or use startup scheme form [1]. Anyway, let's assume the value $y_{1/2}$ is known,

$$L_0^2 f(\tau) = Q_0^1(\tau) f(t_0, y(t_0)) + Q_0^2(\tau) f(t_{1/2}, y(t_{1/2})) + Q_0^3(\tau) f(t_1, y(t_1)), \tag{II.21}$$

where

$$Q_0^1(\tau) = \frac{(t_{1/2}-\tau)(t_1-\tau)}{(t_{1/2}-t_0)(t_1-t_0)},\ Q_0^2(\tau) = \frac{(t_0-\tau)(t_1-\tau)}{(t_0-t_{1/2})(t_1-t_{1/2})},\ Q_0^3(\tau) = \frac{(t_0-\tau)(t_{1/2}-\tau)}{(t_0-t_1)(t_{1/2}-t_1)},$$

Thus, $y(t_{n+1})$ will be below form.

$$\begin{aligned} y(t_{n+1}) &= g(t_{n+1}) + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}(t_{n+1}-\tau)^{\alpha-1}\left(L_j^2 f(\tau) + R_j^2(f(\tau))\right)d\tau, \\ &= g(t_{n+1}) + \frac{1}{\Gamma(\alpha)}\Bigg[A_{n+1}^{1,0}f(t_0,y(t_0)) + A_{n+1}^{2,0}f(t_{1/2},y(t_{1/2})) + A_{n+1}^{3,0}f(t_1,y(t_1)) \\ &\qquad + \sum_{j=1}^{n}\Big(A_{n+1}^{1,j}f(t_{j-1},y(t_{j-1})) + A_{n+1}^{2,j}f(t_j,y(t_j)) + A_{n+1}^{3,j}f(t_{j+1},y(t_{j+1}))\Big)\Bigg] \\ &\qquad + T_{n+1}. \end{aligned} \tag{II.22}$$

where

$$\begin{aligned} A_{n+1}^{1,0} &= \frac{2}{h^2}\int_{t_j}^{t_{j+1}}(t_{n+1}-\tau)^{\alpha-1}(t_{1/2}-\tau)(t_1-\tau)d\tau, \\ A_{n+1}^{2,0} &= -\frac{4}{h^2}\int_{t_j}^{t_{j+1}}(t_{n+1}-\tau)^{\alpha-1}(t_0-\tau)(t_1-\tau)d\tau, \\ A_{n+1}^{3,0} &= \frac{2}{h^2}\int_{t_j}^{t_{j+1}}(t_{n+1}-\tau)^{\alpha-1}(t_0-\tau)(t_{1/2}-\tau)d\tau, \\ A_{n+1}^{1,j} &= \frac{1}{2h^2}\int_{t_j}^{t_{j+1}}(t_{n+1}-\tau)^{\alpha-1}(t_j-\tau)(t_{j+1}-\tau)d\tau, \\ A_{n+1}^{2,j} &= -\frac{1}{h^2}\int_{t_j}^{t_{j+1}}(t_{n+1}-\tau)^{\alpha-1}(t_{j-1}-\tau)(t_{j+1}-\tau)d\tau, \\ A_{n+1}^{3,j} &= \frac{1}{2h^2}\int_{t_j}^{t_{j+1}}(t_{n+1}-\tau)^{\alpha-1}(t_{j-1}-\tau)(t_j-\tau)d\tau, \\ T_{n+1} &= \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}(t_{n+1}-\tau)^{\alpha-1}R_j^2(f(\tau))d\tau. \end{aligned}$$

Similarly, When $j = n$, it need $y(t_{n+1})$. But that term doesn't known. Thus, the $f(t_{n+1}, y(t_{n+1}))$ is approximated to $f_{n+1}^P$ is needed like Linear corrector. Thus, also that term is predicted, and it will be called "predictor".

if $n = 1$ case,

$$y_1 = g(t_1) + \frac{1}{\Gamma(\alpha)}\left(A_{n+1}^{1,0}f_0 + A_{n+1}^{2,0}f_{1/2} + A_{n+1}^{3,0}f_1^P\right), \tag{II.23}$$

and other case, i.e $n \neq 1$,

$$\begin{aligned} y_{n+1} = g(t_{n+1}) + \frac{1}{\Gamma(\alpha)}\Bigg[&A_{n+1}^{1,0}f_0 + A_{n+1}^{2,0}f_{1/2} + A_{n+1}^{3,0}f_1 \\ &+ \sum_{j=1}^{n-1}\Big(A_{n+1}^{1,j}f_{j-1} + A_{n+1}^{2,j}f_j + A_{n+1}^{3,j}f_{j+1}\Big) + A_{n+1}^{1,n}f_{n-1} + A_{n+1}^{2,n}f_n + A_{n+1}^{3,n}f_{n+1}^P\Bigg]. \end{aligned} \tag{II.24}$$

Predictor of Third-order High-Order Predictor Corrector Method(THOM)" is induced as bellow.

$$y^P(t_{n+1}) = g(t_{n+1}) + G(t_{n+1}) + a^1_{n+1} f_{n-2} + a^2_{n+1} f_{n-1} + a^3_{n+1} f_n, \tag{II.25}$$

where

$$G(t_{n+1}) = \frac{1}{\Gamma(\alpha)} \left[ A^{1,0}_{n+1} f_0 + A^{2,0}_{n+1} f_{1/2} + A^{3,0}_{n+1} f_1 + \sum_{j=0}^{n-1} \left( A^{1,j}_{n+1} f_{j-1} + A^{2,j}_{n+1} f_j + A^{3,j}_{n+1} f_{j+1} \right) \right], \tag{II.26}$$

$$a^1_{n+1} = \frac{1}{2h^2} \int_{t_n}^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} (t_{n-1} - \tau)(t_n - \tau) d\tau,$$

$$a^2_{n+1} = -\frac{1}{h^2} \int_{t_n}^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} (t_{n-2} - \tau)(t_n - \tau) d\tau,$$

$$a^3_{n+1} = \frac{1}{2h^2} \int_{t_n}^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} (t_{n-2} - \tau)(t_{n-1} - \tau) d\tau.$$

The Global error $E_{n+1}$ of THOM is

$$\max_{j=1,\cdots,n} |y(t_j) - y_j| = O(h^3). \tag{II.27}$$

Either (II.27) is guaranteed by the below theorem from [1].

**Theorem 2.2.2.** *Global Error of Third-order HOM Suppose $f(\cdot, y(\cdot)) \in C^3[a,b]$ and is Lipschitz continuous in the second argument in Theorem* (1.2.6) *then we have*

$$E_{n+1} \le O(h^3) \tag{II.28}$$

*given $E_1, E_2 \le O(h^3)$ and $E_{1/2} \le O(h^{3-\alpha})$, $(0 < \alpha < 1)$, $O(h^2)$, $(\alpha > 1)$.*

### 2.2.3 Shooting methods

To approximate $y'(t_0) = s$, using shooting method. First, let's start to define $y(s) := y(t,s)|_{t=b}$. And error function $F(s)$ is defined by,

$$F(s) = a_2 y(s) + b_2 y'(s) - \gamma_2. \tag{II.29}$$

Then, to find zero of error function $F(s)$, Newton's and Halley's methods are used. For convenience of writing, let's notate two variable function $f(t,s)$,

$$\frac{\partial f(t,s)}{\partial s} = f_s, \quad \frac{\partial^2 f(t)}{\partial s^2} = f_{ss}. \tag{II.30}$$

**Newton's Method with Robin Boundary Condition**

The general Newton's Method is

$$s_{k+1} = s_k - \frac{F(s_k)}{F'(s_k)}, \quad k = 0, 1, 2, \cdots, \tag{II.31}$$

and it need to get

$$F'(s) = y_s(t,s)\big|_{t=b}. \tag{II.32}$$

Therefore the operator $\frac{\partial}{\partial s}$ is applied to (II.7).

$$\begin{cases} D_a^{\alpha_1} y_s(t) = w_s(t), & y_s(t_0) = (\gamma_1 - b_1)/a_1 \\ D_a^{1-\alpha_1} w_s(t) = z_s(t), & w_s(t_0) = 0 \\ D_a^{\alpha_2 - 1} z_s(t) = f_s(t, y(t), w(t)), & z_s(t_0) = 1 \end{cases} \tag{II.33}$$

$t$ and $s$ are independent, so $f_s(t, y(t), w(t))$ can be written as,

$$f_s(t, y(t), w(t)) = f_y \cdot y_s(t) + f_w \cdot w_s(t), \tag{II.34}$$

Let, define

$$y_s = \hat{y},\ w_s = \hat{w},\ z_s = \hat{z}. \tag{II.35}$$

Then, (II.33) can be rewritten as

$$\begin{cases} D_a^{\alpha_1} \hat{y}(t) = \hat{w}(t), & \hat{y}(t_0) = (\gamma_1 - b_1)/a_1 \\ D_a^{1-\alpha_1} \hat{w}(t) = \hat{z}(t), & \hat{w}(t_0) = 0 \\ D_a^{\alpha_2 - 1} \hat{z}(t) = f_y \cdot \hat{y}(t) + f_w \cdot \hat{w}(t), & \hat{z}(t_0) = 1 \end{cases} \tag{II.36}$$

Thus, $s$ can be approximated by solving above system and it will be used as initial condition to solve the system (**??**). Also, approximated value $y(s, b)$ is gotten, then $F(s)$ can be calculated. By that process will show amount of error, and if that is bigger, then Newton's Method will be iterated enough when $F(s)$ becomes almost zero.

**Halley's Method with Robin Boundary Condition**

The general Halley's Method is

$$s_{k+1} = s_k - \frac{2F(s_k)F'(s_k)}{2F'^2(s_k) - F(s_k)F''(s_k)}, \quad k = 0, 1, 2, \cdots, \tag{II.37}$$

Similarly, to get $F''(s_k)$, operator $\frac{\partial^2}{\partial s^2}$ is applied to (II.7).

$$\begin{cases} D_a^{\alpha_1} y_{ss}(t) = w_{ss}(t), & y_{ss}(t_0) = 0 \\ D_a^{1-\alpha_1} w_{ss}(t) = z_{ss}(t), & w_{ss}(t_0) = 0 \\ D_a^{\alpha_2 - 1} z_{ss}(t) = f_{ss}(t, y(t), w(t)), & z_{ss}(t_0) = 0 \end{cases} \tag{II.38}$$

$t$ and $s$ are independent, so $f_{ss}(t, y(t), w(t))$ can be written as,

$$f_y \cdot y_{ss}(t) + f_w \cdot w_{ss}(t) + f_{yy} \cdot y_s(t)^2 + f_{ww} \cdot w_s(t)^2 + f_{wy} \cdot w_s(t) y_s(t), \tag{II.39}$$

Let, define

$$y_{ss} = \tilde{y},\ w_{ss} = \tilde{w},\ z_{ss} = \tilde{z}. \tag{II.40}$$

$$\begin{cases} D_a^{\alpha_1} \tilde{y}(t) = \tilde{w}(t), & \tilde{y}(t_0) = 0 \\ D_a^{1-\alpha_1} \tilde{w}(t) = \tilde{z}(t), & \tilde{w}(t_0) = 0 \\ D_a^{\alpha_2 - 1} \tilde{z}(t) = f_y \cdot \tilde{y}(t) + f_w \cdot \tilde{w}(t) + f_{yy} \cdot \hat{y}(t)^2 + f_{ww} \cdot \hat{w}(t)^2 + f_{wy} \cdot \hat{w}(t) \hat{y}(t), & \hat{z}(t_0) = 0 \end{cases} \tag{II.41}$$

and $y'_s$ can be founded like below.

$$z_s = D_a^{1-\alpha_1} D_a^{\alpha_1} y_s(t) = D^1 y_s(t) = y'_s(t). \tag{II.42}$$

In general, $y$, $w$ and $z$ are solved by PECE, SHOM or THOM. Conventional methods need predictor for $y, w$ and $z$ for each iteration. But this research will suggest new method that comes from specific structure of nonlinear system. When $y_{n+1}$ term is calculated then it must need $w_{n+1}$. So $w_{n+1}$ term is calculated then, $z_{n+1}$ must be needed also. However $z_{n+1}$ needs $y_{n+1}$ and $w_{n+1}$ term. Therefore a system of FIVP have special iterative structure. Thus, first, predictor $y_{n+1}^P$ and $w_{n+1}^P$ are found by PECE or SHOM or THOM(in this paper SHOM and THOM is used), then $z_{n+1}$ can be calculated. Second, discovered $z_{n+1}$ is used as predictor to find $w_{n+1}$, and finally $y_{n+1}$ can be found by using $w_{n+1}$ as predictor. This method is called "Hybrid Method"(HM). Also above structure can be iterated, because if one circular is accomplished then, there are $y_{n+1}$ and $w_{n+1}$. Thus, above process can be iterated. But it must stop sometime, so it will use below stopping criterion.

$$z_i^c = z_0 + J_a^{\alpha_2 - 1} f_i \tag{II.43}$$

whose $z_0 = y'_0$ is initial value of $z$, then

$$E^i = \frac{z_i^c - z_0}{J_a^{\alpha_2-1} f_i}, \quad E^i \approx 1, \Rightarrow \frac{|E^i - E^{i+1}|}{E^i} << Tol, \ i \le I_{\max}. \tag{II.44}$$

This scheme will be called "Iterated Hybrid Method"(IHM). Figure.2 shows a process of HM and IHM. And similar to SHOM and THOM there are SHM, THM, SIHM and TIHM according to kind of interpolation function.

## 2.3 Numerical Results

This section will show the accuracy and efficiency of new methods. There are two kinds of table. First one is approximated error(II.45). It will illustrate shooting methods are how well it work, Second table is convergence rate of maximum error. It will show accuracy of methods. $h$ is step size and $N$ is number of grid on closed interval $[0, 1]$. $s$ is assumed start initial value for $y'(t_0)$. Approximated error choose $h = 0.01$, and Convergence rate chooses $s = 0.2$. For convenience let's write it as shown in the table below.

$$|F(s)| = |a_2 y(s) + b_2 y'(s) - \gamma_2| \quad \text{Approximated error} \tag{II.45}$$

IHM have 5 maximum iteration number and tolerance is $10^{-6}$.

**Example 2.3.1.** *Polynomial type example*

$$\begin{cases} D_0^{\alpha_2} y(t) = \frac{\Gamma(5)}{\Gamma(5-\alpha_2)} t^{4-\alpha_2} + \frac{\Gamma(5)}{\Gamma(5-\alpha_1)} t^{4-\alpha_1} - t^8 + y^2 - D_0^{\alpha_1} y(t), \\ a_1 y(0) + b_1 y'(0) = \gamma_1, \ a_2 y(1) + b_2 y'(1) = \gamma_2. \end{cases} \tag{II.46}$$

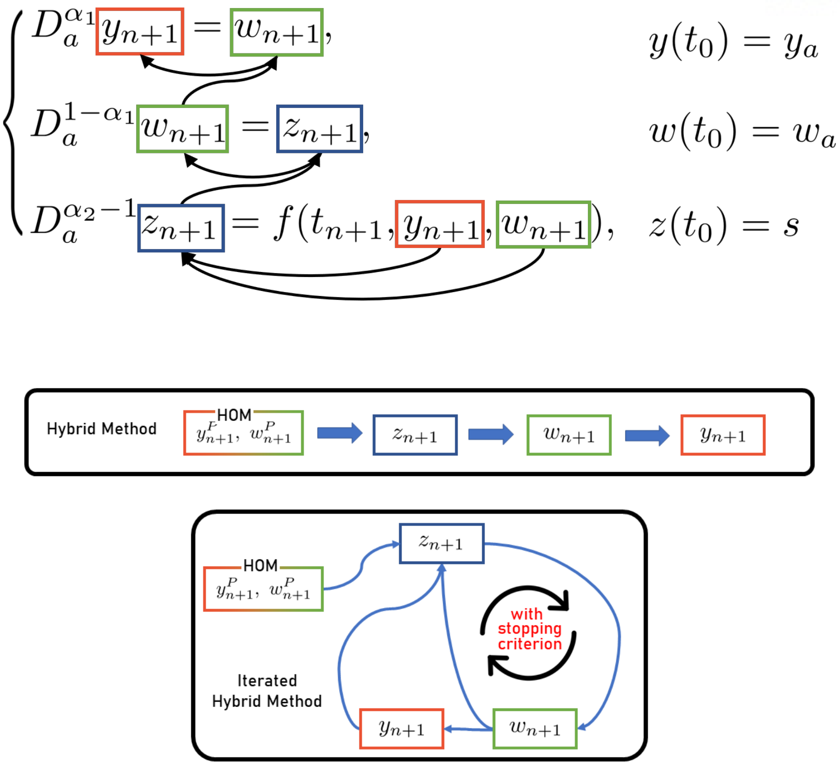



Figure 2: Structure of HM and IHM

*Where* $\alpha_2 = 1.7$, $\alpha_1 = 0.4$. *exact solution is*

$$y(t) = y^4. \tag{II.47}$$

First, $a_1 = 1,\ b_1 = 0,\ a_2 = 1,\ b_2 = 0.$ That case is called 'Dirichlet Boundary Condition'.

| original | notation |
|---|---|
| PECE method with Newton's method | NPECE |
| PECE method with Halley's method | HPECE |
| SHOM with Newton's method | NHOM |
| THOM with Halley's method | HHOM |
| SHM with Newton's method | NHM |
| THM with Halley's method | HHM |
| SIHMwith Newton's method | NIHM |
| TIHM with Halley's method | HIHM |

