Doctoral Thesis

Solitary waves of the Euler-Poisson system

Junsik Bae

Department of Mathematical Sciences

Graduate School of UNIST

2019

ULSAN NATIONAL INSTITUTE OF SCIENCE AND TECHNOLOGY

Solitary waves of the Euler-Poisson system

Junsik Bae

Department of Mathematical Sciences

Graduate School of UNIST

Solitary waves of the Euler-Poisson system

A thesis/dissertation
submitted to the Graduate School of UNIST
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Junsik Bae

6. 14. 2019

Approved by

Advisor

Bongsuk Kwon

Solitary waves of the Euler-Poisson system

Junsik Bae

This certifies that the thesis/dissertation of Junsik Bae is approved.

6/14/2019

Advisor: Bongsuk Kwon

Chang Yeol Jung

Hantaek Bae

Jinmyoung Seok

Injo Hur

For everyone who loves someone.

Abstract

We study solitary waves of the Euler-Poisson (EP) system. More precisely, we study the asymptotic behavior and linear stability of small amplitude solitary wave solutions to the EP system.

The first main result states that in a stretched moving frame, small amplitude solitary waves of the EP system converge to the KdV solitary waves. The proper choice for the speed of moving frame and the associated KdV equation is crucially used in the derivation of the remainder equation. To overcome a difficulty, arising from indefinite signs of the remainder equation, we divide the interval and conduct the analysis separately. We obtain the uniform estimates for the remainder near the peak using the Gronwall-type inequality. For the far-field region, we obtain the uniform decay estimates of the remainder by estimating uniform lower bound of the speed of trajectory curves.

The second main result states that solitary waves of the EP system linearly asymptotically stable modulo two non-decaying modes. The solution of the linearized EP system will be represented by the semigroup generated by the linearized operator around the solitary wave solutions to the EP system. Introducing $e^{\eta x}$-weighted L^2 norm, we perturb the operator in such a way that the essential spectrum of the perturbed operator lies on the open left-half plane of the complex plane. The zero eigenvalue of the operator, resulting from the translation invariance and the speed parameter, is then isolated with algebraic multiplicity two. We study the eigenvalue problem applying the Evans function, which is particularly useful for detecting eigenvalues and their algebraic multiplicity. While calculating the Evans function is not simple in general, the Evans function for the KdV equation is explicitly known. Considering a special scaling, we show that the Evans function for the EP system converges to that for the KdV equation.

Contents

1 Overview

The study of 'solitary waves' is an important subject not only in mathematics but also in many other scientific scenes. We concern the mathematical study of solitary waves of the Euler-Poisson (EP) system. More precisely, we study the asymptotic behavior and the linear stability of small amplitude solitary wave solutions to the EP system.

The EP system is a fluid model which describes the dynamics of ions in electrostatic plasmas. In Section 2, we briefly introduce the EP system and its physical meaning. Also some basic properties of the EP system as well as its derivation will be presented.

In Section 3, we study the asymptotic behavior of small amplitude solitary wave solutions to the EP system. In particular, in a stretched moving frame, they converge to the KdV solitary waves (Bae and Kwon [1]).

The KdV equation is one of the most celebrated partial differential equations. Historically, the KdV equation was first derived from the inviscid Euler equation by Boussinesq (1871, [4]) and Korteweg and de Vries (1895, [21]), to describe Russell's observation (1844, [30]) of traveling solitary waves along a narrow channel. The KdV equation was not much studied until it was discovered in the early 1960s that it is also derived from the other fields such as the study of hydromagnetic waves (Gardner and Morikawa [15]). Among others, the formal connection between the EP system and the KdV equation were found by plasma physicists (Sagdeev [31], Washimi and Taniuti [34]). Later on, the formation and propagation of solitary waves in electrostatic plasma were experimentally observed (Ikezi et al [19]).

We study the asymptotic similarity of the EP solitary waves to the KdV solitary waves in a rigorous manner. The proper choice for the speed of moving frame (the ion sound speed) and the associated KdV equation is crucially used in the derivation of the remainder equation. A difficulty in the analysis of the remainder system stems from indefinite signs of the remainder equation near the peak and near the far-field. Hence we divide the interval and conduct the analysis separately. We obtain the uniform estimates of the remainder near the peak using the Gronwall-type inequality. For the uniform estimates of the remainder near the far-field region, we obtain the uniform decay estimates of the remainder by estimating uniform lower bound of the speed of trajectory curves.

In Section 4, we study the linear stability of solitary wave solutions to the EP system. The terminology 'stability' is somewhat vague; in which sense are solitary waves stable? We need to establsih suitable notions of stability, and it would depend on the properties of the waves and the underlying structures of PDEs. On the other hand, the stability of the KdV solitary waves has been extensively studied (Benjamin [2], Bona [3], Pego and Weinstein [26], [27]). We will investigate those notions of stability in the context of our problem. Then we study the linear stability of solitary waves as the first step toward the nonlinear asymptotic stability (Bae and Kwon, unpublished). We remark that the global existence for the initial value problem of the one-dimensional EP system is not known yet.

The solution of the linearized EP system will be represented by the semigroup generated by the linearized operator around the solitary wave solutions to the EP system. The spectral information of the generator gives the asymptotic behavior of the associated semigroup. The zero eigenvalue of the operator, resulting from the translation invariance and the speed parameter, is embedded in the essential spectrum in L^2 space. However, by introducing $e^{\eta x}$-weighted L^2 norm, one can perturb the operator in such a way that the essential spectrum of the perturbed operator lies on the open left-half plane of the complex plane, and the zero eigenvalue of the operator is then isolated with algebraic multiplicity two. We show that small amplitude solitary waves of the EP system linearly asymptotically stable modulo two non-decaying modes, the generalized eigenvectors corresponding to the zero eigenvalue. We study the eigenvalue problem applying the Evans function, which is particularly useful for detecting eigenvalues and their algebraic multiplicity. While calculating the Evans function is not simple in general, the Evans function for the KdV equation is explicitly known. The approach here is to show that in a special scaling, the Evans function for the EP system converges to that for the KdV equation.

The Evans function was first introduced by Evans ([11], [12], [13], [14]) in the study of stability of some class of traveling waves. It is a complex analytic function in the spectral paramter, and its zeros are related to the eigenvalues of the linearized operator around the nonlinear wave under consideration. In Section 5, we introduce the construction and properties of the Evans function as well as its application to the instability of solitary waves for the generalized KdV equation given in [26]. Also, the general description on the linear asymptotic stability of nonlinear waves and some prerequisites such as the spectral and semigroup theory will be covered referring to textbooks of Coppel ([6], [7]) for the asymptotic behavior of ODEs, Engel and Nagel [10], Kato [20], Pazy [25] for the spectral and semigroup theory, Pego and Weinstein [26], Sandstede [32], Kapitula and Promislow [35] for the Evans function and its applications.

2 Fluid Description of Electrostatic Plasma

2.1 The Euler-Poisson System

The ion dynamics in an electrostatic plasma is described by *the Euler-Poisson system for ions*: in the nondimensionalized form

$$\begin{cases} \partial_t n + \nabla \cdot (nu) = 0, & (2.1a) \\ n\left(\partial_t u + (u \cdot \nabla)u\right) + K\nabla n = -n\nabla\phi, & (2.1b) \\ -\Delta\phi = n - e^{\phi}, & (2.1c) \end{cases}$$

where $n, \phi : (t, x) \in \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ are unknown functions for the ion number density and the electric potential, $u = (u_1, u_2, u_3)^T : (t, x) \in \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ is the velocity vector field of the ions, and $K = T_i/T_e \geq 0$ is a constant representing the ratio of the ion temperature T_i and the electron temperature T_e. Here, $u \cdot \nabla := \sum_{k=1}^{3} u_k \partial_{x_k}$. The system (2.1) is called the *pressureless* if $K = 0$, and the *isothermal* if $K > 0$.[1] In the model (2.1), the electron density is determined by ϕ via the *Boltzmann relation* (2.18). The system (2.1) is a common fluid model for ions in a plasma, and it well describes a variety of phenomena arising in plasma physics such as the formation of double layers and plasma sheaths. For more physicality of (2.1), we refer readers to [5, 9].

The system (2.1) is derived from *the two-fluid Euler-Maxwell for ions and electrons* under the following major assumptions[2]:

- electrostatic – there is no magnetic field;
- isothermal pressures – the temperatures for the ions and the electrons are constant;
- massless electron – the mass of the electrons is zero.

The derivation will be presented in Section 2.2. We remark that for the adiabatic ions, the isothermal pressure term Kn of (2.1) is replaced by the adiabatic pressure law

$$p(n) = An^{\gamma},$$

where $A > 0$ and $\gamma > 1$ are constants.