Notations

| | NPECE | | | | | HPECE | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.31906 | 0.66534 | 1.03376 | 1.42592 | 1.84355 | 0.31906 | 0.66534 | 1.03376 | 1.42592 | 1.84355 |
| 2 | 0.00921 | 0.037 | 0.08281 | 0.14629 | 0.22727 | 0.03334 | 0.15103 | 0.38041 | 0.75613 | 1.31981 |
| 3 | 9.78E-06 | 0.00014 | 0.00067 | 0.00203 | 0.00478 | 0.00036 | 0.00712 | 0.04389 | 0.16613 | 0.47578 |
| 4 | 1.69E-09 | 2.55E-08 | 1.57E-07 | 7.46E-07 | 3.03E-06 | 1.01E-07 | 1.72E-05 | 0.00061 | 0.00859 | 0.0679 |
| 5 | 2.89E-13 | 4.37E-12 | 2.70E-11 | 1.28E-10 | 5.21E-10 | 1.73E-11 | 3.05E-09 | 2.24E-07 | 2.48E-05 | 0.00146 |
| 6 | 2.22E-16 | 4.44E-16 | 4.44E-15 | 2.22E-14 | 8.97E-14 | 2.89E-15 | 5.22E-13 | 3.85E-11 | 4.45E-09 | 9.23E-07 |
| 7 | 2.22E-16 | 0 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 4.44E-16 | 6.55E-15 | 7.63E-13 | 1.58E-10 |
| 8 | 2.22E-16 | 0 | 0 | 0 | 2.22E-16 | 2.22E-16 | 0 | 2.22E-16 | 2.22E-16 | 2.71E-14 |
| 9 | 2.22E-16 | 0 | 0 | 0 | 2.22E-16 | 2.22E-16 | 0 | 2.22E-16 | 2.22E-16 | 2.22E-16 |
| 10 | 2.22E-16 | 0 | 0 | 0 | 2.22E-16 | 2.22E-16 | 0 | 2.22E-16 | 2.22E-16 | 0 |

Table 1: Approximated error of PECE with Polynomial type example and Dirichlet Boundary condition in example 2.3.1.

| | NPECE | | HPECE | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 0.03929 | - | 0.03929 | - |
| 20 | 0.01501 | 1.38817 | 0.01501 | 1.38817 |
| 40 | 0.00558 | 1.42644 | 0.00558 | 1.42644 |
| 80 | 0.00205 | 1.44705 | 0.00205 | 1.44705 |
| 160 | 7.47E-04 | 1.45535 | 7.47E-04 | 1.45535 |
| 320 | 2.72E-04 | 1.45765 | 2.72E-04 | 1.45765 |

Table 2: Convergence rate of PECE methods with Polynomial type example and Dirichlet Boundary condition in example 2.3.1.

| | NHOM | | | | | HHOM | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.32715 | 0.67498 | 1.04536 | 1.43997 | 1.86061 | 0.32682 | 0.67466 | 1.04505 | 1.43965 | 1.86028 |
| 2 | 0.00891 | 0.03629 | 0.0816 | 0.1445 | 0.22488 | 2.50E-05 | 0.00042 | 0.0018 | 0.00456 | 0.00916 |
| 3 | 9.12E-06 | 5.73E-05 | 0.00047 | 0.00167 | 0.00418 | 8.08E-09 | 1.35E-07 | 5.79E-07 | 1.47E-06 | 2.95E-06 |
| 4 | 1.71E-08 | 1.07E-07 | 8.69E-07 | 2.86E-06 | 6.18E-06 | 2.61E-12 | 4.36E-11 | 1.87E-10 | 4.75E-10 | 9.53E-10 |
| 5 | 3.22E-11 | 2.02E-10 | 1.63E-09 | 5.38E-09 | 1.16E-08 | 8.88E-16 | 1.40E-14 | 6.05E-14 | 1.53E-13 | 3.08E-13 |
| 6 | 6.04E-14 | 3.79E-13 | 3.07E-12 | 1.01E-11 | 2.18E-11 | 0 | 0 | 0 | 2.22E-16 | 4.44E-16 |
| 7 | 1.11E-16 | 6.66E-16 | 5.77E-15 | 1.89E-14 | 4.09E-14 | 0 | 0 | 0 | 2.22E-16 | 2.22E-16 |
| 8 | 1.11E-16 | 1.11E-16 | 1.11E-16 | 1.11E-16 | 1.11E-16 | 0 | 0 | 0 | 2.22E-16 | 0 |
| 9 | 2.22E-16 | 0 | 0 | 2.22E-16 | 2.22E-16 | 0 | 0 | 0 | 2.22E-16 | 0 |
| 10 | 1.11E-16 | 0 | 0 | 1.11E-16 | 1.11E-16 | 0 | 0 | 0 | 2.22E-16 | 0 |

Table 3: Approximated error of HOMs with Polynomial type example and Dirichlet Boundary condition in example 2.3.1.

| | NHOM | | HHOM | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 0.005 | - | 1.07E-03 | - |
| 20 | 0.00143 | 1.80632 | 8.36E-05 | 3.67563 |
| 40 | 0.00037 | 1.96494 | 5.81E-06 | 3.8473 |
| 80 | 9.18E-05 | 1.99605 | 3.84E-07 | 3.92012 |
| 160 | 2.29E-05 | 2.00064 | 2.49E-08 | 3.94409 |
| 320 | 5.74E-06 | 2.0003 | 1.67E-09 | 3.89623 |

Table 4: Convergence rate of HOMs with Polynomial type example and Dirichlet Boundary condition in example 2.3.1.

| | NHM | | | | | HHM | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.32715 | 0.67499 | 1.04537 | 1.43998 | 1.86063 | 0.32682 | 0.67466 | 1.04505 | 1.43965 | 1.86028 |
| 2 | 0.00968 | 0.03818 | 0.08496 | 0.14968 | 0.23225 | 0.00011 | 0.00075 | 0.00241 | 0.00556 | 0.01064 |
| 3 | 9.17E-06 | 1.41E-04 | 0.00069 | 0.00211 | 0.00499 | 8.98E-12 | 1.97E-10 | 1.75E-09 | 9.13E-09 | 3.44E-08 |
| 4 | 3.42E-11 | 2.35E-09 | 4.86E-08 | 4.42E-07 | 2.44E-06 | 2.22E-16 | 2.22E-16 | 0 | 1.11E-15 | 1.55E-15 |
| 5 | 2.22E-16 | 6.66E-15 | 1.38E-13 | 1.27E-12 | 7.50E-12 | 2.22E-16 | 2.22E-16 | 0 | 4.44E-16 | 2.22E-16 |
| 6 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 0 | 2.22E-16 | 2.22E-16 |
| 7 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 0 | 2.22E-16 | 2.22E-16 |
| 8 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 0 | 2.22E-16 | 2.22E-16 |
| 9 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 0 | 2.22E-16 | 2.22E-16 |
| 10 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 2.22E-16 | 0 | 2.22E-16 | 2.22E-16 |

Table 5: Approximated error of HMs with Polynomial type example and Dirichlet Boundary condition in example 2.3.1.

| | NHM | | HHM | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 0.00513 | - | 9.36E-04 | - |
| 20 | 0.00141 | 1.86003 | 7.40E-05 | 3.66082 |
| 40 | 0.00036 | 1.96275 | 5.16E-06 | 3.84305 |
| 80 | 9.14E-05 | 1.98976 | 3.41E-07 | 3.91796 |
| 160 | 2.29E-05 | 1.99673 | 2.20E-08 | 3.95315 |
| 320 | 5.73E-06 | 1.99838 | 1.73E-09 | 3.6722 |

Table 6: Convergence rate of HMs with Polynomial type example and Dirichlet Boundary condition in example 2.3.1.

| | NIHM | | | | | HIHM | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.32718 | 0.67502 | 1.04542 | 1.44005 | 1.86072 | 0.32682 | 0.67466 | 1.04505 | 1.43965 | 1.86028 |
| 2 | 0.00968 | 0.03819 | 0.08497 | 0.1497 | 0.23228 | 0.00011 | 0.00075 | 0.00241 | 0.00555 | 0.01064 |
| 3 | 9.14E-06 | 1.41E-04 | 0.00069 | 0.00211 | 0.00499 | 3.50E-12 | 1.59E-10 | 1.63E-09 | 8.86E-09 | 3.39E-08 |
| 4 | 8.98E-12 | 1.85E-09 | 4.65E-08 | 4.36E-07 | 2.43E-06 | 0 | 0 | 2.22E-16 | 2.22E-16 | 2.22E-16 |
| 5 | 2.22E-16 | 2.22E-16 | 4.66E-15 | 5.71E-14 | 7.91E-13 | 0 | 0 | 2.22E-16 | 2.22E-16 | 2.22E-16 |
| 6 | 0 | 0 | 5.55E-16 | 2.22E-16 | 0 | 0 | 0 | 2.22E-16 | 2.22E-16 | 2.22E-16 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2.22E-16 | 2.22E-16 | 2.22E-16 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2.22E-16 | 2.22E-16 | 2.22E-16 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2.22E-16 | 2.22E-16 | 2.22E-16 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2.22E-16 | 2.22E-16 | 2.22E-16 |

Table 7: Approximated error of IHMs with Polynomial type example and Dirichlet Boundary condition in example 2.3.1.

| | NIHM | | HIHM | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 0.00563 | - | 0.00026 | - |
| 20 | 0.00143 | 1.97608 | 1.89E-05 | 3.76896 |
| 40 | 0.00036 | 1.98682 | 1.30E-06 | 3.86003 |
| 80 | 9.07E-05 | 1.99257 | 9.26E-08 | 3.81318 |
| 160 | 2.28E-05 | 1.99343 | 1.46E-08 | 2.6653 |
| 320 | 5.71E-06 | 1.99582 | 2.10E-09 | 2.79573 |

Table 8: Convergence rate of IHMs with Polynomial type example and Dirichlet Boundary condition in example 2.3.1.

Second, $a_1 = 1,\ b_1 = 1,\ a_2 = 1,\ b_2 = 1.$ That case is called 'Robin Boundary Condition'.

| | NPECE | | | | | HPECE | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.58894 | 1.28888 | 2.08715 | 2.98674 | 3.9931 | 0.58894 | 1.28888 | 2.08715 | 2.98674 | 3.9931 |
| 2 | 0.04035 | 0.14513 | 0.29984 | 0.50086 | 0.74941 | 0.07202 | 0.25004 | 0.50828 | 0.84308 | 1.25752 |
| 3 | 2.65E-04 | 0.00315 | 0.01219 | 0.03049 | 0.06063 | 0.00155 | 0.01613 | 0.05621 | 0.12933 | 0.24089 |
| 4 | 5.07E-08 | 2.08E-06 | 2.59E-05 | 1.53E-04 | 5.86E-04 | 9.81E-07 | 8.32E-05 | 0.00096 | 0.00474 | 0.01507 |
| 5 | 7.51E-12 | 3.09E-10 | 3.95E-09 | 2.66E-08 | 1.43E-07 | 1.46E-10 | 1.45E-08 | 4.30E-07 | 7.75E-06 | 7.29E-05 |
| 6 | 8.88E-16 | 4.62E-14 | 5.84E-13 | 3.93E-12 | 2.12E-11 | 2.13E-14 | 2.15E-12 | 6.37E-11 | 1.17E-09 | 1.25E-08 |
| 7 | 0 | 1.78E-15 | 8.88E-16 | 8.88E-16 | 2.66E-15 | 0 | 0 | 9.77E-15 | 1.74E-13 | 1.85E-12 |
| 8 | 0 | 8.88E-16 | 0 | 8.88E-16 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 9: Approximated error of PECE with Polynomial type example and Robin Boundary condition in example 2.3.1.

| | NPECE | | HPECE | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 0.11617 | - | 0.11617 | - |
| 20 | 0.04685 | 1.31023 | 0.04685 | 1.31023 |
| 40 | 0.0177 | 1.40446 | 0.0177 | 1.40446 |
| 80 | 0.00648 | 1.45003 | 0.00648 | 1.45003 |
| 160 | 2.34E-03 | 1.47055 | 2.34E-03 | 1.47055 |
| 320 | 8.42E-04 | 1.47363 | 8.42E-04 | 1.47363 |

Table 10: Convergence rate of PECE methods with Polynomial type example and Robin Boundary condition in example 2.3.1.

| | NHOM | | | | | HHOM | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.60401 | 1.30554 | 2.10577 | 3.00776 | 4.01705 | 0.60319 | 1.3047 | 2.10484 | 3.00664 | 4.01565 |
| 2 | 0.04223 | 0.14833 | 0.30385 | 0.50516 | 0.75349 | 0.05433 | 0.19029 | 0.3923 | 0.65925 | 0.99521 |
| 3 | 2.74E-04 | 3.25E-03 | 0.01249 | 0.03101 | 0.06134 | 0.0006 | 0.00684 | 0.02572 | 0.06314 | 0.12416 |
| 4 | 7.31E-08 | 7.33E-07 | 2.17E-05 | 1.46E-04 | 5.78E-04 | 9.11E-08 | 8.24E-06 | 0.00013 | 0.00081 | 0.00304 |
| 5 | 2.28E-11 | 2.29E-10 | 6.71E-09 | 4.22E-08 | 1.25E-07 | 2.58E-11 | 2.32E-09 | 3.41E-08 | 8.56E-08 | 1.15E-06 |
| 6 | 7.11E-15 | 7.19E-14 | 2.10E-12 | 1.32E-11 | 3.91E-11 | 6.22E-15 | 6.55E-13 | 9.64E-12 | 2.42E-11 | 3.26E-10 |
| 7 | 0 | 0 | 8.88E-16 | 4.44E-15 | 1.07E-14 | 8.88E-16 | 0 | 2.66E-15 | 6.22E-15 | 9.15E-14 |
| 8 | 0 | 0 | 0 | 0 | 1.78E-15 | 8.88E-16 | 0 | 8.88E-16 | 0 | 8.88E-16 |
| 9 | 0 | 0 | 0 | 0 | 0 | 8.88E-16 | 0 | 8.88E-16 | 0 | 8.88E-16 |
| 10 | 0 | 0 | 0 | 0 | 0 | 8.88E-16 | 0 | 8.88E-16 | 0 | 8.88E-16 |

Table 11: Approximated error of HOMs with Polynomial type example and Robin Boundary condition in example 2.3.1.

| | NHOM | | HHOM | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 0.01173 | - | 1.02E-03 | - |
| 20 | 0.00563 | 1.05861 | 1.83E-04 | 2.48635 |
| 40 | 0.00178 | 1.66467 | 3.63E-05 | 2.33187 |
| 80 | 4.92E-04 | 1.85351 | 5.47E-06 | 2.73014 |
| 160 | 1.29E-04 | 1.93106 | 7.51E-07 | 2.8661 |
| 320 | 3.30E-05 | 1.96608 | 9.88E-08 | 2.92502 |

Table 12: Convergence rate of HOMs with Polynomial type example and Robin Boundary condition in example 2.3.1.

| | NHM | | | | | HHM | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.60402 | 1.30555 | 2.10578 | 3.00777 | 4.01705 | 0.6032 | 1.30473 | 2.10489 | 3.00674 | 4.0158 |
| 2 | 0.0423 | 0.14851 | 0.30441 | 0.50657 | 0.75634 | 0.05452 | 0.19077 | 0.39317 | 0.66064 | 0.99726 |
| 3 | 2.87E-04 | 3.29E-03 | 0.01258 | 0.03123 | 0.06181 | 0.00062 | 0.00693 | 0.02594 | 0.06358 | 0.12492 |
| 4 | 1.39E-08 | 1.79E-06 | 2.59E-05 | 1.58E-04 | 6.04E-04 | 8.50E-08 | 1.04E-05 | 0.00014 | 0.00084 | 0.00311 |
| 5 | 1.60E-14 | 2.55E-12 | 1.40E-10 | 4.27E-09 | 6.09E-08 | 0 | 2.38E-11 | 4.55E-09 | 1.55E-07 | 2.11E-06 |
| 6 | 8.88E-16 | 0 | 0 | 6.22E-15 | 6.93E-14 | 0 | 8.88E-16 | 0 | 2.66E-15 | 9.57E-13 |
| 7 | 0 | 0 | 0 | 8.88E-16 | 8.88E-16 | 0 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 13: Approximated error of HMs with Polynomial type example and Robin Boundary condition in example 2.3.1.

| | NHM | | HHM | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 0.01695 | - | 6.52E-04 | - |
| 20 | 0.00626 | 1.43727 | 2.07E-04 | 1.6579 |
| 40 | 0.00185 | 1.76021 | 3.78E-05 | 2.4509 |
| 80 | 4.99E-04 | 1.88733 | 5.56E-06 | 2.76481 |
| 160 | 1.30E-04 | 1.94437 | 7.55E-07 | 2.87921 |
| 320 | 3.31E-05 | 1.97151 | 9.91E-08 | 2.93034 |

Table 14: Convergence rate of HMs with Polynomial type example and Robin Boundary condition in example 2.3.1.

| | NIHM | | | | | TIHM | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.60407 | 1.30561 | 2.10585 | 3.00786 | 4.01718 | 0.60319 | 1.30472 | 2.10488 | 3.00673 | 4.01578 |
| 2 | 0.04231 | 0.14852 | 0.30443 | 0.50659 | 0.75637 | 0.05452 | 0.19077 | 0.39317 | 0.66064 | 0.99725 |
| 3 | 2.87E-04 | 3.29E-03 | 0.01259 | 0.03123 | 0.06181 | 0.00062 | 0.00693 | 0.02594 | 0.06358 | 0.12492 |
| 4 | 1.32E-08 | 1.79E-06 | 2.59E-05 | 1.58E-04 | 6.04E-04 | 8.50E-08 | 1.04E-05 | 0.00014 | 0.00084 | 0.00311 |
| 5 | 0 | 5.52E-13 | 1.11E-10 | 3.74E-09 | 5.96E-08 | 2.66E-15 | 2.39E-11 | 4.55E-09 | 1.55E-07 | 2.11E-06 |
| 6 | 0 | 0 | 8.88E-16 | 0 | 1.78E-15 | 0 | 1.78E-15 | 8.88E-16 | 4.44E-15 | 9.75E-13 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8.88E-16 | 0 | 8.88E-16 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8.88E-16 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8.88E-16 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8.88E-16 | 0 | 0 |

Table 15: Approximated error of IHMs with Polynomial type example and Robin Boundary condition in example 2.3.1.

| | NIHM | | TIHM | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 0.03342 | - | 2.26E-03 | - |
| 20 | 0.00834 | 2.00179 | 3.30E-04 | 2.77993 |
| 40 | 0.0021 | 1.98766 | 4.52E-05 | 2.86784 |
| 80 | 5.30E-04 | 1.98821 | 6.00E-06 | 2.91218 |
| 160 | 1.33E-04 | 1.99021 | 7.82E-07 | 2.94025 |
| 320 | 3.35E-05 | 1.99305 | 1.01E-07 | 2.95705 |

Table 16: Convergence rate of IHMs with Polynomial type example and Robin Boundary condition in example 2.3.1.

**Example 2.3.2.** *Exponential type example.*

$$\begin{cases} D_0^{\alpha_2}y(t) = F(t) + y^2 + tD_0^{\alpha_1}y(t), \\ a_1y(0) + b_1y'(0) = \gamma_1,\ a_2y(1) + b_2y'(1) = \gamma_2. \end{cases} \tag{II.48}$$

*whose* $\alpha_2 = 1.7,\ \alpha_1 = 0.4,$

$$\begin{aligned} F(t) = {} & t^{2-\alpha_2}E_{1,3-\alpha_2}(t) - \left(\frac{\Gamma(3)}{2\Gamma(3-\alpha_2)}t^{2-\alpha_2} + \frac{\Gamma(4)}{6\Gamma(4-\alpha_2)}t^{3-\alpha_2}\right) \\ & - \left\{e^t - \left(1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3\right)\right\}^2 \\ & - t\left\{t^{1-\alpha_1}E_{1,2-\alpha_1}(t) - \left(\frac{\Gamma(2)}{\Gamma(2-\alpha_1)}t^{1-\alpha_1} + \frac{\Gamma(3)}{2\Gamma(3-\alpha_1)}t^{2-\alpha_1} + \frac{\Gamma(4)}{6\Gamma(4-\alpha_1)}t^{3-\alpha_1}\right)\right\} \end{aligned}$$

*and exact solution is*

$$y(t) = e^t - \left(1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3\right). \tag{II.49}$$

First, $a_1 = 1,\ b_1 = 0,\ a_2 = 1,\ b_2 = 0.$ That case is called 'Dirichlet Boundary Condition'.

| | NPECE | | | | | HPECE | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.24714 | 0.50854 | 0.78474 | 1.07665 | 1.38528 | 0.24714 | 0.50854 | 0.78474 | 1.07665 | 1.38528 |
| 2 | 0.00646 | 0.02556 | 0.0569 | 0.10022 | 0.15538 | 0.02653 | 0.11799 | 0.29617 | 0.5898 | 1.03555 |
| 3 | 4.78E-06 | 7.37E-05 | 0.00036 | 0.00111 | 0.00262 | 0.00029 | 0.00566 | 0.03457 | 0.13035 | 0.37355 |
| 4 | 4.57E-11 | 1.28E-09 | 1.81E-08 | 1.49E-07 | 8.01E-07 | 3.78E-08 | 1.33E-05 | 0.00049 | 0.0069 | 0.05426 |
| 5 | 4.02E-16 | 1.16E-14 | 1.63E-13 | 1.35E-12 | 7.30E-12 | 3.42E-13 | 1.94E-10 | 1.06E-07 | 1.98E-05 | 0.00121 |
| 6 | 1.39E-17 | 0 | 1.39E-17 | 2.78E-17 | 6.25E-17 | 0 | 1.75E-15 | 9.57E-13 | 3.41E-10 | 6.20E-07 |
| 7 | 0 | 0 | 0 | 0 | 1.39E-17 | 0 | 0 | 0 | 3.07E-15 | 5.75E-12 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4.16E-17 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 17: Approximated error of PECE with Exponential type example and Dirichlet Boundary condition.

| | NPECE | | HPECE | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 0.00253 | - | 0.00253 | - |
| 20 | 0.00098 | 1.36571 | 0.00098 | 1.36571 |
| 40 | 0.00037 | 1.4066 | 0.00037 | 1.4066 |
| 80 | 0.00014 | 1.42948 | 0.00014 | 1.42948 |
| 160 | 5.07E-05 | 1.43951 | 5.07E-05 | 1.43951 |
| 320 | 1.86E-05 | 1.44369 | 1.86E-05 | 1.44369 |

Table 18: Convergence rate of PECE methods with Exponential type example and Dirichlet Boundary condition

| | NHOM | | | | | HHOM | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.24779 | 0.50964 | 0.78647 | 1.07923 | 1.38896 | 0.24777 | 0.50962 | 0.78645 | 1.0792 | 1.38892 |
| 2 | 0.00639 | 0.02528 | 0.05632 | 0.09925 | 0.15394 | 4.51E-05 | 0.00038 | 0.0013 | 0.00308 | 0.006 |
| 3 | 3.92E-06 | 6.85E-05 | 0.00034 | 0.00106 | 0.00252 | 7.63E-10 | 6.49E-09 | 2.20E-08 | 5.21E-08 | 1.01E-07 |
| 4 | 3.88E-10 | 6.28E-09 | 2.09E-08 | 2.07E-08 | 4.62E-07 | 1.29E-14 | 1.10E-13 | 3.73E-13 | 8.82E-13 | 1.71E-12 |
| 5 | 3.85E-14 | 6.24E-13 | 2.08E-12 | 2.06E-12 | 4.58E-11 | 6.94E-18 | 6.94E-18 | 6.94E-18 | 1.39E-17 | 1.39E-17 |
| 6 | 6.94E-18 | 6.94E-17 | 2.08E-16 | 2.15E-16 | 4.55E-15 | 0 | 0 | 0 | 6.94E-18 | 6.94E-18 |
| 7 | 6.94E-18 | 0 | 0 | 6.94E-18 | 6.94E-18 | 0 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 6.94E-18 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 19: Approximated error of HOMs with Exponential type example and Dirichlet Boundary condition.

| | NHOM | | HHOM | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 0.00029 | - | 5.00E-05 | - |
| 20 | 9.66E-05 | 1.59343 | 4.64E-06 | 3.43146 |
| 40 | 2.67E-05 | 1.85252 | 4.52E-07 | 3.35999 |
| 80 | 6.99E-06 | 1.9361 | 4.92E-08 | 3.19831 |
| 160 | 1.78E-06 | 1.96891 | 5.70E-09 | 3.11099 |
| 320 | 4.51E-07 | 1.98378 | 6.82E-10 | 3.06297 |

Table 20: Convergence rate of HOMs with Exponential type example and Dirichlet Boundary condition.

| | NHM | | | | | HHM | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.24779 | 0.50964 | 0.78647 | 1.07923 | 1.38896 | 0.24777 | 0.50962 | 0.78645 | 1.0792 | 1.38892 |
| 2 | 0.00653 | 0.02582 | 0.05747 | 0.10126 | 0.15704 | 6.47E-05 | 0.00046 | 0.00147 | 0.00339 | 0.0065 |
| 3 | 4.86E-06 | 7.55E-05 | 0.00037 | 0.00114 | 0.00269 | 1.27E-12 | 5.42E-11 | 5.57E-10 | 3.03E-09 | 1.16E-08 |
| 4 | 3.48E-12 | 6.62E-10 | 1.57E-08 | 1.47E-07 | 8.26E-07 | 6.94E-18 | 6.94E-18 | 0 | 0 | 4.16E-17 |
| 5 | 0 | 9.71E-17 | 2.58E-15 | 2.62E-14 | 2.11E-13 | 0 | 0 | 0 | 0 | 0 |
| 6 | 0 | 0 | 1.39E-17 | 1.39E-17 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 0 | 0 | 1.39E-17 | 1.39E-17 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | 1.39E-17 | 1.39E-17 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 1.39E-17 | 1.39E-17 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 1.39E-17 | 1.39E-17 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 21: Approximated error of HMs with Exponential type example and Dirichlet Boundary condition.

| | NHM | | HHM | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 0.00032 | - | 4.95E-05 | - |
| 20 | 9.97E-05 | 1.7012 | 4.31E-06 | 3.5222 |
| 40 | 2.70E-05 | 1.88283 | 4.36E-07 | 3.30421 |
| 80 | 7.02E-06 | 1.94631 | 4.84E-08 | 3.1713 |
| 160 | 1.79E-06 | 1.97271 | 5.66E-09 | 3.0981 |
| 320 | 4.52E-07 | 1.98524 | 6.79E-10 | 3.05715 |

Table 22: Convergence rate of HMs with Exponential type example and Dirichlet Boundary condition.

| | NIHM | | | | | HIHM | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.24779 | 0.50965 | 0.78648 | 1.07924 | 1.38898 | 0.24777 | 0.50962 | 0.78645 | 1.0792 | 1.38892 |
| 2 | 0.00653 | 0.02582 | 0.05748 | 0.10127 | 0.15705 | 6.47E-05 | 0.00046 | 0.00147 | 0.00339 | 0.0065 |
| 3 | 4.86E-06 | 7.55E-05 | 0.00037 | 0.00114 | 0.00269 | 1.06E-12 | 5.27E-11 | 5.52E-10 | 3.01E-09 | 1.15E-08 |
| 4 | 2.64E-12 | 6.50E-10 | 1.57E-08 | 1.47E-07 | 8.25E-07 | 0 | 0 | 0 | 1.39E-17 | 6.94E-18 |
| 5 | 1.39E-17 | 0 | 1.11E-16 | 3.23E-15 | 8.20E-14 | 0 | 0 | 0 | 0 | 0 |
| 6 | 1.39E-17 | 0 | 1.39E-17 | 0 | 1.39E-17 | 0 | 0 | 0 | 0 | 0 |
| 7 | 1.39E-17 | 0 | 1.39E-17 | 0 | 2.78E-17 | 0 | 0 | 0 | 0 | 0 |
| 8 | 1.39E-17 | 0 | 1.39E-17 | 0 | 1.39E-17 | 0 | 0 | 0 | 0 | 0 |
| 9 | 1.39E-17 | 0 | 1.39E-17 | 0 | 1.39E-17 | 0 | 0 | 0 | 0 | 0 |
| 10 | 1.39E-17 | 0 | 1.39E-17 | 0 | 2.78E-17 | 0 | 0 | 0 | 0 | 0 |

Table 23: Approximated error of IHMs with Exponential type example and Dirichlet Boundary condition.

| | NIHM | | HIHM | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 0.00043 | - | 2.75E-05 | - |
| 20 | 1.12E-04 | 1.95258 | 3.09E-06 | 3.15453 |
| 40 | 2.84E-05 | 1.97845 | 3.64E-07 | 3.08536 |
| 80 | 7.17E-06 | 1.98577 | 4.40E-08 | 3.04977 |
| 160 | 1.81E-06 | 1.98968 | 5.38E-09 | 3.03233 |
| 320 | 4.54E-07 | 1.99294 | 6.62E-10 | 3.02209 |

Table 24: Convergence rate of IHMs with Exponential type example and Dirichlet Boundary condition.

Second, $a_1 = 1,\ b_1 = 1,\ a_2 = 1,\ b_2 = 1.$ That case is called 'Robin Boundary Condition'.

| | NPECE | | | | | HPECE | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.41336 | 0.88951 | 1.42505 | 2.01898 | 2.67196 | 0.41336 | 0.88951 | 1.42505 | 2.01898 | 2.67196 |
| 2 | 0.02735 | 0.09401 | 0.18881 | 0.30849 | 0.45321 | 0.03893 | 0.1323 | 0.26772 | 0.44499 | 0.66775 |
| 3 | 0.00017 | 0.00186 | 0.00686 | 0.01653 | 0.03194 | 0.0005 | 0.00518 | 0.01851 | 0.044 | 0.08479 |
| 4 | 7.27E-09 | 8.10E-07 | 1.09E-05 | 6.24E-05 | 0.00023 | 8.83E-08 | 9.25E-06 | 0.00012 | 0.00063 | 0.00224 |
| 5 | 2.83E-14 | 3.30E-12 | 6.98E-11 | 1.15E-09 | 1.31E-08 | 3.46E-13 | 6.56E-11 | 5.10E-09 | 1.42E-07 | 1.75E-06 |
| 6 | 0 | 0 | 2.78E-16 | 4.44E-15 | 5.08E-14 | 0 | 2.78E-16 | 1.98E-14 | 5.59E-13 | 7.86E-12 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1.11E-16 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5.55E-17 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 25: Approximated error of PECE with Exponential type example and Robin Boundary condition.

| | NPECE | | HPECE | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 0.00674 | - | 0.00674 | - |
| 20 | 0.00269 | 1.32232 | 0.00269 | 1.32232 |
| 40 | 0.00104 | 1.37959 | 0.00104 | 1.37959 |
| 80 | 0.00039 | 1.40863 | 0.00039 | 1.40863 |
| 160 | 1.45E-04 | 1.42363 | 1.45E-04 | 1.42363 |
| 320 | 5.39E-05 | 1.43109 | 5.39E-05 | 1.43109 |

Table 26: Convergence rate of PECE methods with Exponential type example and Robin Boundary condition

| | NHOM | | | | | HHOM | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.41411 | 0.89055 | 1.42639 | 2.02066 | 2.67403 | 0.41405 | 0.89041 | 1.42614 | 2.02026 | 2.67342 |
| 2 | 0.02775 | 0.09509 | 0.19065 | 0.311 | 0.45621 | 0.03142 | 0.10795 | 0.21952 | 0.36568 | 0.54942 |
| 3 | 0.00018 | 0.00193 | 0.00707 | 0.01696 | 0.03265 | 0.00026 | 0.0028 | 0.01039 | 0.02545 | 0.05023 |
| 4 | 1.51E-08 | 9.56E-07 | 1.20E-05 | 6.71E-05 | 0.00024 | 1.61E-08 | 2.10E-06 | 2.89E-05 | 0.00017 | 0.00065 |
| 5 | 6.59E-13 | 4.20E-11 | 5.57E-10 | 3.99E-09 | 2.45E-08 | 1.25E-13 | 1.52E-11 | 1.33E-12 | 6.56E-09 | 1.08E-07 |
| 6 | 0 | 1.89E-15 | 2.44E-14 | 1.75E-13 | 1.07E-12 | 5.55E-17 | 1.67E-16 | 1.11E-16 | 5.11E-14 | 8.40E-13 |
| 7 | 0 | 0 | 0 | 0 | 1.67E-16 | 5.55E-17 | 5.55E-17 | 5.55E-17 | 5.55E-17 | 5.55E-17 |
| 8 | 0 | 0 | 0 | 0 | 5.55E-17 | 5.55E-17 | 5.55E-17 | 5.55E-17 | 5.55E-17 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 5.55E-17 | 5.55E-17 | 5.55E-17 | 5.55E-17 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 5.55E-17 | 5.55E-17 | 5.55E-17 | 5.55E-17 | 0 |

Table 27: Approximated error of HOMs with Exponential type example and Robin Boundary condition.

| | NHOM | | HHOM | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 0.0011 | - | 5.65E-05 | - |
| 20 | 4.05E-04 | 1.43433 | 1.71E-05 | 1.72533 |
| 40 | 1.19E-04 | 1.76458 | 2.82E-06 | 2.59886 |
| 80 | 3.21E-05 | 1.89256 | 3.99E-07 | 2.82439 |
| 160 | 8.32E-06 | 1.94851 | 5.29E-08 | 2.91251 |
| 320 | 2.12E-06 | 1.9743 | 6.84E-09 | 2.95208 |

Table 28: Convergence rate of HOMs with Exponential type example and Robin Boundary condition.

| | NHM | | | | | HHM | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.41411 | 0.89055 | 1.42639 | 2.02066 | 2.67403 | 0.41405 | 0.89043 | 1.42617 | 2.02031 | 2.6735 |
| 2 | 0.02742 | 0.09412 | 0.18898 | 0.30872 | 0.45353 | 0.03143 | 0.108 | 0.21966 | 0.36596 | 0.5499 |
| 3 | 1.69E-04 | 1.86E-03 | 0.00686 | 0.01654 | 0.03195 | 0.00026 | 0.00281 | 0.01041 | 0.02549 | 0.05031 |
| 4 | 6.66E-09 | 8.04E-07 | 1.09E-05 | 6.23E-05 | 2.29E-04 | 1.81E-08 | 2.13E-06 | 2.90E-05 | 0.00017 | 0.00065 |
| 5 | 1.67E-16 | 1.72E-13 | 2.76E-11 | 9.03E-10 | 1.22E-08 | 5.55E-17 | 1.23E-12 | 2.29E-10 | 7.96E-09 | 1.14E-07 |
| 6 | 0 | 5.55E-17 | 0 | 5.55E-17 | 3.33E-16 | 5.55E-17 | 5.55E-17 | 5.55E-17 | 5.55E-17 | 3.50E-15 |
| 7 | 0 | 0 | 0 | 0 | 0 | 5.55E-17 | 5.55E-17 | 0 | 0 | 5.55E-17 |
| 8 | 0 | 0 | 0 | 0 | 0 | 5.55E-17 | 5.55E-17 | 0 | 0 | 5.55E-17 |
| 9 | 0 | 0 | 0 | 0 | 0 | 5.55E-17 | 5.55E-17 | 0 | 0 | 5.55E-17 |
| 10 | 0 | 0 | 0 | 0 | 0 | 5.55E-17 | 5.55E-17 | 0 | 0 | 5.55E-17 |