Unlike the compressible Euler system for neutral gases, the Euler-Poisson system has a dispersive character due to the presence of the electric potential, and this aspect makes the system contain rich and interesting phenomena.

[1] As an ideal case for a plasma with $T_i \ll T_e$, the pressureless model is frequently used in plasma physics.

[2] The assumptions of the isothermal electron pressure and the massless electron are based on the physical fact that for every plasma environment, the mass of the electron is very small and negligible compared to the mass of the ion. The mass of the hydrogen ion is $1836 \times m_e$, where $m_e \approx 9.1 \times 10^{-31} kg$ is the mass of the electron.

Dispersion Relation We consider the linearized system for (2.1) around the uniform state solution $(n,u,\phi)=(1,0,0)$. Substituting the small perturbations $(\varepsilon n_1, \varepsilon u_1, \varepsilon\phi_1) = (n-1, u, \phi)$ into (2.1), and then neglecting ε^2 order terms, we obtain the linearized system for (2.1):

$$\begin{cases} \partial_t n_1 + \nabla\cdot u_1 = 0, \\ \partial_t u_1 + K\nabla n_1 = -\nabla\phi_1, \\ -\Delta\phi_1 = n_1 - \phi_1. \end{cases} \tag{2.2}$$

In what follows, we do not consider the adiabatic ion pressure since the analysis is similar to the case $K>0$, in which K is replaced by $A\gamma > 0$. Taking the Fourier transform of (2.2) in t and x (or simply plugging an Ansatz $(n_1,u_1,\phi_1) = (\hat{n}_1, \hat{u}_1, \hat{\phi}_1)e^{i(k\cdot x - wt)}$ into (2.2)), we obtain the dispersion relation for (2.1):

$$\omega^2(k) = |k|^2\left(K + \frac{1}{1+|k|^2}\right), \quad (k\in\mathbb{R}^3). \tag{2.3}$$

We note that $(1+|k|^2)^{-1}$ term comes from the Poisson equation. In low and high frequency regimes, the behavior of ω is given as follows: $\omega(k) \sim \pm\sqrt{K}|k|$ as $|k|\to+\infty$ and $\omega(k)\sim \pm\sqrt{1+K}|k|$ as $|k|\to 0$. We note that while the asymptotic behavior of ω for $K>0$ are similar to that of the compressible Euler system for neutral gases, $\omega_n^2(k) = |k|^2$, the behavior of ω for $K=0$ is completely different on those regimes (see Figure 1). In plasma physics, the constant $\sqrt{1+K}$ is called the *ion sound speed*, and it will frequently appear throughout this thesis.

Now we consider the plane wave solutions to (2.2) and consider the long-wavelength limit.[3] We choose a smooth branch of (2.3) satisfying $\omega(k)>0$ for $k>0$. Expanding it at $k=0$, we obtain

$$\omega(k) = k\sqrt{K+\frac{1}{1+k^2}} = \sqrt{1+K}k - \frac{k^3}{2\sqrt{1+K}} + O(k^5). \tag{2.4}$$

Up to the third order, (2.4) is the dispersion relation of the linear KdV equation

$$\partial_t v + \sqrt{1+K}\partial_x v + (2\sqrt{1+K})^{-1}\partial_x^3 v = 0.$$

We shall investigate some similarities in the asymptotic behaviors of the one-dimensional EP system and the KdV equation in a certain scale.

Conserved Quantities For the smooth solutions to (2.1) such that as $|x|\to\infty$, $(n,u,\phi)\to(1,0,0)$ and their derivatives converge to 0, we have the following invariants:

$$N(t) := \int_{\mathbb{R}^3} (n-1)(t,x)\,dx = N(0), \tag{2.5a}$$

$$M(t) := \int_{\mathbb{R}^3} (nu)(t,x)\,dx = M(0), \tag{2.5b}$$

$$H(t) := \int_{\mathbb{R}^3} \left(\frac{n|u|^2}{2} + P(n) + \frac{|\nabla\phi|^2}{2} + (\phi-1)e^{\phi} + 1\right)(t,x)\,dx = H(0), \tag{2.5c}$$

[3] Long waves compared to *the Debye length* $\lambda_D = \sqrt{K_B T_e/4\pi n_{e0}e^2}$. See the non-dimensionalization (2.20).

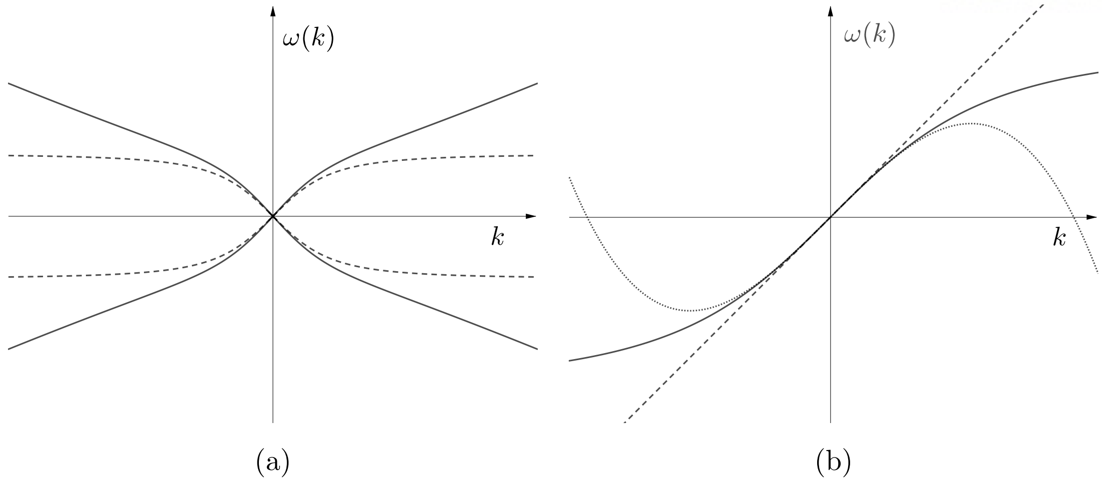


Figure 1: (a) The graphs of $\omega^2(k) = k^2(0.2 + \frac{1}{1+k^2})$ (solid) and $\omega^2(k) = \frac{k^2}{1+k^2}$ (dashed). (b) The graphs of $\omega(k) = \frac{k}{\sqrt{1+k^2}}$ (solid), $\omega(k) = k$ (dashed) and $\omega(k) = k - \frac{k^3}{2}$ (dotted).

where

$$P(n) = \begin{cases} K(n\ln n - n + 1) & \text{for } p(n) = Kn,\ K \geq 0, \\ \dfrac{An^\gamma}{\gamma - 1} - \dfrac{A}{\gamma - 1} & \text{for } p(n) = An^\gamma,\ A > 0, \gamma > 1. \end{cases}$$

Using the Poisson equation (2.1c), we note that H can be written in another form

$$H(t) = \int_{\mathbb{R}^3} \left(\frac{n|u|^2}{2} + P(n) - \frac{|\nabla\phi|^2}{2} + n\phi - e^\phi + 1 \right)(t,x)\,dx. \tag{2.6}$$

These two forms of H have their own advantages. The form (2.5c) clearly shows that $H(t) \geq 0$ for $n > 0$. On the other hand, one can easily derive the Euler-Poisson (2.1) from the form of H in (2.6). Indeed, it is a Hamiltonian of the EP system (2.1). For simplicity, we consider the 1-dimensional case. Let $H_\phi(t) := \int -\frac{|\partial_x\phi|^2}{2} + n\phi - e^\phi + 1\,dx$. By taking the variational derivatives[4] of (2.6), we have

$$\begin{cases} \partial_t n = -\partial_x \dfrac{\delta H}{\delta u} = -\partial_x(nu), \\ \partial_t u = -\partial_x \dfrac{\delta H}{\delta n} = -\partial_x \left(\dfrac{u^2}{2} + K\ln n + \dfrac{\delta H_\phi}{\delta n} \right). \end{cases}$$

We recall that $\phi = \phi(n)$ is determined by n through the Poisson equation (2.1c). By a formal chain rule, we obain

$$\frac{\delta H_\phi}{\delta n} = \frac{\delta H_\phi}{\delta n} + \frac{\delta H_\phi}{\delta \phi}\partial_n\phi(n) = \phi + \frac{\delta H_\phi}{\delta\phi}\partial_n\phi(n) \tag{2.8}$$

where $\partial_n\phi(n)$ is a formal derivative of the operator $n \mapsto \phi$ in n. On the other hand, we see that

$$\frac{\delta H_\phi}{\delta\phi} = \int \partial_x^2\phi + n - e^\phi\,dx = 0. \tag{2.9}$$

[4] $\lim_{\varepsilon\to 0} \dfrac{H(u+\varepsilon h) - H(u)}{\varepsilon} = \int \dfrac{\delta H(u)}{\delta u} h\,dx$ for each function h.

Hence, the 1D Euler-Poisson system can be written as

$$\partial_t \begin{pmatrix} n \\ u \end{pmatrix} = \mathcal{J} \begin{pmatrix} \delta H/\delta n \\ \delta H/\delta u \end{pmatrix} = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix} \begin{pmatrix} \delta H/\delta n \\ \delta H/\delta u \end{pmatrix},$$

where $\mathcal{J}$ is a skew-symmetric operator in $L^2 \times L^2$ space, for instance. Now it is easy to see that the Hamiltonian H in (2.6) is conserved:

$$\partial_t H(n,u) = \int \frac{\delta H}{\delta n}\partial_t n + \frac{\delta H}{\delta u}\partial_t u \, dx = -\int \frac{\delta H}{\delta n}\partial_x \frac{\delta H}{\delta u} + \frac{\delta H}{\delta u}\partial_x \frac{\delta H}{\delta n}\, dx = 0.$$

In what follows, we derive (2.5) in another way. It is easy to see that (2.5a) follows integrating (2.1a) in x over $\mathbb{R}^3$ and then applying the divergence theorem. From (2.1a) and (2.1b), we obtain

$$\partial_t(nu) = -\nabla\cdot((nu)\otimes u) - \nabla p - n\nabla\phi, \tag{2.10}$$

where

$$\nabla\cdot((nu)\otimes u) = \left(\nabla\cdot(nu_1 u), \nabla\cdot(nu_2 u), \nabla\cdot(nu_3 u)\right)^T.$$

Using (2.1c) and integrating by parts, one has that

$$\begin{aligned} -n\nabla\phi &= (\Delta\phi - e^{\phi})\nabla\phi \\ &= -\phi\nabla\Delta\phi + \nabla(\phi\Delta\phi) - \nabla e^{\phi} \\ &= \nabla\left(\frac{|\nabla\phi|^2}{2}\right) - \begin{pmatrix} \nabla\cdot(\phi\nabla\partial_{x_1}\phi) \\ \nabla\cdot(\phi\nabla\partial_{x_2}\phi) \\ \nabla\cdot(\phi\nabla\partial_{x_3}\phi) \end{pmatrix} + \nabla(\phi\Delta\phi) - \nabla e^{\phi}. \end{aligned} \tag{2.11}$$

Now (2.5b) follows from (2.11) into (2.10) by applying the divergence theorem.

Taking the dot product of (2.1b) with u, and then using (2.1a), it is straightforward to see that for $p(n) = Kn$,

$$\partial_t\left(\frac{n|u|^2}{2} + K\left(n\ln n - (n-1)\right)\right) + \nabla\cdot\left(\frac{n|u|^2}{2}u + Knu\ln n\right) = -nu\cdot\nabla\phi, \tag{2.12}$$

and for $p(n) = An^{\gamma}$,

$$\partial_t\left(\frac{n|u|^2}{2} + \frac{p}{\gamma-1}\right) + \nabla\cdot\left(\frac{n|u|^2}{2}u + \frac{\gamma}{\gamma-1}pu\right) = -nu\cdot\nabla\phi. \tag{2.13}$$

Using (2.1a) and (2.1c), we obtain

$$\begin{aligned} -nu\cdot\nabla\phi &= \nabla\cdot(nu)\phi - \nabla\cdot(nu\phi) \\ &= -\partial_t n\,\phi - \nabla\cdot(nu\phi) \\ &= \Delta\partial_t\phi\,\phi - e^{\phi}\partial_t\phi\,\phi - \nabla\cdot(nu\phi) \\ &= -\nabla\partial_t\phi\cdot\nabla\phi + \nabla\cdot(\nabla\phi_t\phi) - \partial_t\left((\phi-1)e^{\phi}\right) - \nabla\cdot(nu\phi) \\ &= -\partial_t\left(\frac{|\nabla\phi|^2}{2}\right) - \partial_t\left((\phi-1)e^{\phi}\right) + \nabla\cdot(\nabla\phi_t\phi) - \nabla\cdot(nu\phi) \end{aligned} \tag{2.14}$$

From (2.12)–(2.14), we get (2.5c).

2.2 Model Derivations

We start from the following fundamental model of plasmas:

The two-fluid Euler-Maxwell system

$$
\begin{cases}
\partial_T n_i + \nabla \cdot (n_i u_i) = 0, & (2.15a)\\
\partial_T n_e + \nabla \cdot (n_e u_e) = 0, & (2.15b)\\
m_i n_i(\partial_T u_i + (u_i \cdot \nabla) u_i) = -\nabla p_i + n_i e \left(E + \dfrac{u_i \times B}{c} \right), & (2.15c)\\
m_e n_e(\partial_T u_e + (u_e \cdot \nabla) u_e) = -\nabla p_e - n_e e \left(E + \dfrac{u_e \times B}{c} \right), & (2.15d)\\
\nabla \times B = \dfrac{4\pi e}{c}(n_i u_i - n_e u_e) + \dfrac{1}{c}\partial_T E, & (2.15e)\\
\nabla \times E = -\dfrac{1}{c}\partial_T B, & (2.15f)\\
\nabla \cdot E = 4\pi e(n_i - n_e), & (2.15g)\\
\nabla \cdot B = 0, & (2.15h)
\end{cases}
$$

where $n_i, n_e : (T, X) \in \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ are the density functions for the ion and the electron, $u_i, u_e : (T, X) \in \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ are the velocity fields of the ion and electron, $E, B : (T, X) \in \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ are the electric field and the magnetic field, and c and $\pm e$ are physical constants representing the speed of light and the charge of an ion and an electron. Here, p_i and p_e are the pressure of the ion and electron, respectively.

Before we proceed, we briefly discuss some underlying assumptions in (2.15) and the meaning of equations of (2.15). First of all, the model (2.15) assumes that the plasma is fully ionized (there is no neutral particle) and that it is composed of electrons and singly charged ions (the charge of an ion particle is e). A neutral gas may interact with the components of a plasma through collisions.

The equations (2.15a)–(2.15b), called *the mass conservation law for ions and electrons*, say that there is no net creation (or loss) of ions and electrons. If the recombination rate of a plasma is not negligible, one needs to consider the relevant source (or sink) terms. The equations (2.15c)–(2.15d), called *the momentum balance equations for ions and electrons*, are simply the fluid version of Newton's second law. It is assumed that the net force applied to the infinitesimal volume elements of the ions (or the electrons) following the flow is the sum of the electromagnetic force and the pressure gradient force. The ions and the electrons interact with each other through the electromagnetic forces generated by *Maxwell's equations* (2.15e)–(2.15h). For the adiabatic flow, the pressure is given by $p(n) = An^\gamma$, where $A > 0$ and $\gamma > 1$ are constants. For the isothermal flow, $p(n) = k_B T n$, where k_B is the Boltzmann constant and T is a constant temperature. The choice of the pressure law depends on the physical situations.

We remark that the system (2.15) has 14 unknown functions and 16 equations. By taking divergence of (2.15e) and (2.15f), however, we see that (2.15a)–(2.15b) implies (2.15g)–(2.15h).

The two-fluid Euler-Poisson system If we assume that a plasma is electrostatic[5], we have $\nabla \times E = 0$ from (2.15f). Hence, there exists the electrostatic potential function $\Phi : (T, X) \in \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ satisfying $E = -\nabla\Phi$, and (2.15) becomes *the two-fluid Euler-Poisson system for ions and electrons*:

$$\begin{cases} \partial_T n_i + \nabla \cdot (n_i u_i) = 0, & (2.16a) \\ \partial_T n_e + \nabla \cdot (n_e u_e) = 0, & (2.16b) \\ m_i n_i(\partial_T u_i + (u_i \cdot \nabla)u_i) + \nabla p_i = -n_i e \nabla\Phi, & (2.16c) \\ m_e n_e(\partial_T u_e + (u_e \cdot \nabla)u_e) + \nabla p_e = n_e e \nabla\Phi, & (2.16d) \\ -\Delta\Phi = 4\pi e(n_i - n_e). & (2.16e) \end{cases}$$

Here, $(n_i, u_i, n_e, u_e, \Phi) = (n_{i0}, 0, n_{e0}, 0, 0)$ with a positive constant $n_{i0} = n_{e0} > 0$ is a uniform state solution of (2.16). We remark that by taking divergence of (2.15e) and using (2.15a)–(2.15b), we obtain $\partial_T (4\pi e(n_i - n_e) + \Delta\Phi) = 0$, which is equivalent to (2.16e).