Table 29: Approximated error of HMs with Exponential type example and Robin Boundary condition.

| | NHM | | HHM | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 0.00135 | - | 8.43E-05 | - |
| 20 | 4.38E-04 | 1.62581 | 1.88E-05 | 2.16215 |
| 40 | 1.23E-04 | 1.83097 | 2.92E-06 | 2.68841 |
| 80 | 3.25E-05 | 1.9189 | 4.04E-07 | 2.85479 |
| 160 | 8.37E-06 | 1.95937 | 5.32E-08 | 2.92421 |
| 320 | 2.12E-06 | 1.97881 | 6.85E-09 | 2.95677 |

Table 30: Convergence rate of HMs with Exponential type example and Robin Boundary condition.

| | NIHM | | | | | HIHM | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.41412 | 0.89055 | 1.4264 | 2.02067 | 2.67405 | 0.41405 | 0.89043 | 1.42617 | 2.02031 | 2.67349 |
| 2 | 0.02742 | 0.09412 | 0.18898 | 0.30872 | 0.45353 | 0.03143 | 0.108 | 0.21966 | 0.36596 | 0.54989 |
| 3 | 0.00017 | 0.00186 | 0.00686 | 0.01654 | 0.03196 | 0.00026 | 0.00281 | 0.01041 | 0.02549 | 0.05031 |
| 4 | 6.66E-09 | 8.04E-07 | 1.09E-05 | 6.23E-05 | 0.00023 | 1.81E-08 | 2.13E-06 | 2.90E-05 | 0.00017 | 0.00065 |
| 5 | 5.55E-17 | 1.39E-15 | 2.72E-11 | 9.01E-10 | 1.22E-08 | 1.67E-16 | 1.23E-12 | 2.29E-10 | 7.96E-09 | 1.14E-07 |
| 6 | 5.55E-17 | 5.55E-17 | 1.11E-16 | 5.55E-17 | 1.11E-16 | 1.11E-16 | 1.11E-16 | 1.11E-16 | 1.11E-16 | 3.55E-15 |
| 7 | 5.55E-17 | 5.55E-17 | 0 | 5.55E-17 | 5.55E-17 | 1.11E-16 | 1.11E-16 | 1.67E-16 | 1.11E-16 | 1.11E-16 |
| 8 | 5.55E-17 | 5.55E-17 | 0 | 5.55E-17 | 5.55E-17 | 1.11E-16 | 1.11E-16 | 1.11E-16 | 1.11E-16 | 1.67E-16 |
| 9 | 5.55E-17 | 5.55E-17 | 0 | 5.55E-17 | 5.55E-17 | 1.11E-16 | 1.11E-16 | 1.11E-16 | 1.11E-16 | 1.11E-16 |
| 10 | 5.55E-17 | 5.55E-17 | 0 | 5.55E-17 | 5.55E-17 | 1.11E-16 | 1.11E-16 | 1.11E-16 | 1.11E-16 | 1.11E-16 |

Table 31: Approximated error of IHMs with Exponential type example and Robin Boundary condition.

| | NIHM | | HIHM | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 2.07E-03 | - | 0.00018 | - |
| 20 | 5.30E-04 | 1.96918 | 2.49E-05 | 2.85921 |
| 40 | 1.34E-04 | 1.97928 | 3.29E-06 | 2.92012 |
| 80 | 3.39E-05 | 1.9861 | 4.26E-07 | 2.94907 |
| 160 | 8.53E-06 | 1.99039 | 5.46E-08 | 2.96587 |
| 320 | 2.14E-06 | 1.99343 | 6.94E-09 | 2.97582 |

Table 32: Convergence rate of IHMs with Exponential type example and Robin Boundary condition.

**Example 2.3.3.** *Sin type example*

$$\begin{cases} D_0^{\alpha_2} y(t) = F(t) - y(t)^2 + D_0^{\alpha_1} y(t), \\ a_1 y(0) + b_1 y'(0) = \gamma_1,\ a_2 y(1) + b_2 y'(1) = \gamma_2. \end{cases} \tag{II.50}$$

*whose* $\alpha_2 = 1.7,\ \alpha_1 = 0.4,$

$$\begin{aligned} F(t) = & -\frac{1}{2} i(i)^{\lceil \alpha_2 \rceil} t^{(\lceil \alpha_2 \rceil - \alpha_2)} (E_{\lceil \alpha_2 \rceil - \alpha_2 + 1}(it) - (-1)^{\lceil \alpha_2 \rceil} E_{1, \lceil \alpha_2 \rceil - \alpha_2 + 1}(-it)) \\ & + \frac{\Gamma(4)}{6\Gamma(4-\alpha_2)} t^{3-\alpha_2} + \left( \sin(t) - t + \frac{t^3}{6} \right)^2 \\ & + \frac{1}{2} i(i)^{\lceil \alpha_1 \rceil} t^{(\lceil \alpha_1 \rceil - \alpha_1)} (E_{\lceil \alpha_1 \rceil - \alpha_1 + 1}(it) - (-1)^{\lceil \alpha_1 \rceil} E_{1, \lceil \alpha_1 \rceil - \alpha_1 + 1}(-it)) \\ & + \frac{\Gamma(2)}{\Gamma(2-\alpha_1)} t^{1-\alpha_1} - \frac{\Gamma(4)}{6\Gamma(4-\alpha_1)} t^{3-\alpha_1}. \end{aligned}$$

*and exact solution is*

$$y(t) = \sin(t) - t + \frac{t^3}{6}. \tag{II.51}$$

First, $a_1 = 1,\ b_1 = 0,\ a_2 = 1,\ b_2 = 0$. That case is called 'Dirichlet Boundary Condition'.

| | NPECE | | | | | HPECE | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.2823 | 0.54788 | 0.79772 | 1.03281 | 1.25406 | 0.2823 | 0.54788 | 0.79772 | 1.03281 | 1.25406 |
| 2 | 0.02554 | 0.09517 | 0.19987 | 0.33159 | 0.48335 | 0.01042 | 0.04038 | 0.08616 | 0.14272 | 0.20371 |
| 3 | 2.34E-04 | 2.97E-03 | 0.01288 | 0.03515 | 0.07425 | 2.53E-05 | 0.00026 | 0.00106 | 0.00278 | 0.00554 |
| 4 | 2.67E-07 | 5.98E-06 | 6.64E-05 | 4.30E-04 | 1.83E-03 | 2.71E-08 | 2.82E-07 | 1.27E-06 | 3.99E-06 | 9.94E-06 |
| 5 | 2.85E-10 | 6.39E-09 | 7.22E-08 | 5.17E-07 | 3.01E-06 | 2.89E-11 | 3.01E-10 | 1.36E-09 | 4.26E-09 | 1.06E-08 |
| 6 | 3.04E-13 | 6.82E-12 | 7.70E-11 | 5.52E-10 | 3.22E-09 | 3.09E-14 | 3.21E-13 | 1.45E-12 | 4.54E-12 | 1.13E-11 |
| 7 | 3.23E-16 | 7.28E-15 | 8.22E-14 | 5.89E-13 | 3.43E-12 | 3.12E-17 | 3.43E-16 | 1.55E-15 | 4.85E-15 | 1.21E-14 |
| 8 | 1.73E-18 | 6.94E-18 | 8.67E-17 | 6.28E-16 | 3.66E-15 | 0 | 1.73E-18 | 1.73E-18 | 5.20E-18 | 1.21E-17 |
| 9 | 0 | 0 | 1.73E-18 | 1.73E-18 | 3.47E-18 | 0 | 0 | 0 | 1.73E-18 | 1.73E-18 |
| 10 | 0 | 0 | 1.73E-18 | 1.73E-18 | 1.73E-18 | 0 | 0 | 0 | 1.73E-18 | 1.73E-18 |

Table 33: Approximated error of PECE with Sin type example and Dirichlet Boundary condition.

| | NPECE | | HPECE | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 0.00054 | - | 0.00054 | - |
| 20 | 0.00022 | 1.3193 | 0.00022 | 1.3193 |
| 40 | 8.31E-05 | 1.38846 | 8.31E-05 | 1.38846 |
| 80 | 3.11E-05 | 1.42029 | 3.11E-05 | 1.42029 |
| 160 | 1.15E-05 | 1.43462 | 1.15E-05 | 1.43462 |
| 320 | 4.23E-06 | 1.4411 | 4.23E-06 | 1.4411 |

Table 34: Convergence rate of PECE methods with Sin type example and Dirichlet Boundary condition

| | NHOM | | | | | HHOM | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.2826 | 0.54824 | 0.79801 | 1.03294 | 1.25395 | 2.83E-01 | 5.48E-01 | 7.98E-01 | 1.03E+00 | 1.25E+00 |
| 2 | 0.02564 | 0.09552 | 0.20056 | 0.33264 | 0.48474 | 1.80E-02 | 6.96E-02 | 1.51E-01 | 2.58E-01 | 3.87E-01 |
| 3 | 0.00024 | 0.003 | 0.01299 | 0.03545 | 0.07484 | 8.90E-05 | 1.12E-03 | 5.14E-03 | 1.51E-02 | 3.42E-02 |
| 4 | 2.67E-07 | 6.03E-06 | 6.73E-05 | 0.00044 | 0.00186 | 9.67E-08 | 1.46E-06 | 1.11E-05 | 6.46E-05 | 2.87E-04 |
| 5 | 2.82E-10 | 6.39E-09 | 7.26E-08 | 5.22E-07 | 3.06E-06 | 1.03E-10 | 1.56E-09 | 1.19E-08 | 6.99E-08 | 3.24E-07 |
| 6 | 2.98E-13 | 6.75E-12 | 7.67E-11 | 5.52E-10 | 3.24E-09 | 1.10E-13 | 1.67E-12 | 1.27E-11 | 7.47E-11 | 3.46E-10 |
| 7 | 3.16E-16 | 7.13E-15 | 8.11E-14 | 5.83E-13 | 3.42E-12 | 1.16E-16 | 1.78E-15 | 1.36E-14 | 7.98E-14 | 3.69E-13 |
| 8 | 0 | 6.94E-18 | 8.50E-17 | 6.18E-16 | 3.62E-15 | 1.73E-18 | 1.73E-18 | 1.56E-17 | 8.50E-17 | 3.97E-16 |
| 9 | 0 | 0 | 0 | 0 | 3.47E-18 | 1.73E-18 | 0 | 1.73E-18 | 1.73E-18 | 1.73E-18 |
| 10 | 0 | 0 | 0 | 0 | 0 | 1.73E-18 | 0 | 1.73E-18 | 1.73E-18 | 1.73E-18 |

Table 35: Approximated error of HOMs with Sin type example and Dirichlet Boundary condition.

| | NHOM | | HHOM | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 5.42E-05 | - | 1.14E-05 | - |
| 20 | 2.28E-05 | 1.25086 | 1.58E-06 | 2.84513 |
| 40 | 6.84E-06 | 1.73458 | 1.90E-07 | 3.05613 |
| 80 | 1.85E-06 | 1.88864 | 2.26E-08 | 3.07058 |
| 160 | 4.79E-07 | 1.94779 | 2.74E-09 | 3.04644 |
| 320 | 1.22E-07 | 1.97352 | 3.36E-10 | 3.02582 |

Table 36: Convergence rate of HOMs with Sin type example and Dirichlet Boundary condition.

| | NHM | | | | | HHM | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.2826 | 0.54824 | 0.79801 | 1.03294 | 1.25395 | 0.28264 | 0.54832 | 0.79813 | 1.03309 | 1.25413 |
| 2 | 0.02581 | 0.09608 | 0.20163 | 0.33428 | 0.48692 | 0.01807 | 0.0697 | 0.15129 | 0.25868 | 0.38745 |
| 3 | 0.00024 | 0.00306 | 0.01321 | 0.03602 | 0.07596 | 8.93E-05 | 0.00112 | 0.00516 | 0.01511 | 0.03426 |
| 4 | 2.76E-07 | 6.26E-06 | 7.00E-05 | 0.00045 | 0.00193 | 9.73E-08 | 1.47E-06 | 1.12E-05 | 6.51E-05 | 0.00029 |
| 5 | 2.96E-10 | 6.72E-09 | 7.66E-08 | 5.52E-07 | 3.25E-06 | 1.04E-10 | 1.58E-09 | 1.20E-08 | 7.06E-08 | 3.27E-07 |
| 6 | 3.17E-13 | 7.20E-12 | 8.20E-11 | 5.91E-10 | 3.49E-09 | 1.12E-13 | 1.69E-12 | 1.29E-11 | 7.56E-11 | 3.50E-10 |
| 7 | 3.42E-16 | 7.71E-15 | 8.79E-14 | 6.33E-13 | 3.74E-12 | 1.20E-16 | 1.81E-15 | 1.38E-14 | 8.10E-14 | 3.75E-13 |
| 8 | 0 | 8.67E-18 | 9.37E-17 | 6.80E-16 | 4.00E-15 | 1.73E-18 | 1.73E-18 | 1.56E-17 | 8.67E-17 | 3.99E-16 |
| 9 | 0 | 0 | 1.73E-18 | 1.73E-18 | 3.47E-18 | 1.73E-18 | 1.73E-18 | 1.73E-18 | 1.73E-18 | 1.73E-18 |
| 10 | 0 | 0 | 1.73E-18 | 0 | 0 | 1.73E-18 | 1.73E-18 | 1.73E-18 | 1.73E-18 | 1.73E-18 |

Table 37: Approximated error of HMs with Sin type example and Dirichlet Boundary condition.

| | NHM | | HHM | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 7.03E-05 | - | 1.17E-05 | - |
| 20 | 2.47E-05 | 1.5099 | 1.55E-06 | 2.92347 |
| 40 | 7.04E-06 | 1.81089 | 1.86E-07 | 3.0556 |
| 80 | 1.87E-06 | 1.91461 | 2.24E-08 | 3.05846 |
| 160 | 4.81E-07 | 1.95744 | 2.72E-09 | 3.03785 |
| 320 | 1.22E-07 | 1.97732 | 3.35E-10 | 3.02124 |

Table 38: Convergence rate of HMs with Sin type example and Dirichlet Boundary condition.

| | NIHM | | | | | HIHM | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.2826 | 0.54824 | 0.79801 | 1.03294 | 1.25395 | 0.28264 | 0.54831 | 0.79813 | 1.03309 | 1.25413 |
| 2 | 0.02581 | 0.09608 | 0.20164 | 0.33429 | 0.48694 | 0.01807 | 0.0697 | 0.15129 | 0.25868 | 0.38745 |
| 3 | 0.00024 | 0.00306 | 0.01321 | 0.03602 | 0.07597 | 8.93E-05 | 0.00112 | 0.00516 | 0.01511 | 0.03425 |
| 4 | 2.77E-07 | 6.26E-06 | 7.00E-05 | 0.00045 | 0.00193 | 9.73E-08 | 1.47E-06 | 1.12E-05 | 6.51E-05 | 0.00029 |
| 5 | 2.96E-10 | 6.72E-09 | 7.66E-08 | 5.52E-07 | 3.25E-06 | 1.04E-10 | 1.58E-09 | 1.20E-08 | 7.06E-08 | 3.27E-07 |
| 6 | 3.17E-13 | 7.20E-12 | 8.21E-11 | 5.91E-10 | 3.49E-09 | 1.12E-13 | 1.69E-12 | 1.29E-11 | 7.56E-11 | 3.50E-10 |
| 7 | 3.40E-16 | 7.72E-15 | 8.79E-14 | 6.33E-13 | 3.74E-12 | 1.21E-16 | 1.81E-15 | 1.38E-14 | 8.10E-14 | 3.75E-13 |
| 8 | 0 | 1.04E-17 | 9.37E-17 | 6.80E-16 | 4.01E-15 | 1.73E-18 | 1.73E-18 | 1.56E-17 | 8.67E-17 | 4.02E-16 |
| 9 | 0 | 3.47E-18 | 1.73E-18 | 1.73E-18 | 5.20E-18 | 0 | 0 | 1.73E-18 | 1.73E-18 | 0 |
| 10 | 0 | 0 | 0 | 0 | 1.73E-18 | 0 | 0 | 1.73E-18 | 0 | 0 |

Table 39: Approximated error of IHMs with Sin type example and Dirichlet Boundary condition.

| | NIHM | | HIHM | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 1.17E-04 | - | 1.13E-05 | - |
| 20 | 3.01E-05 | 1.96038 | 1.37E-06 | 3.03823 |
| 40 | 7.67E-06 | 1.97379 | 1.70E-07 | 3.01491 |
| 80 | 1.94E-06 | 1.98228 | 2.11E-08 | 3.00464 |
| 160 | 4.90E-07 | 1.98723 | 2.64E-09 | 3.00171 |
| 320 | 1.23E-07 | 1.99128 | 3.30E-10 | 3.00015 |

Table 40: Convergence rate of IHMs with Sin type example and Dirichlet Boundary condition.

Second, $a_1 = 1,\ b_1 = 1,\ a_2 = 1,\ b_2 = 1.$ That case is called 'Robin Boundary Condition'.

| | NPECE | | | | | HPECE | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.48704 | 0.89047 | 1.20133 | 1.40555 | 1.4823 | 0.48704 | 0.89047 | 1.20133 | 1.40555 | 1.4823 |
| 2 | 0.10143 | 0.34375 | 0.65281 | 0.97315 | 1.25738 | 0.11394 | 0.37116 | 0.68561 | 1.00193 | 1.27576 |
| 3 | 0.00447 | 0.05055 | 0.18268 | 0.41346 | 0.72588 | 0.00659 | 0.06702 | 0.22204 | 0.4699 | 0.78405 |
| 4 | 1.26E-05 | 0.00113 | 0.01435 | 0.07309 | 0.22639 | 2.81E-05 | 0.00232 | 0.02448 | 0.10625 | 0.28875 |
| 5 | 1.19E-08 | 1.60E-06 | 0.0001 | 0.00234 | 0.02199 | 2.67E-08 | 4.89E-06 | 0.00033 | 0.00574 | 0.04099 |
| 6 | 1.11E-11 | 1.50E-09 | 9.87E-08 | 4.51E-06 | 0.00023 | 2.49E-11 | 4.58E-09 | 3.58E-07 | 2.20E-05 | 0.00089 |
| 7 | 1.04E-14 | 1.40E-12 | 9.23E-11 | 4.22E-09 | 2.33E-07 | 2.33E-14 | 4.28E-12 | 3.35E-10 | 2.08E-08 | 1.22E-06 |
| 8 | 6.94E-18 | 1.31E-15 | 8.63E-14 | 3.95E-12 | 2.18E-10 | 0 | 3.98E-15 | 3.13E-13 | 1.95E-11 | 1.15E-09 |
| 9 | 0 | 0 | 8.33E-17 | 3.71E-15 | 2.04E-13 | 0 | 0 | 3.05E-16 | 1.82E-14 | 1.07E-12 |
| 10 | 0 | 0 | 0 | 0 | 1.94E-16 | 0 | 0 | 1.39E-17 | 1.39E-17 | 1.01E-15 |

Table 41: Approximated error of PECE with Sin type example and Robin Boundary condition.

| | NPECE | | HPECE | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 0.00119 | - | 0.00119 | - |
| 20 | 0.00048 | 1.31165 | 0.00048 | 1.31165 |
| 40 | 1.85E-04 | 1.3724 | 1.85E-04 | 1.3724 |
| 80 | 6.99E-05 | 1.40339 | 6.99E-05 | 1.40339 |
| 160 | 2.61E-05 | 1.41954 | 2.61E-05 | 1.41954 |
| 320 | 9.72E-06 | 1.42774 | 9.72E-06 | 1.42774 |

Table 42: Convergence rate of PECE methods with Sin type example and Robin Boundary condition

| | NHOM | | | | | HHOM | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.48762 | 0.89157 | 1.20299 | 1.40781 | 1.48511 | 0.48772 | 0.89183 | 1.20347 | 1.40858 | 1.48624 |
| 2 | 0.10168 | 0.34427 | 0.65343 | 0.97388 | 1.2585 | 0.10626 | 0.35557 | 0.66898 | 0.98986 | 1.27077 |
| 3 | 0.00449 | 0.05073 | 0.18298 | 0.41358 | 0.72559 | 0.00519 | 0.05676 | 0.19931 | 0.44048 | 0.75788 |
| 4 | 1.26E-05 | 0.00114 | 0.01441 | 0.07316 | 0.2261 | 1.69E-05 | 0.00151 | 0.01801 | 0.08681 | 0.25589 |
| 5 | 1.13E-08 | 1.57E-06 | 0.0001 | 0.00234 | 0.02195 | 1.54E-08 | 2.39E-06 | 0.00016 | 0.00348 | 0.02956 |
| 6 | 1.01E-11 | 1.41E-09 | 9.49E-08 | 4.43E-06 | 0.00022 | 1.39E-11 | 2.16E-09 | 1.59E-07 | 8.62E-06 | 0.00042 |
| 7 | 9.06E-15 | 1.26E-12 | 8.48E-11 | 3.97E-09 | 2.22E-07 | 1.25E-14 | 1.95E-12 | 1.44E-10 | 7.82E-09 | 4.62E-07 |
| 8 | 6.94E-18 | 1.12E-15 | 7.58E-14 | 3.55E-12 | 1.99E-10 | 6.94E-18 | 1.76E-15 | 1.30E-13 | 7.06E-12 | 4.17E-10 |
| 9 | 0 | 1.39E-17 | 6.94E-17 | 3.16E-15 | 1.78E-13 | 6.94E-18 | 6.94E-18 | 1.25E-16 | 6.37E-15 | 3.77E-13 |
| 10 | 0 | 6.94E-18 | 6.94E-18 | 6.94E-18 | 1.67E-16 | 6.94E-18 | 0 | 6.94E-18 | 6.94E-18 | 3.33E-16 |

Table 43: Approximated error of HOMs with Sin type example and Robin Boundary condition.

| | NHOM | | HHOM | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 1.73E-04 | - | 1.19E-05 | - |
| 20 | 6.85E-05 | 1.33816 | 3.97E-06 | 1.57784 |
| 40 | 2.07E-05 | 1.72711 | 6.62E-07 | 2.58621 |
| 80 | 5.63E-06 | 1.87683 | 9.28E-08 | 2.83353 |
| 160 | 1.47E-06 | 1.94177 | 1.22E-08 | 2.92648 |
| 320 | 3.74E-07 | 1.97138 | 1.56E-09 | 2.96587 |

Table 44: Convergence rate of HOMs with Sin type example and Robin Boundary condition.

| | NHM | | | | | HHM | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.48762 | 0.89157 | 1.20299 | 1.4078 | 1.48509 | 0.48772 | 0.89182 | 1.20343 | 1.4085 | 1.48612 |
| 2 | 0.10144 | 0.34381 | 0.653 | 0.97366 | 1.25848 | 0.10628 | 0.35563 | 0.66908 | 0.98997 | 1.27085 |
| 3 | 0.00445 | 0.05045 | 0.18238 | 0.4129 | 0.72517 | 0.00519 | 0.05679 | 0.19941 | 0.44067 | 0.75816 |
| 4 | 1.24E-05 | 1.12E-03 | 1.43E-02 | 0.07273 | 0.22543 | 1.69E-05 | 0.00151 | 0.01803 | 0.0869 | 0.25613 |
| 5 | 1.13E-08 | 1.55E-06 | 9.92E-05 | 2.31E-03 | 2.18E-02 | 1.54E-08 | 2.40E-06 | 0.00016 | 0.00349 | 0.02962 |
| 6 | 1.02E-11 | 1.40E-09 | 9.36E-08 | 4.34E-06 | 2.20E-04 | 1.40E-11 | 2.18E-09 | 1.60E-07 | 8.66E-06 | 0.00042 |
| 7 | 9.17E-15 | 1.26E-12 | 8.44E-11 | 3.92E-09 | 2.19E-07 | 1.27E-14 | 1.97E-12 | 1.45E-10 | 7.88E-09 | 4.65E-07 |
| 8 | 0 | 1.14E-15 | 7.61E-14 | 3.54E-12 | 1.98E-10 | 6.94E-18 | 1.78E-15 | 1.32E-13 | 7.15E-12 | 4.22E-10 |
| 9 | 0 | 0 | 6.94E-17 | 3.18E-15 | 1.78E-13 | 1.39E-17 | 6.94E-18 | 1.18E-16 | 6.47E-15 | 3.82E-13 |
| 10 | 0 | 0 | 0 | 0 | 1.60E-16 | 1.39E-17 | 1.39E-17 | 1.39E-17 | 1.39E-17 | 3.33E-16 |

Table 45: Approximated error of HMs with Sin type example and Robin Boundary condition.

| | NHM | | HHM | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 2.33E-04 | - | 2.23E-05 | - |
| 20 | 7.64E-05 | 1.60895 | 4.64E-06 | 2.2632 |
| 40 | 2.16E-05 | 1.82124 | 7.00E-07 | 2.72951 |
| 80 | 5.74E-06 | 1.91417 | 9.49E-08 | 2.883 |
| 160 | 1.48E-06 | 1.95714 | 1.23E-08 | 2.9457 |
| 320 | 3.75E-07 | 1.97776 | 1.57E-09 | 2.97362 |

Table 46: Convergence rate of HMs with Sin type example and Robin Boundary condition.

| | NIHM | | | | | HIHM | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 | s=0.2 | s=0.4 | s=0.6 | s=0.8 | s=1.0 |
| 1 | 0.48762 | 0.89157 | 1.203 | 1.40782 | 1.48512 | 0.48772 | 0.89181 | 1.20343 | 1.40849 | 1.4861 |
| 2 | 0.10144 | 0.34381 | 0.653 | 0.97365 | 1.25848 | 0.10628 | 0.35563 | 0.66908 | 0.98997 | 1.27084 |
| 3 | 0.00445 | 0.05045 | 0.18238 | 0.41289 | 0.72515 | 0.00519 | 0.05679 | 0.19941 | 0.44068 | 0.75816 |
| 4 | 1.24E-05 | 0.00112 | 0.01427 | 0.07272 | 0.22542 | 1.69E-05 | 0.00151 | 0.01804 | 0.0869 | 0.25613 |
| 5 | 1.13E-08 | 1.55E-06 | 9.92E-05 | 0.00231 | 0.02175 | 1.54E-08 | 2.40E-06 | 0.00016 | 0.00349 | 0.02962 |
| 6 | 1.02E-11 | 1.40E-09 | 9.36E-08 | 4.34E-06 | 0.00022 | 1.40E-11 | 2.18E-09 | 1.60E-07 | 8.66E-06 | 0.00042 |
| 7 | 9.15E-15 | 1.26E-12 | 8.44E-11 | 3.92E-09 | 2.19E-07 | 1.27E-14 | 1.97E-12 | 1.45E-10 | 7.88E-09 | 4.65E-07 |
| 8 | 6.94E-18 | 1.14E-15 | 7.61E-14 | 3.53E-12 | 1.96E-10 | 2.78E-17 | 1.78E-15 | 1.32E-13 | 7.15E-12 | 4.22E-10 |
| 9 | 0 | 1.39E-17 | 4.86E-17 | 3.20E-15 | 1.77E-13 | 6.94E-18 | 6.94E-18 | 1.11E-16 | 6.47E-15 | 3.82E-13 |
| 10 | 0 | 1.39E-17 | 6.94E-18 | 1.39E-17 | 1.53E-16 | 0 | 6.94E-18 | 6.94E-18 | 6.94E-18 | 3.61E-16 |

Table 47: Approximated error of IHMs with Sin type example and Robin Boundary condition.

| | NIHM | | HIHM | |
|---|---|---|---|---|
| N | error | roc | error | roc |
| 10 | 3.66E-04 | - | 4.47E-05 | - |
| 20 | 9.35E-05 | 1.96851 | 6.05E-06 | 2.8839 |
| 40 | 2.37E-05 | 1.97858 | 7.85E-07 | 2.94664 |
| 80 | 5.99E-06 | 1.98542 | 1.00E-07 | 2.97273 |
| 160 | 1.51E-06 | 1.98992 | 1.26E-08 | 2.98554 |
| 320 | 3.79E-07 | 1.99312 | 1.59E-09 | 2.99187 |

Table 48: Convergence rate of IHMs with Sin type example and Robin Boundary condition.

PECEs seems to have $p = \min\{2,\, 1-\alpha_2,\, 1-\alpha_1\}$ convergence rate. And convergence rates of HOMs, HMs and AHM are 2 and 3 independently of $\alpha_2$ and $\alpha_1$. Their main methods to solve the system comes from the following papers [14], [1]. Therefore, it can be seen that the convergence rate will be derived as the above result.

Due to structural of main solving scheme, Covnergence rates of HOMs, HMs and IHMs are better then PECE's. And Convergence rate for HOMs, HMs and IHMs are looks similar, but, the approximated error is different. Dirichlet boundary condition examples exactly shows that HMs and IHMs are little better than HOMs. However approximated error of Robin boundary condition examples looks like similar each other.

To begin with, transformed system of FIVP have a special iterate structure. So, this paper suggest the numerical method using that constitution which is called "Iterated Hybrid Method". But, Numerical result show that methods don't have impressive efficiency than "Hybrid Method". Especially, Dirichlet boundary condition cases show us better results about approximated error than that of Robin boundary condition, but those things are not noticeable. And in numerical results of IHMs have slightly stable convergence rate, nevertheless, its error almost same to HMs. Maybe in iterate process, numerical error is accumulated at each time.

# III Fractional Two-Point Boundary Value Problem with 1st order Derivative Term

And also below case can be considered.

$$\begin{cases} D_a^{\alpha_2} y(t) = f(t, y(t), y'(t)) \\ g(y(t), y'(t)) = \gamma_t \big|_{t=a,b}, \end{cases} \tag{III.1}$$

where $1 < \alpha_2 < 2,\ \alpha_2, \in \mathbb{R}$ and g be linear function. Now, let's talk about how to solve above case. Below Figure.3 briefly show outline of solving process. But, there are two questions about

$$\begin{cases} D_0^{\alpha_2} y(t) = f(t, y, y') \\ g(y(t), y'(t)) = \gamma_t|_{t=a,b} \end{cases}$$

$$\Big\Downarrow ?$$

$$\begin{cases} D_0^{\alpha_2} y(t) = f(t, y, D_0^{1-\epsilon} y), \\ g(y(t), y'(t)) = \gamma_t|_{t=a,b} \end{cases}$$

$$\Big\Downarrow$$

$$\begin{cases} D_0^{1-\epsilon} y(t) = w(t), & y(t_0) = y_a \\ D_0^{\epsilon} w(t) = z(t), & w(t_0) = w_a \\ D_0^{\alpha_2 - 1} z(t) = f(t, y, w), & z(t_0) = s \end{cases}$$

$$\Big\Downarrow ?$$

$$\begin{cases} D_0^{1-\epsilon} y(t) = z(t), & y(t_0) = y_a \\ D_0^{\alpha_2 - 1} z(t) = f(t, y, z), & z(t_0) = s \end{cases}$$

Figure 3: Structure of solver for BVP with ODE

above process. First, how to replace 1st derivative term to fractional one? The intuitive answer of the first question is that "almost 1" derivative nearly same as "1" derivative. Second, how to reduce the number of system? The intuitive answer is "almost 0" derivative virtually same "0" derivative that means "no derivative". Thus, below contents will talk about that precisely.

## 3.1 Description of Fractional Boundary Value Problem containing 1st order Derivative

Let's answer first question.

**Lemma 3.1.1.** (First Gronwall inequality for two-term equations [10]).
*Let $\alpha_2 > 0$ and $\alpha_1, \tilde{\alpha}_1 \in (0, \alpha_2)$ be chosen so that the equation*

$$D_0^{\alpha_2} y(t) = f(t, y(t), D_0^{\alpha_1} y(t)), \tag{III.2}$$

*subject to be initial conditions*

$$y(0) = y_0,\ y'(0) = y_0',\ \cdots,\ y^{(\lceil\alpha_2\rceil-1)}(0) = y_0^{(\lceil\alpha_2\rceil-1)}$$

*and*

$$D_0^{\alpha_2} z(t) = f(t, z(t), D_0^{\tilde{\alpha}_1} \tilde{z}(t))$$

*subejct to the same initial condition*

$$z(0) = y_0,\ z'(0) = y_0',\ \cdots,\ z^{(\lceil\alpha_2\rceil-1)}(0) = y_0^{(\lceil\alpha_2\rceil-1)}$$

*(where $f$ satisfies a Lipschitz condition in its second and third arguments on a suitable domain) have unique continuous solutions $y, z : [0, T] \to \mathbb{R}$. We assume further that $[\alpha_1] = [\tilde{\alpha_1}]$. Then there exist constants $K_1$ and $K_2$ such that*

$$|y(t) - z(t)| \leq K_1 |\alpha_1 - \tilde{\alpha}_1| E_{\alpha_n}(K_2 T^{\alpha_2}), \quad \forall t \in [0, T]. \tag{III.3}$$

Thus, above equation (III.1) can be approximated to below equation by Lemma (3.1.1).

$$\begin{cases} D_a^{\alpha_2} y(t) = f(t, y(t), D_a^{1-\epsilon} y(t)) \\ g(y(t), y'(t)) = \gamma_t \big|_{t=a,b}, \end{cases} \tag{III.4}$$

where $1 < \alpha_2 < 2$, $\epsilon \to 0+$ and $g$ be linear function. $\alpha_2$ can be variable order. Then, Two Point boundary value problem can be similarly transformed to system of initial value problem.

$$\begin{cases} D_a^{1-\epsilon} y(t) = w(t), & y(t_0) = y_a \\ D_a^{\epsilon} w(t) = z(t), & w(t_0) = 0 \\ D_a^{\alpha_2 - 1} z(t) = f(t, y(t), w(t)). & z(t_0) = s \end{cases} \tag{III.5}$$

If $\epsilon$ is really small constant then $w$ and $z$ intuitively is almost same. Actually, there are examples (2.3.1),(2.3.2) and (2.3.3) whose data are shown the naive idea is in a measure correct. But, there isn't any guarantee then the idea needs proof.