The Euler-Poisson system for ions We assume that the electron mass m_e is zero and that the electron pressure is isothermal, $p_e(n_e) = k_B T_e n_e$. Then, (2.16d) becomes

$$k_B T_e \nabla n_e = n_e e \nabla\Phi. \tag{2.17}$$

Dividing (2.17) by n_e, integrating the resulting equation, and then imposing that $\Phi \to 0$ and $n_e \to n_{e0}$ as $|X| \to \infty$, we derive the *Boltzmann relation* for electrons:

$$n_e = n_{e0} \exp\left(\frac{e\Phi}{k_B T_e}\right). \tag{2.18}$$

The physical meaning of the Boltzmann relation is that due to their small inertia, electrons almost instantaneously react to the plasma fluctuation so that the pressure gradient and electrostatic forces acting on them are balanced (see (2.17)).

Assuming that the isothermal pressure for the ion, $p_i(n_i) = k_B T_e n_i$, we have *the one-fluid Euler-Poisson system for ions*

$$\begin{cases} \partial_T n_i + \nabla \cdot (n_i u_i) = 0, \\ m_i n_i(\partial_T u_i + (u_i \cdot \nabla)u_i) + k_B T_i \nabla n_i = -n_i e \nabla\Phi, \\ -\Delta\Phi = 4\pi e \left[n_i - n_{e0} \exp\left(\frac{e\Phi}{k_B T_e}\right)\right]. \end{cases} \tag{2.19}$$

Now, (2.1) is obtained upon an appropriate non-dimensionalization

$$x = \frac{X}{\sqrt{k_B T_e / 4\pi n_{e0} e^2}}, \quad t = T\sqrt{\frac{4\pi n_{e0} e^2}{m_i}}, \quad n = \frac{n_i}{n_{e0}}, \quad u = \frac{u_i}{\sqrt{k_B T_e / m_i}}, \quad \phi = \frac{e\Phi}{k_B T_e}. \tag{2.20}$$

For the adiabatic pressure, (2.20) leads $p(n) := \dfrac{p_i(n_{e0} n)}{n_{e0} k_B T_e} = \dfrac{A_i (n_{e0} n)^\gamma}{n_{e0} k_B T_e} =: A n^\gamma$.

[5]Indeed, electrostatic assumption only requires that $\partial_T B$ is zero or negligible. On the other hand, if the motion of a plasma is one-dimensional (plane wave), then the magnetic force terms in the momentum equations become zero.

3 Small Amplitude Limit of Solitary Waves for the Euler-Poisson System

3.1 Introduction

We consider the one-dimensional EP system

$$\begin{cases} \partial_t n + \partial_s(nu) = 0, \\ \partial_t u + u\partial_s u + K\dfrac{\partial_s n}{n} = -\partial_s \phi, \\ \partial_s^2 \phi = e^{\phi} - n, \end{cases} \tag{1DEP}$$

where $K \geq 0$ is a constant, with the far-field conditions

$$n \to 1, \quad u \to 0, \quad \phi \to 0 \quad \text{as} \quad |s| \to \infty, \tag{FC}$$

We aim to show that for $V = \sqrt{1+K}$, *the ion sound speed*, there holds

$$\sup_{\xi\in\mathbb{R}} \left[\left(|n - 1 - \varepsilon n_{\text{KdV}}| + |u - \varepsilon V n_{\text{KdV}}| + |\phi - \varepsilon n_{\text{KdV}}| \right) e^{\alpha|\xi|/2} \right] \leq C\varepsilon^2 \tag{3.1}$$

for all small $\varepsilon > 0$, where

$$\xi := \varepsilon^{1/2} \left(s - (V + \gamma\varepsilon)t \right), \tag{3.2}$$

and

$$n_{\text{KdV}}(\xi) := \frac{3\gamma}{V} \text{sech}^2 \left(\sqrt{\frac{V\gamma}{2}} \xi \right). \tag{3.3}$$

is the solitary wave solution to the KdV equation

$$\partial_{\bar{t}} v + V v \partial_{\bar{x}} v + \frac{1}{2V} \partial_{\bar{x}}^3 v = 0 \tag{KdV}$$

traveling with a speed $\gamma > 0$, that is, $\xi = \bar{x} - \gamma\bar{t}$.

In what follows, we illustrate how (1DEP) is related to (KdV), and then critically discuss some related results. We introduce two approaches: the reductive perturbation method [34] and the Sagdeev potential method [31].

Reductive perturbation[6] We present a formal derivation of (KdV) from (1DEP). This result was first found in [34] for the pressureless case.

By introducing a specific scaling, called the Gardner-Morikawa transformation,[7]

$$\bar{x} = \varepsilon^{1/2}(s - Vt), \quad \bar{t} = \varepsilon^{3/2} t, \tag{GM}$$

[6] As its name indicates, this method can be applied to reduce a complicated system into a simple scalar equation.

[7] For more details on the Gardner-Morikawa transformation, we refer to [15].

we obtain from (1DEP) that

$$\begin{cases} \varepsilon\partial_{\bar{t}}n - V\partial_{\bar{x}}n + \partial_{\bar{x}}(nu) = 0, \\ \varepsilon\partial_{\bar{t}}u - V\partial_{\bar{x}}u + u\partial_{\bar{x}}u + K\dfrac{\partial_{\bar{x}}n}{n} = -\partial_{\bar{x}}\phi, \\ \varepsilon\partial_{\bar{x}}^2\phi = e^{\phi} - n. \end{cases} \tag{3.4}$$

We suppose that the solutions to (3.4) is given by

$$n = 1 + \sum_{k=1}^{\infty}\varepsilon^k n_k, \quad u = \sum_{k=1}^{\infty}\varepsilon^k u_k, \quad \phi = \sum_{k=1}^{\infty}\varepsilon^k\phi_k. \tag{3.5}$$

Then we substitute (3.5) into (3.4) and set the coefficients of ε^n zero.

The coefficients of ε^0: We do not have ε^0 order terms.

The coefficients of ε^1: At the order of ε, we have

$$\begin{pmatrix} -V & 1 & 0 \\ K & -V & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \partial_{\bar{x}}n_1 \\ \partial_{\bar{x}}u_1 \\ \partial_{\bar{x}}\phi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.6}$$

Hence, in order to have a non-trivial first order profiles, V must satisfy $V^2 = 1 + K$. We choose $V = \sqrt{1+K}$. From (3.6), we have the relation[8]

$$u_1 = Vn_1, \quad \phi_1 = n_1. \tag{3.7}$$

The coefficients of ε^2: At the order of ε^2, we get

$$\begin{cases} \partial_{\bar{t}}n_1 - V\partial_{\bar{x}}n_2 + \partial_{\bar{x}}u_2 + n_1\partial_{\bar{x}}u_1 + \partial_{\bar{x}}n_1u_1 = 0, & \text{(3.8a)} \\ \partial_{\bar{t}}u_1 - V\partial_{\bar{x}}u_2 + u_1\partial_{\bar{x}}u_1 + K(\partial_{\bar{x}}n_2 - n_1\partial_{\bar{x}}n_1 = -\partial_{\bar{x}}\phi_2, & \text{(3.8b)} \\ \partial_{\bar{x}}^2\phi_1 = \phi_2 + \dfrac{1}{2}(\phi_1)^2 - n_2. & \text{(3.8c)} \end{cases}$$

We multiply (3.8a) by V, differentiate (3.8c) in x, and then add each equation of (3.8) together. Then the second order profiles (n_2, u_2, ϕ_2) are canceled since $V = \sqrt{1+K}$. Using the relation (3.7), it is straightforward to see that n_1 satisfies the KdV equation (KdV).

A mathematical validity of the formal expansion (3.5) has been studied in [16], where it is shown that on any fixed time interval, the solutions to (3.4) with some *well-prepared* initial data converge to the solutions to (KdV) as ε tends to zero.

Sagdeev potential We present a formal approximation of the EP solitary wave solutions to in terms of the KdV solitary wave solutions. This result was first introduced in [31]. We also

[8] $V = \sqrt{1+K}$ and (3.7) imply that (3.6) is not overdetermined.

refer to [5] and [9].