**Lemma 3.1.2.** *[10] Let $f \in C[a, b]$ and $m \geq 0$. Moreover assume that $\alpha_k$ is a sequence of positive numbers such that $\alpha_k \to \alpha$ as $k \to \infty$. Then, for every $\epsilon > 0$,*

$$\lim_{k \to \infty} \sup_{t \in [a+\epsilon, b]} |J_a^{\alpha_k} f(t) - J_a^{\alpha} f(t)| = 0. \tag{III.6}$$

**Lemma 3.1.3.** *Let,* $0 < \gamma \leq \alpha \leq \beta$ *the,*

$$|J_a^\alpha y(t) + J_a^\beta y(t)| \leq \left[\frac{\Gamma(\gamma)}{\Gamma(\alpha)}T^{\alpha-\gamma} + \frac{\Gamma(\gamma)}{\Gamma(\beta)}T^{\beta-\gamma}\right] J_a^\gamma |y(t)|. \tag{III.7}$$

*Proof.*

$$\begin{aligned} J_a^\alpha y(t) + J_a^\beta y(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} y(\tau) d\tau + \frac{1}{\Gamma(\beta)}\int_a^t (t-\tau)^{\beta-1} y(\tau) d\tau \\ &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\gamma-1+(\alpha-\gamma)} y(\tau) d\tau + \frac{1}{\Gamma(\beta)}\int_a^t (t-\tau)^{\gamma-1+\beta+\gamma} y(\tau) ds \end{aligned} \tag{III.8}$$

Then,

$$\begin{aligned} |J_a^\alpha y(t) + J_a^\beta y(t)| &\leq \frac{T^{\alpha-\gamma}}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\gamma-1} |y(\tau)| d\tau + \frac{T^{\beta-\gamma}}{\Gamma(\beta)}\int_a^t (t-\tau)^{\gamma-1} |y(\tau)| d\tau \\ &\leq \left[\frac{\Gamma(\gamma)}{\Gamma(\alpha)}T^{\alpha-\gamma} + \frac{\Gamma(\gamma)}{\Gamma(\beta)}T^{\beta-\gamma}\right] \frac{1}{\Gamma(\gamma)}\int_a^t (t-\tau)^{\gamma-1} |y(\tau)| d\tau \\ &\leq \left[\frac{\Gamma(\gamma)}{\Gamma(\alpha)}T^{\alpha-\gamma} + \frac{\Gamma(\gamma)}{\Gamma(\beta)}T^{\beta-\gamma}\right] J^\gamma |y(t)|. \end{aligned} \tag{III.9}$$

□

Now, let' answer to Second question.

**Lemma 3.1.4.** *Let* $1 < \alpha_2 < 2$, $\forall T \in \mathbb{R}^+$ *and* $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is satisfies a Lipsichiz condition in its second and third arguments on a suitable domain. as*

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq L\left(|x_2 - x_1| + |y_2 - y_1|\right) \tag{III.10}$$

*where* $\forall t \in [0, T]$, $x_1, x_2, y_1, y_2 : [0, T] \to \mathbb{R}$ *and* $0 < L$. *if for any* $0 < \epsilon << 1$ *and* $\hat{y}$ *and* $\tilde{y}$ *are solutions of below systems,*

$$\begin{cases} D_a^{1-\epsilon}\hat{y}(t) = w(t), & y(t_0) = y_a \\ D_a^{\epsilon} w(t) = z(t), & w(t_0) = 0 \\ D_a^{\alpha_2-1} z(t) = f(t, \hat{y}(t), w(t)). & z(t_0) = s \end{cases} \quad \textit{and} \quad \begin{cases} D_a^{1-\epsilon}\tilde{y}(t) = \tilde{z}(t), & \tilde{y}(t_0) = y_a \\ D_a^{\alpha_2-1}\tilde{z}(t) = f(t, \tilde{y}(t), \tilde{z}(t)). & \tilde{z}(t_0) = s \end{cases} \tag{III.11}$$

*then,*

$$|\hat{y}(t) - \tilde{y}(t)| \to 0, \quad \textit{as } \epsilon \to 0. \tag{III.12}$$

*Proof.*

$$\hat{y}(t) = y_a + J_a^{1-\epsilon} w(t), \quad \tilde{y}(t) = y_a + J_a^{1-\epsilon}\tilde{z}(t), \tag{III.13}$$

$$\Rightarrow \hat{y}(t) - \tilde{y}(t) = J_a^{1-\epsilon}(w(t) - \tilde{z}(t)), \tag{III.14}$$

$$w(t) - \tilde{z}(t) = w(t) - J_a^\epsilon \tilde{z}(t) + J_a^\epsilon \tilde{z}(t) - \tilde{z}(t). \tag{III.15}$$

Since, $w(t) = J^\epsilon z(t)$.

$$\Rightarrow |w(t) - \tilde{z}(t)| \leq J_a^\epsilon |z(t) - \tilde{z}(t)| + |J_a^\epsilon \tilde{z}(t) - \tilde{z}(t)|. \tag{III.16}$$

Also, $z(t) = s + J_a^{\alpha_2-1} f(t, \hat{y}(t), w(t))$ and $\tilde{z}(t) = s + J_a^{\alpha_2-1} f(t, \tilde{y}(t), w(t))$.

$$\begin{aligned}
|z(t) - \tilde{z}(t)| &= |J_a^{\alpha_2-1} f(t, \hat{y}(t), w(t)) - f(t, \tilde{y}(t), \tilde{z}(t))| \\
&\le J_a^{\alpha_2-1} |f(t, \hat{y}(t), w(t)) - f(t, \tilde{y}(t), \tilde{z}(t))| \\
&\le L J_a^{\alpha_2-1} \left( |\hat{y}(t) - \tilde{y}(t)| + |w(t) - \tilde{z}(t)| \right) \qquad \text{by Lipschitz condition.} \\
&\le L J_a^{\alpha_2-1} \left[ J_a^{1-\epsilon} |w(t) - \tilde{z}(t)| + J_a^{\epsilon} |z(t) - \tilde{z}(t)| + |J_a^{\epsilon} \tilde{z}(t) - \tilde{z}(t)| \right] \\
&\qquad \text{by (III.14) and (III.16).} \\
&\le L \left[ J_a^{\alpha_2-\epsilon} |w(t) - \tilde{z}(t)| + J_a^{\alpha_2-1+\epsilon} |z(t) - \tilde{z}(t)| + \frac{T^{\alpha_2-1}}{\Gamma(\alpha_2)} \| J_a^{\epsilon} \tilde{z}(t) - \tilde{z}(t) \|_\infty \right].
\end{aligned} \tag{III.17}$$

Now, we need to calculate $J_a^{\alpha_2-\epsilon} |w(t) - \tilde{z}(t)|$.

$$\begin{aligned}
J_a^{\alpha_2-\epsilon} |w(t) - \tilde{z}(t)| &\le J_a^{\alpha_2-\epsilon} \left( J_a^{\epsilon} |z(t) - \tilde{z}(t)| + |J_a^{\epsilon} \tilde{z}(t) - \tilde{z}(t)| \right) \\
&\le J_a^{\alpha_2} |z(t) - \tilde{z}(t)| + \frac{T^{\alpha_2-\epsilon}}{\Gamma(\alpha_2 - \epsilon + 1)} \| J_a^{\epsilon} \tilde{z}(t) - \tilde{z}(t) \|_\infty
\end{aligned} \tag{III.18}$$

Using (III.18), we have

$$\begin{aligned}
|z(t) - \tilde{z}(t)| &\le L \Bigg[ J_a^{\alpha_2} |z(t) - \tilde{z}(t)| + \frac{T^{\alpha_2-\epsilon}}{\Gamma(\alpha_2 - \epsilon + 1)} \| J_a^{\epsilon} \tilde{z}(t) - \tilde{z}(t) \|_\infty \\
&\qquad + J_a^{\alpha_2-1+\epsilon} |z(t) - \tilde{z}(t)| + \frac{T^{\alpha_2-1}}{\Gamma(\alpha_2)} \| J_a^{\epsilon} \tilde{z}(t) - \tilde{z}(t) \|_\infty \Bigg] \\
&= L \Bigg[ J_a^{\alpha_2-\epsilon} |z(t) - \tilde{z}(t)| + J_a^{\alpha_2-1+\epsilon} |z(t) - \tilde{z}(t)| \\
&\qquad + \left\{ \frac{T^{\alpha_2-\epsilon}}{\Gamma(\alpha_2 - \epsilon + 1)} + \frac{T^{\alpha_2-1}}{\Gamma(\alpha_2)} \right\} \| J_a^{\epsilon} \tilde{z}(t) - \tilde{z}(t) \|_\infty \Bigg] \\
&\qquad \text{by Lemma (3.1.3), let } \gamma = \alpha_2 - 1,\ \alpha = \alpha_2 - 1 + \epsilon, \text{ and } \beta = \alpha_2 \text{ then,} \\
&\le L \Bigg[ \left\{ \frac{\Gamma(\alpha_2 - 1)}{\Gamma(\alpha_2 - 1 + \epsilon)} T^{\epsilon} + \frac{\Gamma(\alpha_2 - 1)}{\Gamma(\alpha_2)} T \right\} J_a^{\alpha_2-1} |z(t) - \tilde{z}(t)| \\
&\qquad + \left\{ \frac{T^{\alpha_2-\epsilon}}{\Gamma(\alpha_2 - \epsilon + 1)} + \frac{T^{\alpha_2-1}}{\Gamma(\alpha_2)} \right\} \| J_a^{\epsilon} \tilde{z}(t) - \tilde{z}(t) \|_\infty \Bigg]
\end{aligned} \tag{III.19}$$

Let,

$$C_\epsilon^1 \equiv \frac{\Gamma(\alpha_2 - 1)}{\Gamma(\alpha_2 - 1 + \epsilon)} T^{\epsilon} + \frac{\Gamma(\alpha_2 - 1)}{\Gamma(\alpha_2)} T, \quad C_\epsilon^2 \equiv \frac{T^{\alpha_2-\epsilon}}{\Gamma(\alpha_2 - \epsilon + 1)} + \frac{T^{\alpha_2-1}}{\Gamma(\alpha_2)} \tag{III.20}$$

So, we have

$$|z(t) - \tilde{z}(t)| \le L C_\epsilon^2 \| J_a^{\epsilon} \tilde{z}(t) - \tilde{z}(t) \|_\infty + L C_\epsilon^1 J_a^{\alpha_2-1} |z(t) - \tilde{z}(t)|. \tag{III.21}$$

Applying the Lemma 6.19 of [10],

$$|z(t) - \tilde{z}(t)| \le L C_\epsilon^2 \| J_a^{\epsilon} \tilde{z}(t) - \tilde{z}(t) \|_\infty E_{\alpha_2-1}[L C_\epsilon^1 T^{\alpha_2-1}]. \tag{III.22}$$

Therefore,

$$\begin{aligned}
|\tilde{y}(t) - \hat{y}(t)| &\le J_a^{1-\epsilon}|w(t) - \tilde{z}(t)| \\
&\le J_a|z(t) - \tilde{z}(t)| + J_a^{1-\epsilon}|J_a^{\epsilon}\tilde{z}(t) - \tilde{z}(t)| \\
&\le T|z(t) - \tilde{z}(t)| + \frac{T^{1-\epsilon}}{\Gamma(2-\epsilon)}\|J_a^{\epsilon}\tilde{z}(t) - \tilde{z}(t)\|_\infty \\
&\le TLC_\epsilon^2\|J_a^{\epsilon}\tilde{z}(t) - \tilde{z}(t)\|_\infty E_{\alpha_2-1}[LC_\epsilon^1 T^{\alpha_2-1}] + \frac{T^{1-\epsilon}}{\Gamma(2-\epsilon)}\|J_a^{\epsilon}\tilde{z}(t) - \tilde{z}(t)\|_\infty \\
&= \left[TLC_\epsilon^2 E_{\alpha_2-1}[LC_\epsilon^1 T^{\alpha_2-1}] + \frac{T^{1-\epsilon}}{\Gamma(2-\epsilon)}\right]\|J_a^{\epsilon}\tilde{z}(t) - \tilde{z}(t)\|_\infty
\end{aligned} \tag{III.23}$$

Let, $C_\epsilon \equiv TLC_\epsilon^2 E_{\alpha_2-1}[LC_\epsilon^1 T^{\alpha_2-1}] + \frac{T^{1-\epsilon}}{\Gamma(2-\epsilon)}$. Then,

$$|\tilde{y}(t) - \hat{y}(t)| \le C_\epsilon \|J_a^{\epsilon}\tilde{z}(t) - \tilde{z}(t)\|_\infty. \tag{III.24}$$

Thus, by Lemma (3.1.2).

$$|\tilde{y}(t) - \hat{y}(t)| \le C_\epsilon \|J_a^{\epsilon}\tilde{z}(t) - \tilde{z}(t)\|_\infty \to 0, \text{ as } \epsilon \to 0. \tag{III.25}$$

□

## 3.2 Numerical Results

Bellow tables and figures guarantee correctness of Lemma (3.1.4). Those show that the maximum error go to zero when epsilon decrease to zero also. In this section HM is used.

**Example 3.2.1.** *Polynomial type example*

$$\begin{cases} D_0^{\alpha_2} y(t) = \frac{\Gamma(4-\alpha_2)}{\Gamma(5-\alpha_2)} t^{4-\alpha_2(t)} - \frac{\Gamma(8)}{\Gamma(8-\alpha_2)} t^{7-\alpha_2} - \exp(t^4) + \exp(y) - \sin(4t^3) + \sin(y') \\ y(0) - \frac{1}{\alpha_2 - 1} y'(0) = \gamma_1, \ \ y(1) + y'(1) = \gamma_2. \end{cases} \tag{III.26}$$

*whose* $\alpha_2 = 1.6$, $\epsilon = 10^{-i}$, $i = 1, 2, \cdots, 10$ *and exact solution is*

$$y(t) = t^4. \tag{III.27}$$

| $i$ | NHM | HHM |
|---|---|---|
| 1 | 0.050506 | 0.05124 |
| 2 | 0.004765 | 0.004734 |
| 3 | 0.000545 | 0.00047 |
| 4 | 0.000136 | 4.74E-05 |
| 5 | 9.63E-05 | 5.11E-06 |
| 6 | 9.25E-05 | 1.14E-06 |
| 7 | 9.21E-05 | 8.54E-07 |
| 8 | 9.20E-05 | 8.26E-07 |
| 9 | 9.20E-05 | 8.23E-07 |
| 10 | 9.20E-05 | 8.23E-07 |

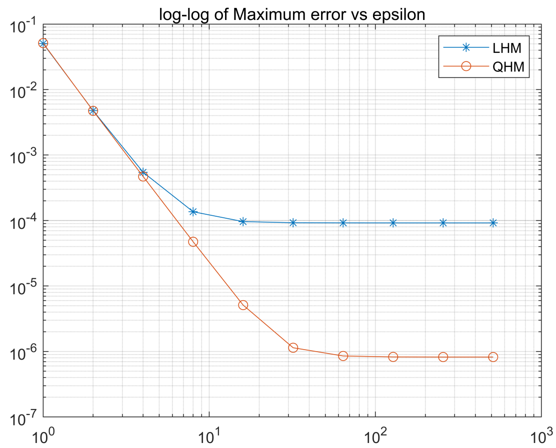



Figure 4: Maximum error vs $\epsilon$ in Example.(3.2.1)

**Example 3.2.2.** *exponential type example*

$$\begin{cases} D_0^{\alpha_2} y(t) = F(t) + \exp(y) + \sin(y'), \\ y(0) - \frac{1}{\alpha_2 - 1} y'(0) = \gamma_1, \ y(1) + y'(1) = \gamma_2. \end{cases} \tag{III.28}$$

*where* $\alpha_2 = 1.6$, $\epsilon = 10^{-i}$, $i = 1, 2, \cdots, 10$,

$$F(t) = t^{2-\alpha_2} E_{1,3-\alpha_2}(t) - \left( \frac{\Gamma(3)}{2\Gamma(3-\alpha_2)} t^{2-\alpha_2} + \frac{\Gamma(4)}{6\Gamma(4-\alpha_2)} t^{3-\alpha_2} \right) - \exp\left( e^t - \left( 1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3 \right) \right) - \sin\left( e^t - \left( 1 + t + \frac{1}{2}t^2 \right) \right),$$

*and exact solution is*

$$y(t) = e^t - \left( 1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3 \right). \tag{III.29}$$

| $i$ | NHM | HHM |
|---|---|---|
| 1 | 0.004241 | 0.004194 |
| 2 | 0.000403 | 0.000401 |
| 3 | 3.79E-05 | 3.99E-05 |
| 4 | 7.52E-06 | 3.97E-06 |
| 5 | 6.21E-06 | 3.87E-07 |
| 6 | 6.08E-06 | 6.91E-08 |
| 7 | 6.06E-06 | 5.99E-08 |
| 8 | 6.06E-06 | 5.90E-08 |
| 9 | 6.06E-06 | 5.89E-08 |
| 10 | 6.06E-06 | 5.89E-08 |

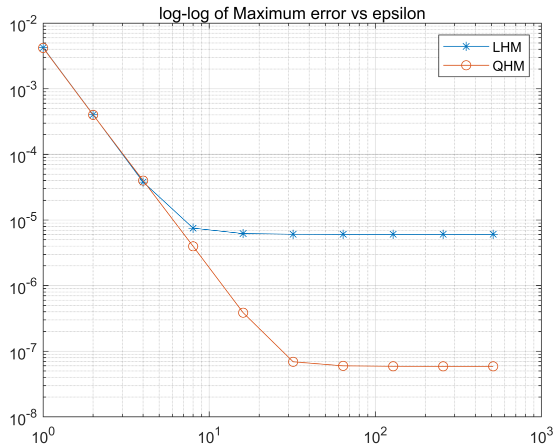



Figure 5: Maximum error vs $\epsilon$ in Example.(3.2.2)

**Example 3.2.3.** *Sin type example*

$$\begin{cases} D_0^{\alpha_2} y(t) = F(t) + \exp(y) + \sin(y'), \\ y(0) - \frac{1}{\alpha_2 - 1} y'(0) = \gamma_1,\ y(1) + y'(1) = \gamma_2. \end{cases} \tag{III.30}$$

*whose* $\alpha_2 = 1.6$, $\epsilon = 10^{-i}$, $i = 1, 2, \cdots, 10$,

$$\begin{aligned} F(t) = &-\frac{1}{2} i(i)^{\lceil \alpha_2 \rceil} t^{(\lceil \alpha_2 \rceil - \alpha_2)} (E_{\lceil \alpha_2 \rceil - \alpha_2 + 1}(it) - (-1)^{\lceil \alpha_2 \rceil} E_{1, \lceil \alpha_2 \rceil - \alpha_2 + 1}(-it)) \\ &- \exp\left(e^t - \left(1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3\right)\right) - \sin\left(e^t - \left(1 + t + \frac{1}{2}t^2\right)\right), \end{aligned}$$

*and exact solution is*

$$y(t) = \sin(t) - t + \frac{t^3}{6}. \tag{III.31}$$

| $i$ | NHM | HHM |
|---|---|---|
| 1 | 0.000763 | 0.000757 |
| 2 | 7.16E-05 | 7.18E-05 |
| 3 | 6.43E-06 | 7.13E-06 |
| 4 | 1.83E-06 | 7.09E-07 |
| 5 | 1.60E-06 | 6.70E-08 |
| 6 | 1.58E-06 | 1.89E-08 |
| 7 | 1.58E-06 | 1.68E-08 |
| 8 | 1.58E-06 | 1.66E-08 |
| 9 | 1.58E-06 | 1.65E-08 |
| 10 | 1.58E-06 | 1.65E-08 |

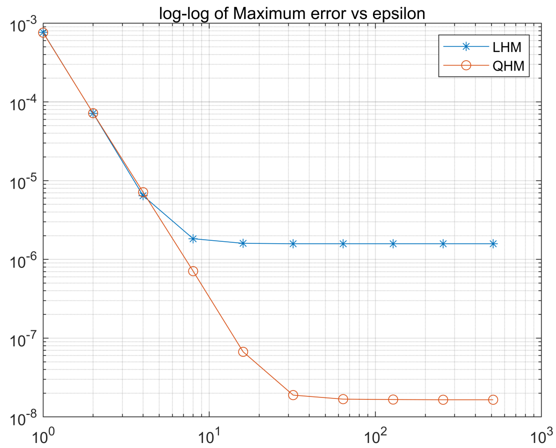



Figure 6: Maximum error vs $\epsilon$ in Example.(3.2.3)

Next numerical results will compare between conventional method [15] and new schemes(NHOM and HHOM with 1st Derivative).