We assume that $(n, u, \phi)(\bar{\xi})$, where $\bar{\xi} = s - Mt$ for a constant $M > 0$, is a solution to (1DEP). By imposing $n - 1, u, \phi \to 0$ as $\bar{\xi} \to -\infty$, we have

$$\begin{cases} (M-u)\,n = M, & (3.9a) \\ (M-u)^2 + 2K\ln n = M^2 - 2\phi, & (3.9b) \\ \partial_{\bar{\xi}}^2 \phi = e^{\phi} - n. & (3.9c) \end{cases}$$

If $n > 0$, then $M - u > 0$ since $M > 0$. Hence, n is explicitly expressed in terms of ϕ, and we obtain from (3.9) that

$$\partial_{\bar{\xi}}^2 \phi = e^{\phi} - \frac{M}{\sqrt{M^2 - 2\phi}} =: \frac{\partial}{\partial \phi} U(\phi), \tag{3.10}$$

where $-U(\phi)$ is called the *Sagdeev potential.* Multiplying (3.10) by $\partial_{\bar{\xi}}\phi$ and imposing $\partial_{\bar{\xi}}\phi \to 0$ as $\bar{\xi} \to -\infty$, we have

$$\frac{1}{2}(\partial_{\bar{\xi}}\phi)^2 = e^{\phi} + M\sqrt{M^2 - 2\phi} - (1 + M^2) = U(\phi). \tag{3.11}$$

From (3.11), we see that $U(\phi)$ must be positive at least for small $0 < \phi < \frac{M^2}{2}$. By expanding $U(\phi)$ around $\phi = 0$, we find

$$U(\phi) = \left(\frac{1}{2} - \frac{1}{2M^2}\right)\phi^2 + O(\phi^3),$$

and thus, we must have $M^2 > 1$. Since we assumed that ϕ is a solitary wave, there is some $\xi_* \in \mathbb{R}$ such that $\partial_{\bar{\xi}}\phi(\xi_*) = 0$. Hence, we must have $U(\phi) = 0$ for some $0 < \phi < \frac{M^2}{2}$, and this happens only if

$$\exp\left(\frac{M^2}{2}\right) - (1 + M^2) = U\left(\frac{M^2}{2}\right) < 0.$$

Thus $1 < M < \zeta_0$, where ζ_0 is a unique positive root of (3.25), is a necessary condition for the existence of solitary wave solutions to (3.10). In fact, via a phase plane analysis, it is shown in [22] that $1 < M < \zeta_0$ is a sufficient condition for the existence of solitary wave solutions to (3.10).

Now we let $M = 1 + \gamma\varepsilon$ for sufficiently small $\gamma\varepsilon > 0$ and assume that ϕ is small. By expanding the RHS of (3.10) for, one can obtain

$$\partial_{\bar{\xi}}^2 \phi - 2\gamma\varepsilon\phi + \phi^2 = O\left(|\phi|(|\gamma\varepsilon|^2 + |\phi|^2)\right).$$

By neglecting the RHS terms, we formally obtains

$$\phi \approx 3\gamma\varepsilon\,\mathrm{sech}^2\left(\sqrt{2^{-1}\gamma\varepsilon}\,\bar{\xi}\right) = 3\gamma\varepsilon\,\mathrm{sech}^2\left(\sqrt{2^{-1}\gamma\varepsilon}\,[s - (1 + \gamma\varepsilon)t]\right). \tag{3.12}$$

Here we can rewrite the argument of (3.12) as

$$(\sqrt{2})^{-1}\left[(\gamma\varepsilon)^{1/2}(s - t) - (\gamma\varepsilon)^{3/2}t\right], \tag{3.13}$$

and this suggests that (GM) is a suitable transformation for (1DEP) to detect solitary waves with the amplitude of order $O(\gamma\varepsilon)$. By introducing the scaling (3.2), we see that the RHS of (3.12) is nothing but $\varepsilon n_{\text{KdV}}$ for $K = 0$.

Discussion While the result of [16] well describes the asymptotic behavior of small amplitude solutions to (1DEP) up to the order of $t = O(\varepsilon^{-3/2})$, this setting does not give a fully satisfactory answer to the KdV limit of the solitary waves for the EP system. Also, the formal result of [31] for (3.12) has not yet been completely justified in a rigorous manner.

The main results of this section assert that the formal expansion (3.5) is valid in the presence of solitary waves, and also justify the formal approximation (3.12). Moreover our results covers the isothermal case.

Remark 1. In the previous section, we observed that up to third order, the dispersion relation (2.4) of the EP system is that of a linear KdV equation

$$\partial_t v + \sqrt{1+K}\partial_s v + (2\sqrt{1+K})^{-1}\partial_x^3 v = 0,$$

or equivalently,

$$\partial_{\tilde{t}} v + \frac{1}{2\sqrt{1+K}}\partial_{\tilde{x}}^3 v = 0 \tag{3.14}$$

with the change of variable

$$\tilde{x} = s - \sqrt{1+K}t, \quad \tilde{t} = t. \tag{3.15}$$

This suggests that (3.15) is a suitable moving frame to obtain (KdV) from (1DEP). Now we consider the nonlinear term $v\partial_{\tilde{x}} v$ and (3.14) together to find an appropriate time and length scale for a small amplitude $v = O(\varepsilon)$. By setting

$$\tilde{t} = \varepsilon^{\alpha}\bar{t}, \quad \tilde{x} = \varepsilon^{\beta}\bar{x} \tag{3.16}$$

for $\bar{x}, \bar{t} = O(1)$, we get

$$\partial_{\tilde{t}} v \to \varepsilon^{1-\alpha}, \quad v\partial_{\tilde{x}} v \to \varepsilon^{2-\beta}, \quad \partial_{\tilde{x}}^3 v \to \varepsilon^{1-3\beta}.$$

In light of that the existence of solitary wave solutions to the KdV equation is due to the exact balance between nonlinear transport and dispersion effect, we first set $\beta = -1/2$ and, accrdingly, set $\alpha = -3/2$. This choice of α and β, together with (3.15)–(3.16), lead the transformation (GM) with $V = \sqrt{1+K}$.

3.2 Main Results

We plug $(n, u, \phi)(\xi)$ into (1DEP)–(FC), where ξ is given by (3.2). Then we obtains

$$\begin{cases} -(V+\gamma\varepsilon)n' + (nu)' = 0, & (3.17a) \\ -(V+\gamma\varepsilon)u' + uu' + K\dfrac{n'}{n} = -\phi', & (3.17b) \\ \varepsilon\phi'' = e^{\phi} - n, & (3.17c) \end{cases}$$

with the far-field condition

$$n \to 1, \quad u \to 0, \quad \phi \to 0 \ \text{ as } \ |\xi| \to \infty, \tag{3.18}$$

where $'$ denotes the derivative in ξ. We note that (3.17) has the translation invariance.

Definition. The solution (n, u, ϕ) to (3.17)–(3.18) is called *solitary wave* if the following hold:

(i) (symmetry)

$$n(\xi) = n(-\xi), \quad u(\xi) = u(-\xi), \quad \phi(\xi) = \phi(-\xi) \ \text{ for } \xi \in \mathbb{R}, \tag{3.19}$$

(ii) (monotonicity)

$$n'(\xi), \quad u'(\xi), \quad \phi'(\xi) < 0 \quad \text{for } \xi \in (0, \infty). \tag{3.20}$$

We observe that (n, u, ϕ) satisfying (3.18)–(3.20) have their unique maximum values (n_*, u_*, ϕ_*) at $\xi = 0$, that is,

$$(n, u, \phi)(0) = (n_*, u_*, \phi_*), \tag{3.21}$$

and there hold

$$n(\xi) > 1, \quad u(\xi) > 0, \quad \phi(\xi) > 0 \quad \text{for } \ \xi \in \mathbb{R}. \tag{3.22}$$

To present the existence theorem, we define some parameters. When $K > 0$, let ζ_K be a unique root of

$$z^K \left[K(z-1)^2 + 1 \right] = \exp\left(\frac{K}{2} \left(z^2 - 1 \right) \right) \tag{3.23}$$

satisfying $\zeta_K > \sqrt{\dfrac{1+K}{K}} > 1$. We have that

$$z^K \left[K(z-1)^2 + 1 \right] > \exp\left(\frac{K}{2} \left(z^2 - 1 \right) \right) \ \text{ for } z \in (1, \zeta_K). \tag{3.24}$$

For the case $K = 0$, let ζ_0 be the unique positive root of

$$z^2 + 1 = \exp(z^2/2). \tag{3.25}$$

It is east to check that $\zeta_0 > 1$ and

$$z^2 + 1 > \exp(z^2/2) \ \text{ for } z \in (0, \zeta_0). \tag{3.26}$$

We refer to Appendix for (3.23)–(3.24).

Let (V, γ, ε) be the positive numbers satisfying

$$\begin{cases} \sqrt{\dfrac{1+K}{K}} < \dfrac{V + \gamma\varepsilon}{\sqrt{K}} < \zeta_K & \text{when } \ K > 0, \quad (3.27\text{a}) \\ 1 < V + \gamma\varepsilon < \zeta_0 & \text{when } \ K = 0. \quad (3.27\text{b}) \end{cases}$$

Theorem 3.1. *Suppose that* (V, γ, ε) *satisfies* (3.27)*. Then the equation* (3.17)–(3.18) *admits a unique (up to a shift) non-trivial smooth solution* (n, u, ϕ)*. Moreover, it satisfies* (3.19)–(3.22) *upon a suitable shift of phase.*

Remark 2. By inspection, one can check that (3.27) is necessary for the existence of non-trivial smooth solutions to (3.17)–(3.18).

Theorem 3.1 holds true as long as (3.27) is satisfied. However, for Theorem 3.2, we restrict our analysis to the case $V = \sqrt{K+1}$ and consider only $\varepsilon > 0$ as a parameter for fixed V and γ.

For the solitary wave solution $(n^\varepsilon, u^\varepsilon, \phi^\varepsilon)$ to (3.17)–(3.18) satisfying (3.19)–(3.22), we define the remainders as

$$n_R^\varepsilon := n^\varepsilon - 1 - \varepsilon n_{\mathrm{KdV}}, \quad u_R^\varepsilon := u^\varepsilon - \varepsilon V n_{\mathrm{KdV}}, \quad \phi_R^\varepsilon := \phi^\varepsilon - \varepsilon n_{\mathrm{KdV}}. \tag{3.28}$$

We denote the k-th derivative of f in z by $f^{(k)}$. We present the main theorem on the asymptotic behavior of the EP solitary waves.

Theorem 3.2 (Bae and Kwon, [1])**.** *Let* $V = \sqrt{1+K}$*,* $K \geq 0$ *and* $\gamma > 0$ *be fixed. Let* k *be any non-negative integer. Then there exist positive constants* ε_1*,* α*, and* $C_k > 0$ *such that for all* $0 < \varepsilon < \varepsilon_1$*,*

$$\sup_{\xi \in \mathbb{R}} \left| e^{\alpha|\xi|/2} ({n_R^\varepsilon}^{(k)}, {u_R^\varepsilon}^{(k)}, {\phi_R^\varepsilon}^{(k)})(\xi) \right| \leq C_k \varepsilon^2 \tag{3.29}$$

holds. Here α *and* C_k *are independent of* ε*, and* α *is independent of* k*.*

Theorem 3.1–3.2 assert that the EP system admits solitary wave solutions traveling slightly faster than the *ion sound speed* $V = \sqrt{1+K}$, and that they are well approximated by solitary wave solutions to the KdV equation.

For the proof of Theorem 3.1, we derive a system of first-order ODEs, equivalent to (3.17)-(3.18). Then we employ a phase plane analysis in a similar fashion as [8]. To prove Theorem 3.2, we derive the remainder equation for ϕ_R^ε:

$$\begin{cases} {\phi_R^\varepsilon}'' - F_\varepsilon \phi_R^\varepsilon = \mathcal{M}_3^\varepsilon, \\ F_\varepsilon(\xi) = 2V\gamma - 2V^2 n_{\mathrm{KdV}} - V^2 \dfrac{\phi_R^\varepsilon}{\varepsilon}, \end{cases} \tag{3.30}$$

where $\mathcal{M}_3^\varepsilon$ is a function of $n_{\mathrm{KdV}}, n_R^\varepsilon, u_R^\varepsilon$ and ϕ_R^ε. In the derivation of (3.30), the choice of the ion sound speed $V = \sqrt{1+K}$ and the fact that n_{KdV} satisfies the associated KdV equation are crucially used. One of the main difficulties in the analysis of (3.30) stems from the indefinite sign of $F_\varepsilon(\xi)$. Indeed, from a careful observation of the phase plane analysis, we obtain a *sharp* estimate for the peak values of the solitary wave solution. We set $(n_*^\varepsilon, u_*^\varepsilon, \phi_*^\varepsilon) := (n^\varepsilon, u^\varepsilon, \phi^\varepsilon)(0)$.

Proposition 3.3. *Let* $V = \sqrt{1+K}$*,* $K \geq 0$ *and* $\gamma > 0$ *be fixed. Then for all* $0 < \varepsilon < \varepsilon_0$*,*

$$\left|n_*^\varepsilon - 1 - 3\gamma V^{-1}\varepsilon\right| + \left|u_*^\varepsilon - 3\gamma\varepsilon\right| + \left|\phi_*^\varepsilon - 3\gamma V^{-1}\varepsilon\right| \leq \varepsilon^2 C, \tag{3.31}$$

Moreover, $V = \sqrt{1+K}$ *is necessary for* $\lim_{\varepsilon \to 0} n_*^\varepsilon = 1$*.*

From (3.31) and (3.18), we observe that $\lim_{\varepsilon\to 0} F_\varepsilon(0) = -4V\gamma < 0$ while $\lim_{\xi\to\infty} F_\varepsilon(\xi) = 2V\gamma$ for all ε. Hence we need to divide the interval $[0,\infty)$ into two parts and conduct the analysis separately.

We observe that the coefficient $3\gamma/V$ of ε in (3.31) is exactly the peak value of $n_{\mathrm{KdV}}(\xi)$ in (3.3). This implies that at least at $\xi = 0$, $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)$ is $O(\varepsilon^2)$ as $\varepsilon \to 0$. This fact together with Gronwall's inequality, we get *the local uniform estimate* for $\varepsilon^{-2}\phi_R^\varepsilon$ around $\xi = 0$ in Proposition 3.9.

To obtain *the local uniform decay estimate* for $\varepsilon^{-2}\phi_R^\varepsilon$ around $\xi = \infty$, we need a careful analysis. It is not clear if we may choose a uniform $\xi_1 > 0$ in such a way that $F_\varepsilon(\xi)$ has a positive sign on $[\xi_1, \infty)$ for all ε. In other words, as $\varepsilon \to 0$, the time $\xi_\varepsilon > 0$ at which $F_\varepsilon(\xi_\varepsilon) = 0$ is realized can tend to ∞. Verifying existence of such a uniform time ξ_1 is a important step in obtaining the uniform estimate for the remainders in the far-field region. To this end, we obtain the uniform lower bounds for the speed of trajectory curves $\varepsilon^{-1}(n^\varepsilon, E^\varepsilon)(\xi)$ in Lemma 3.5–3.6, where the estimate (3.31) plays a crucial role again. Then we obtain the uniform decay estimate for $\varepsilon^{-1}(n^\varepsilon - 1, u^\varepsilon, \phi^\varepsilon)(\xi)$ (Proposition 3.7), which yields *the local uniform decay estimate* for $\varepsilon^{-2}\phi_R^\varepsilon$ around $\xi = \infty$ (Proposition 3.10). Using that ϕ_R^ε is symmetric about $\xi = 0$ together with Proposition 3.9–3.10, we obtain *the uniform decay estimate* for $\varepsilon^{-2}\phi_R^\varepsilon$ on $\mathbb{R}$. The estimates for $\varepsilon^{-2}n_R^\varepsilon$ and $\varepsilon^{-2}u_R^\varepsilon$ immediately follow from (3.115).

Some numerical tests for the convergence of $\varepsilon^{-1}(n^\varepsilon - 1)$ to n_{KdV} are presented in Figure 2. We employ RK4 to solve the ODE system (3.39) with the suitably chosen initial values by using the first integral (3.43)–(3.44).

Remark 3. Unlike the pressureless case, the isothermal system can not be reduced to the explicit second-order ODE. Instead we consider the explicit system of ODEs (3.39).

Notation: In this section, we set

$$J = J(\varepsilon) := \sqrt{1+K} + \gamma\varepsilon = V + \gamma\varepsilon. \tag{3.32}$$

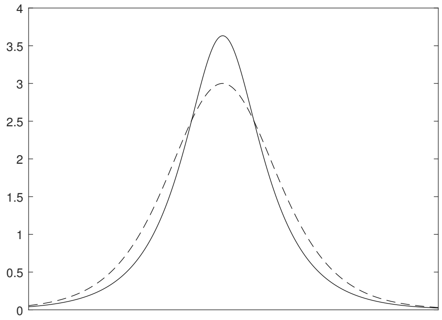
3.3 Existence of Solitary Waves

We reduce (3.17) to a system of first-order ODEs and prove the existence theorem via a phase plane analysis.