**Example 3.2.4.** *Single-term Fractional Orders, Robin BCs, $C^1$ solution*

$$\begin{cases} D_0^\alpha y(t) = F(t) - (2t+6)y'(t) \\ y(0) - \frac{1}{1-\alpha}y'(0) = \gamma_1,\ y(1) + y'(1) = \gamma_2 \end{cases} \tag{III.32}$$

*whose $1 < \alpha < 2$, and*
$F(t) = \frac{\Gamma(\alpha+1)}{\Gamma(1)} + \frac{\Gamma(2\alpha)}{\Gamma(\alpha)}t^{(\alpha-1)} + 4\frac{\Gamma(4)}{\Gamma(4-\alpha)}t^{(3-\alpha)} + \frac{\Gamma(5)}{\Gamma(5-\alpha)}t^{(4-\alpha)} + (2t+6)(\alpha x^{\alpha-1} + (2\alpha-1)t^{2\alpha-2} + 3 + 12t^2 + 3t^2)$.
*and exact solution is*

$$y(t) = x^\alpha + t^{2\alpha-1} + 1 + 3t + 4t^3 + t^4. \tag{III.33}$$

*In numerical algorithm, $F(t)$ term is calculated as analysis and $\epsilon = 10^{-10}$. And in this example, initial vale $s_0$ means $y(0)$.*

| MID | $\alpha = 1.1$ | | $\alpha = 1.3$ | | $\alpha = 1.5$ | | $\alpha = 1.7$ | | $\alpha = 1.9$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| N | error | roc | error | roc | error | roc | error | roc | error | roc |
| 64 | 9.56E-03 | - | 5.14E-03 | - | 3.91E-03 | - | 3.76E-03 | - | 4.07E-03 | - |
| 128 | 3.89E-03 | 1.298 | 1.60E-03 | 1.686 | 1.05E-03 | 1.895 | 9.44E-04 | 1.995 | 1.02E-03 | 1.996 |
| 256 | 1.65E-03 | 1.236 | 5.20E-04 | 1.618 | 2.87E-04 | 1.872 | 2.37E-04 | 1.997 | 2.55E-04 | 1.998 |
| 512 | 7.22E-04 | 1.193 | 1.78E-04 | 1.548 | 8.01E-05 | 1.840 | 5.92E-05 | 1.998 | 6.39E-05 | 1.999 |
| 1024 | 3.22E-04 | 1.165 | 6.36E-05 | 1.484 | 2.30E-05 | 1.803 | 1.51E-05 | 1.971 | 1.60E-05 | 1.999 |
| 2048 | 1.45E-04 | 1.148 | 2.36E-05 | 1.432 | 6.77E-06 | 1.762 | 3.90E-06 | 1.955 | 4.00E-06 | 2.000 |
| NHOM | $\alpha = 1.1$ | | $\alpha = 1.3$ | | $\alpha = 1.5$ | | $\alpha = 1.7$ | | $\alpha = 1.9$ | |
| N | error | roc | error | roc | error | roc | error | roc | error | roc |
| 64 | 4.42E-03 | - | 1.31E-03 | - | 1.14E-03 | - | 1.27E-03 | - | 1.42E-03 | - |
| 128 | 1.89E-03 | 1.225 | 3.83E-04 | 1.775 | 2.95E-04 | 1.955 | 3.17E-04 | 1.999 | 0.00035 | 2.000 |
| 256 | 8.10E-04 | 1.224 | 1.27E-04 | 1.588 | 7.62E-05 | 1.955 | 7.92E-05 | 2.002 | 8.87E-05 | 2.000 |
| 512 | 3.48E-04 | 1.219 | 6.31E-05 | 1.012 | 1.97E-05 | 1.949 | 1.98E-05 | 2.004 | 2.22E-05 | 2.000 |
| 1024 | 1.50E-04 | 1.213 | 2.88E-05 | 1.130 | 5.15E-06 | 1.938 | 4.92E-06 | 2.006 | 5.54E-06 | 2.000 |
| 2048 | 6.91E-05 | 1.120 | 1.26E-05 | 1.191 | 1.36E-06 | 1.922 | 1.22E-06 | 2.008 | 1.38E-06 | 2.001 |

Table 49: Convergence rate of A MID and transformed NHOM in Example (3.2.4). Initial value : MID ($\beta_1 = \gamma_1, \beta_2 = \gamma_2$), NHOM ($s0 = 0.2$)

**Example 3.2.5.** *Single-term Fractional Orders, Robin BCs, $C^\infty$ solution*

$$\begin{cases} D_0^\alpha y(t) = F(t) - \cos(t)y(t) - \sin(t)y'(t) \\ y(0) - \frac{1}{1-\alpha}y'(0) = \gamma_1,\ y(1) + y'(1) = \gamma_2 \end{cases} \tag{III.34}$$

*whose $1 < \alpha < 2$ and exact solution is*

$$y(t) = \sin(\lambda t) - t + \frac{t^3}{6}. \tag{III.35}$$

*In numerical algorithm $\epsilon = 10^{-10}$. And in this example, initial vale $s_0$ means $y(0)$.*

| MID | $\alpha = 1.1$ | | $\alpha = 1.3$ | | $\alpha = 1.5$ | | $\alpha = 1.7$ | | $\alpha = 1.9$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| N | error | roc | error | roc | error | roc | error | roc | error | roc |
| 10 | 3.36E-04 | - | 3.42E-04 | - | 3.81E-04 | - | 4.18E-04 | - | 4.48E-04 | - |
| 20 | 8.57E-05 | 1.971 | 9.08E-05 | 1.911 | 1.03E-04 | 1.880 | 1.14E-04 | 1.871 | 1.23E-04 | 1.869 |
| 40 | 2.13E-05 | 2.010 | 2.33E-05 | 1.966 | 2.69E-05 | 1.942 | 2.99E-05 | 1.936 | 3.21E-05 | 1.936 |
| 80 | 5.20E-06 | 2.033 | 5.84E-06 | 1.993 | 6.86E-06 | 1.973 | 7.63E-06 | 1.969 | 8.20E-06 | 1.968 |
| 160 | 1.26E-06 | 2.045 | 1.45E-06 | 2.005 | 1.73E-06 | 1.988 | 1.93E-06 | 1.984 | 2.07E-06 | 1.984 |
| 320 | 3.04E-07 | 2.051 | 3.61E-07 | 2.010 | 4.34E-07 | 1.995 | 4.85E-07 | 1.992 | 5.21E-07 | 1.992 |
| NHOM | $\alpha = 1.1$ | | $\alpha = 1.3$ | | $\alpha = 1.5$ | | $\alpha = 1.7$ | | $\alpha = 1.9$ | |
| N | error | roc | error | roc | error | roc | error | roc | error | roc |
| 10 | 7.27E-03 | - | 1.97E-03 | - | 9.45E-04 | - | 6.80E-04 | - | 5.80E-04 | - |
| 20 | 1.78E-03 | 2.033 | 3.92E-04 | 2.329 | 1.97E-04 | 2.263 | 1.56E-04 | 2.124 | 1.40E-04 | 2.047 |
| 40 | 3.92E-04 | 2.180 | 7.80E-05 | 2.328 | 4.38E-05 | 2.170 | 3.75E-05 | 2.055 | 3.47E-05 | 2.016 |
| 80 | 8.38E-05 | 2.226 | 1.62E-05 | 2.272 | 1.03E-05 | 2.093 | 9.25E-06 | 2.021 | 8.64E-06 | 2.005 |
| 160 | 1.78E-05 | 2.236 | 3.50E-06 | 2.205 | 2.49E-06 | 2.046 | 2.30E-06 | 2.007 | 2.16E-06 | 2.001 |
| 320 | 3.79E-06 | 2.232 | 7.93E-07 | 2.144 | 6.12E-07 | 2.021 | 5.74E-07 | 2.002 | 5.39E-07 | 2.000 |
| HHOM | $\alpha = 1.1$ | | $\alpha = 1.3$ | | $\alpha = 1.5$ | | $\alpha = 1.7$ | | $\alpha = 1.9$ | |
| N | error | roc | error | roc | error | roc | error | roc | error | roc |
| 10 | 1.08E-03 | - | 2.39E-04 | - | 9.60E-05 | - | 6.16E-05 | - | 5.08E-05 | - |
| 20 | 1.19E-04 | 3.185 | 2.20E-05 | 3.440 | 9.55E-06 | 3.329 | 7.10E-06 | 3.117 | 6.34E-06 | 3.001 |
| 40 | 1.26E-05 | 3.239 | 2.09E-06 | 3.398 | 1.04E-06 | 3.198 | 8.60E-07 | 3.044 | 7.98E-07 | 2.991 |
| 80 | 1.32E-06 | 3.254 | 2.08E-07 | 3.325 | 1.21E-07 | 3.105 | 1.06E-07 | 3.015 | 1.00E-07 | 2.992 |
| 160 | 1.39E-07 | 3.253 | 2.19E-08 | 3.248 | 1.46E-08 | 3.052 | 1.33E-08 | 3.004 | 1.26E-08 | 2.995 |
| 320 | 1.47E-08 | 3.245 | 2.42E-09 | 3.179 | 1.79E-09 | 3.024 | 1.66E-09 | 3.000 | 1.58E-09 | 2.997 |

Table 50: Convergence rate of MID, NHOM and HHOM in Example (3.2.5). Initial value : MID ($\beta_1 = \gamma_1, \beta_2 = \gamma_2$), NHOM ($s0 = 0.2$), HHOM ($s_0 = 0.2$)

**Example 3.2.6.** *exponential type example*

$$\begin{cases} D_0^{\alpha_2} y(t) = F(t) + y^2 + ty', \\ y(0) - \frac{1}{\alpha_2 - 1} y'(0) = \gamma_1, \ y(1) + y'(1) = \gamma_2. \end{cases} \tag{III.36}$$

*where* $\alpha_2 = 1.6$, $\epsilon = 10^{-i}$, $i = 1, 2, \cdots, 10$,

$$\begin{aligned} F(t) = t^{2-\alpha_2} E_{1,3-\alpha_2}(t) &- \left( \frac{\Gamma(3)}{2\Gamma(3-\alpha_2)} t^{2-\alpha_2} + \frac{\Gamma(4)}{6\Gamma(4-\alpha_2)} t^{3-\alpha_2} \right) \\ &- \left( e^t - \left( 1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3 \right) \right)^2 - t \left( e^t - \left( 1 + t + \frac{1}{2}t^2 \right) \right), \end{aligned}$$

*and exact solution is*

$$y(t) = e^t - \left( 1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3 \right). \tag{III.37}$$

*In numerical algorithm $\epsilon = 10^{-10}$. In this example, initial vale $s_0$ means $y(0)$.*

| NHOM | $\alpha = 1.1$ | | $\alpha = 1.3$ | | $\alpha = 1.5$ | | $\alpha = 1.7$ | | $\alpha = 1.9$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| N | error | roc | error | roc | error | roc | error | roc | error | roc |
| 10 | 1.15E-02 | - | 1.65E-03 | - | 5.81E-04 | - | 9.99E-04 | - | 1.03E-03 | - |
| 20 | 3.02E-03 | 1.921 | 2.27E-04 | 2.862 | 2.49E-04 | 1.222 | 2.87E-04 | 1.801 | 2.67E-04 | 1.952 |
| 40 | 7.12E-04 | 2.086 | 1.69E-05 | 3.747 | 7.80E-05 | 1.675 | 7.58E-05 | 1.919 | 6.49E-05 | 2.039 |
| 80 | 1.59E-04 | 2.161 | 1.23E-05 | 0.461 | 2.18E-05 | 1.840 | 1.94E-05 | 1.965 | 1.70E-05 | 1.932 |
| 160 | 3.54E-05 | 2.171 | 3.83E-06 | 1.682 | 5.78E-06 | 1.915 | 4.91E-06 | 1.984 | 4.26E-06 | 1.998 |
| 320 | 8.58E-06 | 2.044 | 9.48E-07 | 2.013 | 1.49E-06 | 1.957 | 1.23E-06 | 1.991 | 1.07E-06 | 1.999 |
| HHOM | $\alpha = 1.1$ | | $\alpha = 1.3$ | | $\alpha = 1.5$ | | $\alpha = 1.7$ | | $\alpha = 1.9$ | |
| N | error | roc | error | roc | error | roc | error | roc | error | roc |
| 10 | 6.18E-04 | - | 1.60E-04 | - | 4.84E-05 | - | 8.78E-05 | - | 9.77E-05 | - |
| 20 | 7.41E-05 | 3.061 | 1.01E-05 | 3.986 | 9.74E-06 | 2.312 | 1.48E-05 | 2.572 | 1.44E-05 | 2.765 |
| 40 | 8.43E-06 | 3.134 | 9.69E-07 | 3.379 | 1.69E-06 | 2.523 | 2.08E-06 | 2.827 | 1.93E-06 | 2.897 |
| 80 | 9.36E-07 | 3.171 | 1.50E-07 | 2.688 | 2.45E-07 | 2.791 | 2.76E-07 | 2.916 | 2.51E-07 | 2.941 |
| 160 | 1.02E-07 | 3.192 | 2.11E-08 | 2.832 | 3.29E-08 | 2.897 | 3.56E-08 | 2.953 | 3.23E-08 | 2.960 |
| 320 | 1.11E-08 | 3.206 | 2.83E-09 | 2.901 | 4.27E-09 | 2.946 | 4.54E-09 | 2.970 | 4.13E-09 | 2.968 |

Table 51: Convergence rate of A NHOM and transformed HHOM in Example (3.2.6). Initial value : NHOM ($s0 = 0.2$) and HHOM ($s_0 = 0.2$).

# IV Application

To transform fixed order $\alpha_2$ and $\alpha_1$ to variable order $\alpha_2(t)$ and $\alpha_1(t)$ was tried. That means $\alpha_2(t)$ and $\alpha_1(t)$ are some function defined on $[a,b] \to \mathbb{R}$. And there are some proper result when $\alpha_2(t)$ is defined by $[a,b] \to (1,2)$, and $\alpha_1$ defined by constant function. Then, that specific variable order case similarly works just like fixed order case. That means,

$$\begin{cases} D_a^{\alpha_2(t)} y(t) = f(t, y(t), D_a^{\alpha_1(t)} y(t)) \\ g(y(t), y'(t)) = \gamma_t\big|_{t=a,b}, \end{cases} \tag{IV.1}$$

where $1 < \alpha_2(t) < 2,\ 0 < \alpha_1(t) < 1,\ \alpha_1 \in \mathbb{R},\ \alpha_2(t) : [a,b] \longrightarrow \mathbb{R}$ and $g$ be linear function.

By similar process, above variable order two-point boundary value problem can be transformed to a variable order system of initial value problem, and also it can use Newton's and Halley's shooting method similarly.

But, if $\alpha_1(t)$ is not constant function, then it doesn't work properly. It will show bellow tables and figures, but it doesn't have any theoretical base. Also the general $\alpha_2(t)$ and $\alpha_1(t)$ case can be considered.

$$\begin{cases} D_a^{\alpha_2(t)} y(t) = f(t, y(t), D_a^{\alpha_1(t)} y(t)) \\ g(y(t), y'(t)) = \gamma_t\big|_{t=a,b}, \end{cases} \tag{IV.2}$$

where $\alpha_2(t) : [a,b] \longrightarrow \mathbb{R}$, $\alpha_1(t) : [a,b] \longrightarrow \mathbb{R}$ and $g$ be linear function.