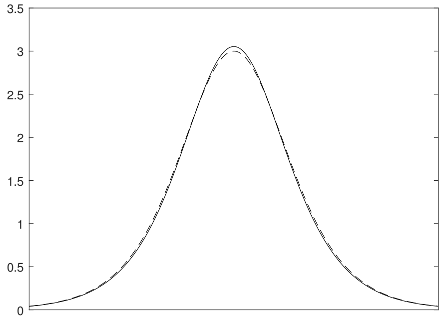
3.3.1 Reduction to the System of First-order ODEs

We assume (n, u, ϕ) is a solution of (3.17)–(3.18). Integrating (3.17a)–(3.17b) in ξ, we get

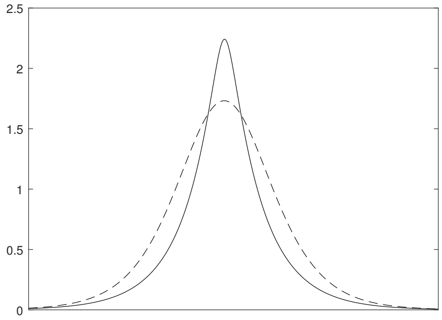
$$\begin{cases} -(V+\gamma\varepsilon)n + nu = -(V+\gamma\varepsilon), & \text{(3.33a)} \\ -(V+\gamma\varepsilon)u + \dfrac{1}{2}u^2 + K\ln n = -\phi. & \text{(3.33b)} \end{cases}$$



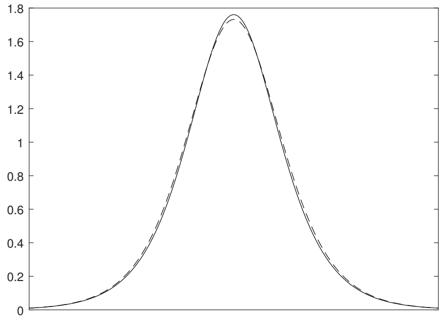
(a) $K = 0$ and $\varepsilon = 0.1$



(b) $K = 0$ and $\varepsilon = 0.01$



(c) $K = 2$ and $\varepsilon = 0.1$



(d) $K = 2$ and $\varepsilon = 0.01$

Figure 2: Numerical tests for comparison of n_{KdV} (dashed) with $\varepsilon^{-1}(n^\varepsilon - 1)$ (solid) in the frame $\xi = \varepsilon^{1/2}(x - (\sqrt{1+K} + \varepsilon t))$.

We note that (3.33a) is solvable for u in terms of n. Hence (3.33) is written as

$$\begin{cases} u = (V + \gamma\varepsilon)\left(1 - \dfrac{1}{n}\right), & \text{(3.34a)} \\ \phi = H(n), & \text{(3.34b)} \end{cases}$$

where

$$H(n) := \frac{(V+\gamma\varepsilon)^2}{2}\left(1 - \frac{1}{n^2}\right) - K \ln n. \tag{3.35}$$

We differentiate (3.34b) in ξ to obtain

$$\phi' = h(n)n', \tag{3.36}$$

where

$$h(n) := \frac{dH(n)}{dn} = \frac{(V+\gamma\varepsilon)^2}{n^3} - \frac{K}{n}. \tag{3.37}$$

We define

$$E(\xi) := -\phi'(\xi). \tag{3.38}$$

From (3.36) and (3.17c), we then obtain an *ODE system for* (n, E):

$$\begin{cases} -h(n)n' = E, & (3.39\text{a}) \\ \varepsilon E' = n - e^{H(n)}. & (3.39\text{b}) \end{cases}$$

Multiplying (3.39b) by E, we have from (3.39a) and (3.37) that

$$\begin{aligned} \frac{\varepsilon}{2}\left(E^2\right)' &= -nh(n)n' + e^{H(n)}h(n)n' \\ &= \left(\frac{(V+\gamma\varepsilon)^2}{n} + Kn + e^{H(n)}\right)'. \end{aligned} \tag{3.40}$$

We integrate (3.40) in ξ to obtain

$$\frac{\varepsilon}{2}E^2 = \frac{(V+\gamma\varepsilon)^2}{n} + Kn + e^{H(n)} + c \tag{3.41}$$

for some constant c. From (3.41) and (3.18), n and E must satisfy the far-field condition

$$n(\xi) \to 1, \quad E(\xi) \to 0, \quad \text{as} \quad |\xi| \to \infty. \tag{3.42}$$

Conversely, we assume that (n, E) satisfies (3.39) and (3.42). Then (n, u, ϕ), where u and ϕ are defined by (3.34), satisfies (3.17a)-(3.17b) and (3.18). Moreover, we have $E = -\phi'$ from (3.36) and (3.39a). Hence, by (3.39b), (n, u, ϕ) also satisfies (3.17c). We remark that (3.41) yields a first integral of (3.39) with (3.42):

$$\frac{\varepsilon}{2}E^2 - g(n) + g(1) = 0, \tag{3.43}$$

where

$$g(n) := \frac{(V+\gamma\varepsilon)^2}{n} + Kn + e^{H(n)}. \tag{3.44}$$

3.3.2 Stationary Points

Next we find the stationary points of (3.39). From (3.37), we observe that when $K > 0$,

$$h(n) \begin{cases} = 0, & (n = n_s), \\ < 0, & (n > n_s), \\ > 0, & (0 < n < n_s, \end{cases} \tag{3.45}$$

where

$$n_s := \frac{V+\gamma\varepsilon}{\sqrt{K}}, \tag{3.46}$$

and when $K = 0$,

$$h(n) > 0, \quad (n > 0). \tag{3.47}$$

We henceforth assume that

(A1) $n < n_s$ when $K > 0$.

It will be shown later that the assumption **(A1)** is valid. Then we have that $h > 0$ for $K \geq 0$, and the points at which

$$\frac{E}{h(n)} = 0, \quad n = e^{H(n)} \tag{3.48}$$

hold are the stationary points of (3.39). Thus the stationary points must lie on n axis. We note that

$$n = e^{H(n)} \quad \text{iff} \quad l(n) := \ln n - H(n) = 0 \tag{3.49}$$

for $n > 0$. It is easy to check that $l(n)$ strictly decreases on the interval $(0, n_c)$ and strictly increases on the interval (n_c, ∞), where

$$n_c := \frac{V + \gamma\varepsilon}{\sqrt{1+K}}. \tag{3.50}$$

The condition (3.27) implies that $1 < n_c$ for $K \geq 0$. Moreover, since $\lim_{n\to\infty} l(n) = \infty$ and $l(1) = 0$, we find that the function l has only two zeros $n = 1$ and $n = n_{ce}$ such that

$$1 < n_c < n_{ce}. \tag{3.51}$$

Therefore, (3.39) has only two stationary points $(n_{ce}, 0)$ and $(1, 0)$. We observe that for $K \geq 0$,

$$\begin{cases} n < e^{H(n)} & \text{for } n \in (1, n_{ce}), \\ n > e^{H(n)} & \text{for } n \in (n_{ce}, \infty). \end{cases} \tag{3.52}$$

3.3.3 Local Behavior

The Jacobian matrix of (3.39) is given by

$$\begin{pmatrix} \dfrac{E}{h^2}\dfrac{dh(n)}{dn} & \dfrac{-1}{h} \\ \dfrac{1}{\varepsilon}\left((1 - he^H\right) & 0 \end{pmatrix}. \tag{3.53}$$

At stationary points $(1, 0)$ and $(n_{ce}, 0)$, the trace of (3.53) is zero. Since $n = 1$ satisfies the equation (3.49), we obtain from (3.45)–(3.47) and (3.27) that

$$\frac{1 - h(1)e^{H(1)}}{\epsilon h(1)} = \frac{1 + K - (V + \gamma\varepsilon)^2}{\epsilon h(1)} < 0.$$

Thus, the stationary point $(1, 0)$ is saddle for $K \geq 0$. Since $n = n_{ce}$ also satisfies (3.49), we see from (3.50)–(3.51) that

$$\begin{aligned} 1 - h(n_{ce})e^{H(n_{ce})} &= 1 + K - \frac{(V + \gamma\varepsilon)^2}{(n_{ce})^2} \\ &= (1 + K)\left(1 - \frac{(n_c)^2}{(n_{ce})^2}\right) > 0. \end{aligned} \tag{3.54}$$

When $K = 0$, the stationary point $(n_{ce}, 0)$ is center by (3.47) and (3.54). On the other hand, when $K > 0$, $(n_{ce}, 0)$ can be center or saddle depending on the location of n_{ce} with respect to n_s (see (3.45)). We will see later that (3.27a) implies that

(A2) $n_{ce} < n_s$ when $K > 0$.

We observe that $(n_{ce}, 0)$ is a center under **(A2)**.

3.3.4 Direction of Vector Fields

From (3.52) and (3.39b), we see that $E' < 0$ in the region where $1 < n < n_{ce}$, and $E' > 0$ in the region where $n > n_{ce}$. Since $h(n) > 0$ by **(A1)**, (3.45) and (3.47), we find from (3.39a) that $n' > 0$ in the region where $E < 0$, and $n' < 0$ in the region where $E > 0$ (see Figure 3).