But, that case doesn't consider in this paper, thus, probably that will be future works.

## 4.1 Variable order Fractional Two-Point Boundary Value Problem

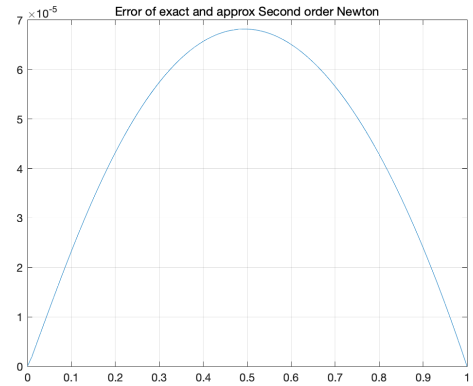
Bellow four examples will show good result of variable order BVP with Accelerated Hybrid Method. Shooting iteration $m$ is 10 and $s = 0.2$.

**Example 4.1.1.** *Dirichlet Boundary Condition with variable order $\alpha_2(t)$.*

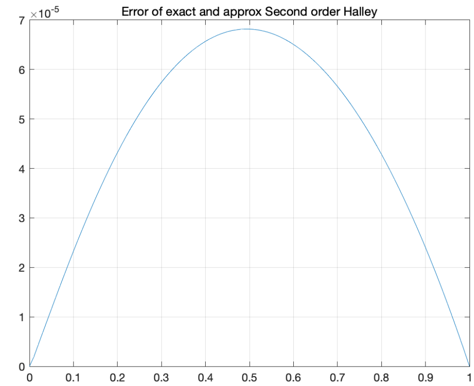
$$\begin{cases} D_0^{\alpha_2(t)} y(t) = \frac{\Gamma(5)}{\Gamma(5-\alpha_2(t))} t^{4-\alpha_2(t)} - t^8 + y^2 - \frac{\Gamma(5)}{\Gamma(5-\alpha_1(t))} t^{4-\alpha_1(t)} + D_0^{\alpha_1(t)} y(t) \\ y(0) = 0,\ y(1) = 1, \end{cases} \tag{IV.3}$$

*whose $\alpha_2(t) = 1.0 + 0.99\sin(\pi t/2),\ \alpha_1(t) = 0.5.$ and exact solution is*

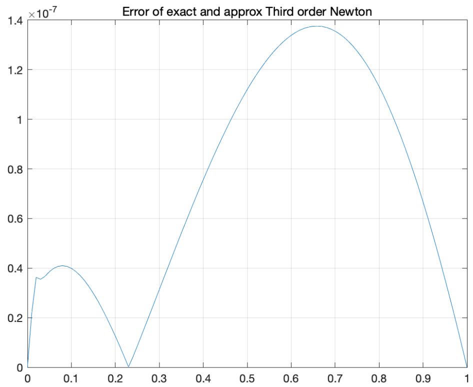
$$y(t) = y^4. \tag{IV.4}$$



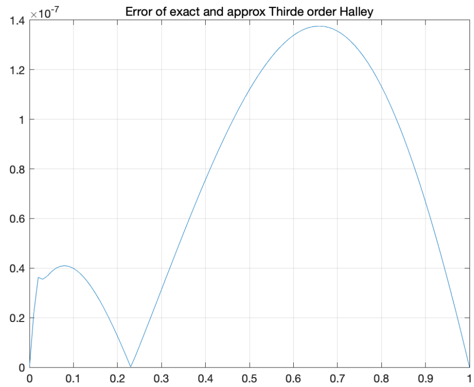

(a) SN Variable order with Dirichlet Boundary Condition




(b) SH Variable order with Dirichlet Boundary Condition




(c) SN Variable order with Dirichlet Boundary Condition



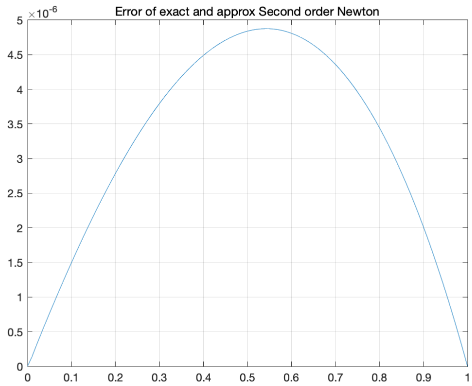

(d) SH Variable order with Dirichlet Boundary Condition

**Example 4.1.2.** *Robin Boundary Condition with variable order* $\alpha_2(t)$.

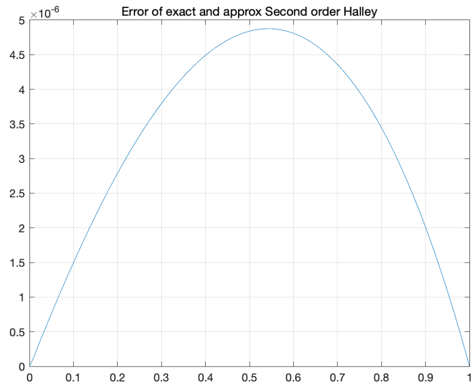
$$\begin{cases} D_0^{\alpha_2(t)} y(t) = \frac{\Gamma(5)}{\Gamma(5-\alpha_2(t))} t^{4-\alpha_2(t)} + \frac{\Gamma(5)}{\Gamma(5-\alpha_1)} t^{4-\alpha_1} + t^8 - y^2 - D_0^{\alpha_1} y(t) \\ y(0) + y'(0) = 0, \ \ y(1) + y'(1) = 5 \end{cases} \tag{IV.5}$$

*whose* $\alpha_2 = 1.0 + 0.99 \sin(\pi t/2)$, $\alpha_1 = 0.5$. *and exact solution is*

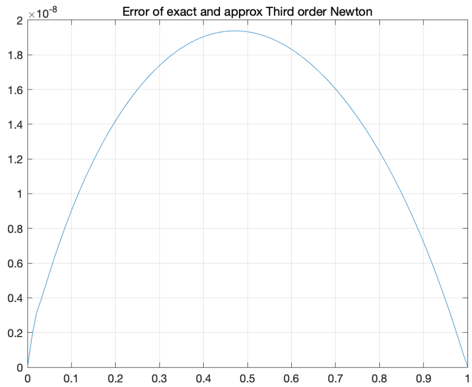
$$y(t) = y^4. \tag{IV.6}$$



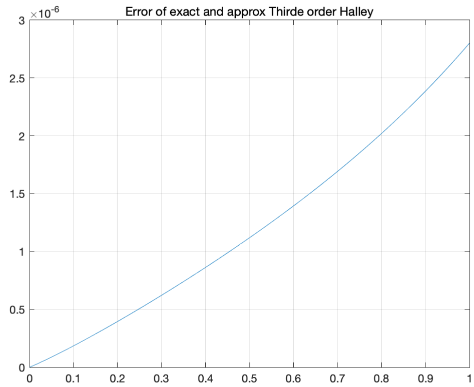

(a) SN Variable order with Robin Boundary Condition




(b) SH Variable order with Robin Boundary Condition




(c) SN Variable order with Robin Boundary Condition



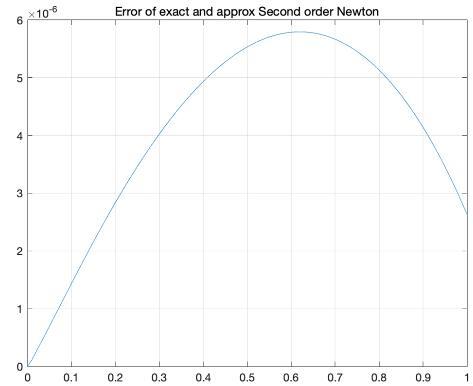

(d) SH Variable order with Robin Boundary Condition

**Example 4.1.3.** *Dirichlet Boundary Condition with variable order* $\alpha_2(t)$

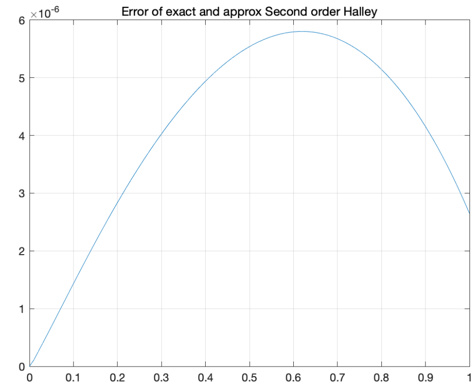
$$\begin{cases} D_0^{\alpha_2(t)}y(t) = \lambda^2 t^{2-\alpha_2(t)}E_{1,3-\alpha_2(t)}(\lambda t) - \left(\lambda^2\frac{\Gamma(3)}{2\Gamma(3-\alpha_2(t))}t^{2-\alpha_2(t)} + \lambda^3\frac{\Gamma(4)}{6\Gamma(4-\alpha_2(t))}t^{3-\alpha_2(t)}\right) \\ \qquad\qquad -A^2 + y^2 - tB + tD_0^{\alpha_1}y(t), \\ y(t_0) = 0, \;\; y(T) \approx 1.0577 \end{cases} \tag{IV.7}$$

*whose* $\alpha_2(t) = 1.0 + 0.99\sin(\pi t/2),\ \alpha_1 = 0.5,\ \lambda = 2,$
$A = e^{\lambda t} - \left(1 + \lambda t + \frac{\lambda^2}{2}t^2 + \frac{\lambda^3}{3!}t^3\right),$
$B = \lambda t^{1-\alpha_1}E_{1,2-\alpha_1}(\lambda t) - \left(\lambda\frac{\Gamma(2)}{\Gamma(2-\alpha_1)}t^{1-\alpha_1} + \lambda^2\frac{\Gamma(3)}{2\Gamma(3-\alpha_1)}t^{2-\alpha_1} + \lambda^3\frac{\Gamma(4)}{6\Gamma(4-\alpha_1)}t^{3-\alpha_1}\right)$
*and exact solution is*

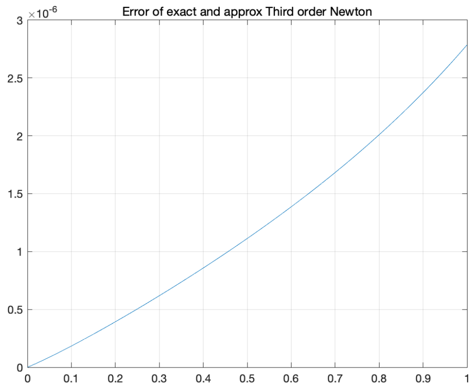
$$y(t) = e^{\lambda t} - \left(1 + \lambda t + \frac{\lambda^2}{2}t^2 + \frac{\lambda^3}{3!}t^3\right). \tag{IV.8}$$



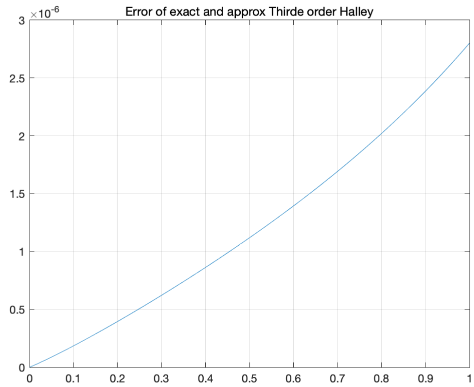

(a) SN Variable order with Dirichlet Boundary Condition




(b) SH Variable order with Dirichlet Boundary Condition




(c) SN Variable order with Dirichlet Boundary Condition



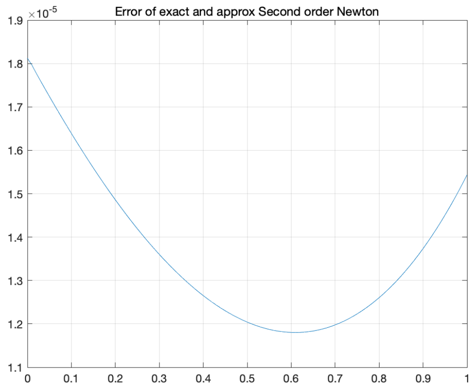

(d) SH Variable order with Dirichlet Boundary Condition

**Example 4.1.4.** *Robin Boundary Condition with variable order* $\alpha_2(t)$

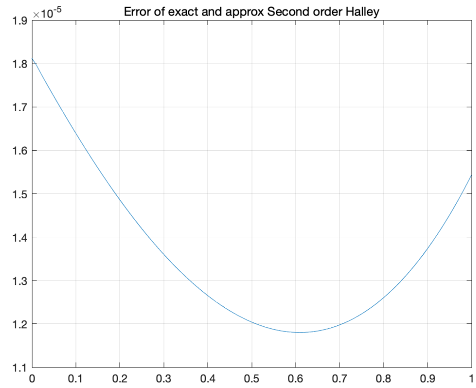
$$\begin{cases} D_0^{\alpha_2(t)} y(t) = \lambda^2 t^{2-\alpha_2(t)} E_{1,3-\alpha_2(t)}(\lambda t) - \left(\lambda^2 \frac{\Gamma(3)}{2\Gamma(3-\alpha_2(t))} t^{2-\alpha_2(t)} + \lambda^3 \frac{\Gamma(4)}{6\Gamma(4-\alpha_2(t))} t^{3-\alpha_2(t)}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -A^2 + y^2 - tB + tD_0^{\alpha_1} y(t), \\ y(0) + y'(0) = 0, \; y(1) + y'(1) = 5 \end{cases} \tag{IV.9}$$

*whose* $\alpha_2(t) = 1.0 + 0.99\sin(\pi t/2)$, $\alpha_1 = 0.5$, $\lambda = 2$,
$A = e^{\lambda t} - \left(1 + \lambda t + \frac{\lambda^2}{2} t^2 + \frac{\lambda^3}{3!} t^3\right)$,
$B = \lambda t^{1-\alpha_1} E_{1,2-\alpha_1}(\lambda t) - \left(\lambda \frac{\Gamma(2)}{\Gamma(2-\alpha_1)} t^{1-\alpha_1} + \lambda^2 \frac{\Gamma(3)}{2\Gamma(3-\alpha_1)} t^{2-\alpha_1} + \lambda^3 \frac{\Gamma(4)}{6\Gamma(4-\alpha_1)} t^{3-\alpha_1}\right)$
*and exact solution is*

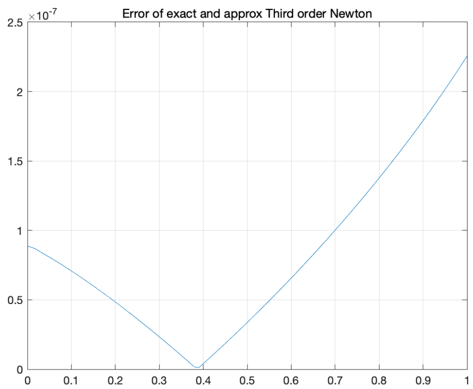
$$y(t) = e^{\lambda t} - \left(1 + \lambda t + \frac{\lambda^2}{2} t^2 + \frac{\lambda^3}{3!} t^3\right). \tag{IV.10}$$



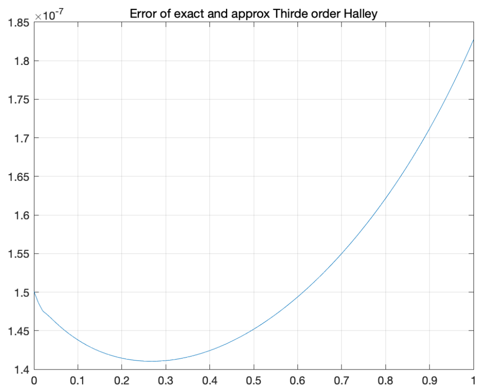

(a) SN Variable order with Robin Boundary Condition




(b) SH Variable order with Robin Boundary Condition




(c) SN Variable order with Robin Boundary Condition




(d) SH Variable order with Robin Boundary Condition

# V Conclusion

In this paper, we suggest several more numerical approaches for solving FBVP under Caputo operator.

Process : The main ideas are described as follows

1. The given FBVP can be transposed into the system of FIVP.
2. Nonlinear shooting methods are applied to find unknown initial value for system of FIVP.
3. Four numerical methods are applied to approximated the system of FIVP.

Convergence rate of three methods which are HOM, HM and IHM is not depended under the differentiation order. But that of PECE is affected by the fractional order and is worse than the former three methods. And some of numerical results from HMs and IHMs are better than those obtained by HOMs, but they are almost the same and that is not remarkable.

And we also deal with FBVP with 1st order derivative term case.

Process :

1. Exchange 1st order term of FBVP to fractional order $\alpha_1 = 1 - \epsilon$ term which $\epsilon$ is very small

number.
2. Converted FBVP can be solved above process.
3. System of FIVP which is transposed from FBVP have $\epsilon$ fractional order term and it can be omitted.

Numerical results of FBVP with 1st order derivative term show that maximum error between exact value and approximated solution gets smaller as $\alpha_1$ gets closer to 1. And suggested method can also solve examples which is come from conventional paper [15] and if the smoothness of given function is guaranteed, then the new method have better convergence rate than MID. Furthermore, conventional scheme can not solve nonlinear problem but, suggested methods can solve the examples.

Finally, although any theoretical consideration is absent and only intuitive idea from fixed order theoretical contents is present, numerical examples of variable order case are included. And the results show that generous interpretation works. Thus, that case must need precise theories, because that numerical results can be just a coincidence.

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