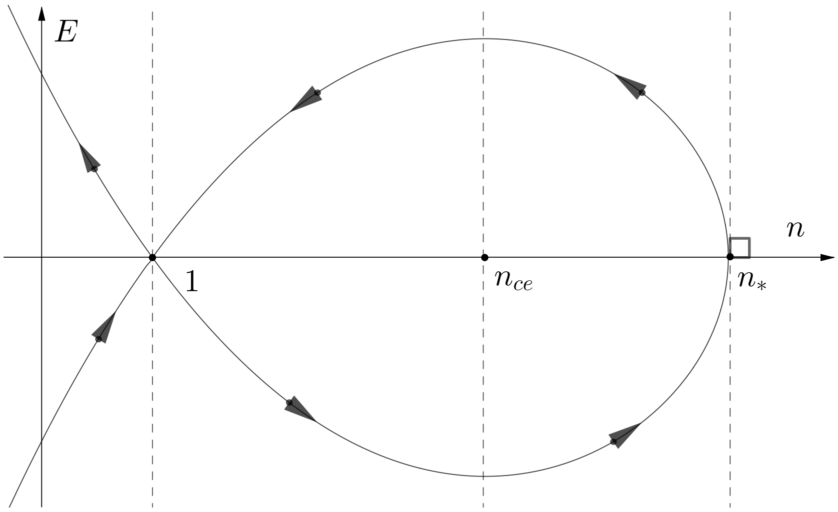


Figure 3: Trajectory curve

3.3.5 First Integral

We see from (3.43)–(3.44) that the trajectory starting from the point $(1, 0)$ with $E < 0$ satisfies $E(n) = -\frac{\sqrt{2(g(n) - g(1))}}{\sqrt{\varepsilon}}$. Taking the derivative in n,

$$\frac{dE}{dn}(n) = -\frac{dg}{dn}(n)\frac{1}{\sqrt{2\varepsilon(g(n) - g(1))}}, \tag{3.55}$$

where

$$\begin{aligned} \frac{dg}{dn}(n) &= -\frac{(V + \gamma\varepsilon)^2}{n^2} + K + h(n)e^{H(n)} \\ &= -h(n)\left(n - e^{H(n)}\right). \end{aligned} \tag{3.56}$$

We recall that $n = 1$ and $n = n_{ce}$ satisfy (3.49) and that they are only such points. Hence, in the case $K > 0$, (3.56) vanishes only at 1, n_{ce} and n_s. From **(A2)**, (3.45) and (3.52), we see that $g(n)$ strictly increases on $(1, n_{ce})$ and strictly decreases on (n_{ce}, n_s). This yields that if

$$g(1) > g(n_s), \tag{3.57}$$

there is a unique n_* such that (see Figure 4)

$$1 < n_{ce} < n_* < n_s, \quad g(n_*) = g(1). \tag{3.58}$$

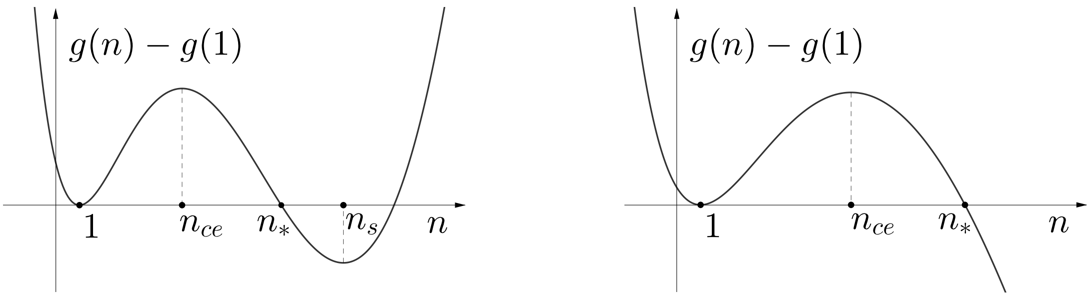
We shall show that (3.27a) implies (3.57) in Lemma 3.4.

When $K = 0$, $g(n)$ is strictly increasing on (n_{ce}, ∞) since $h(n) > 0$ for $n > 0$. By (3.26) and (3.27b), one has

$$\lim_{n\to\infty} g(n) = e^{(V+\gamma\varepsilon)^2/2} < (V+\gamma\varepsilon)^2 + 1 = g(1).$$

Hence there is a unique n_* satisfying

$$1 < n_{ce} < n_*, \quad g(n_*) = g(1). \tag{3.59}$$



(a) $K > 0$ (b) $K = 0$

Figure 4: Graphs of $g(n) - g(1)$

Now we show that the condition (3.27a) implies $g(1) > g(n_s)$.

Lemma 3.4. (3.27a) *implies* (3.57).

Proof of Lemma 3.4. From (3.44), it is enough to show that

$$(V+\gamma\varepsilon)^2 + K + 1 > \frac{(V+\gamma\varepsilon)^2}{n_s} + Kn^s + e^{H(n_s)}. \tag{3.60}$$

By the definition (3.46) of n_s, (3.60) can be written as

$$K(n_s)^2 + K + 1 > 2Kn_s + e^{H(n_s)}. \tag{3.61}$$

On the other hand, from (3.35) and (3.46), we have

$$\begin{aligned} H(n_s) &= \frac{(V+\gamma\varepsilon)^2}{2}\left(1 - \frac{1}{(n_s)^2}\right) - K\ln n_s \\ &= \frac{K}{2}\left((n_s)^2 - 1\right) - K\ln n_s. \end{aligned}$$

Therefore, (3.61) is equivalent to

$$K(n_s)^2 + K + 1 > 2Kn_s + (n_s)^{-K}\exp\left(\frac{K}{2}\left((n_s)^2 - 1\right)\right),$$

and equivalently, we have

$$(n_s)^K\left[K(n_s - 1)^2 + 1\right] > \exp\left(\frac{K}{2}\left((n_s)^2 - 1\right)\right). \tag{3.62}$$

From the definition of n_s, (3.24), and (3.62), we find that (3.27a) implies (3.57). □

Proof of Theorem 3.1. We show that Lemma 3.4 verifies the assumption **(A2)**. If $n_{ce} \geq n_s$, then by (3.45) and (3.52), we see from (3.56) that $g(n)$ is strictly increasing on $(1, n_s)$. This implies that $g(1) < g(n_s)$, which contradicts (3.57).

By the stable manifold theorem, a smooth solution (n, E) exponentially decaying to the saddle point $(1, 0)$ as $\xi \to -\infty$ exists. This solution can be extended until it reaches a neighbourhood of $(n_*, 0)$. By expanding $n(E)$ near $E = 0$, we find that there exists some integer $m > 1$ such that

$$n(E) - n(0) = E^m \left(\frac{d^m n}{dE^m}(0) + O(E) \right)$$

for $E < 0$ since from (3.55)-(3.56) and (3.58)–(3.59), we have

$$\lim_{n \to n_*^-} \frac{dE}{dn}(n) = +\infty \quad \text{for } K \geq 0.$$

Hence we have $E \sim -(n_* - n)^{1/m}$ for sufficiently small E. Using (3.39a), we have

$$\xi - \xi_0 = \int_{\xi_0}^{\xi} -\frac{h(n(\xi))}{E(\xi)} \frac{dn}{d\xi} \, d\xi = \int_{n_{\xi_0}}^{n_\xi} -\frac{h(n)}{E(n)} \, dn \lesssim \int_{n_{\xi_0}}^{n_\xi} \frac{h(n)}{(n_* - n)^{1/m}} \, dn.$$

The integral on the RHS converges as $n_\xi \to n_*$ since $m > 1$. Hence, the trajectory reaches at the point $(n_*, 0)$ in finite time. On the other hand, from (3.43), we observe that the phase portrait is symmetric about n-axis. The symmetric phase portrait together with the direction of vector fields yields that the trajectory follows a homoclinic orbit. Now the assumption **(A1)** is justified since $n_* < n_s$ from (3.58). Thus there exists a non-trivial smooth solution (n, E) of (3.39) satisfying (3.42), and equivalently, a non-trivial smooth solution (n, u, ϕ) of (3.17)–(3.18) exists. From (3.34), it is easy to check that (n, u, ϕ) satisfies (3.19)–(3.22) up to a shift so that (3.21) holds. □

3.4 Peak of Solitary Waves

The proof of Proposition 3.3 consists of four steps. We first prove that

$$\lim_{\varepsilon \to 0} n_*^\varepsilon = 1 \tag{3.63}$$

and that $V = \sqrt{1 + K}$ is a necessary choice for (3.63). We derive a rough estimate of $n_*^\varepsilon - 1$ in Step 2. In Step 3, this estimate will be used to obtain the sharp estimate of n_*^ε. In Step 4, we derive the sharp estimates of u_*^ε and ϕ_*^ε.

Proof of Proposition 3.3. Step 1: By (3.51) and (3.58) (or (3.59) for $K = 0$), we have $1 < n_c^\varepsilon < n_*^\varepsilon$. Together with (3.50), this implies that $V = \sqrt{1 + K}$ is necessary for (3.63).

When $K > 0$, we observe from (3.58)–(3.59) that n_*^ε is defined as a unique root such that $1 < n_*^\varepsilon < n_s^\varepsilon$ and $g(n_*^\varepsilon) = g(1)$ hold. We examine the behavior of the function $g(n)$ as ε tends to 0 considering the limiting case $\varepsilon = 0$. From (3.44), it is clear that for any $L > 1$, $g(n)$ uniformly converges to

$$g_0(n) := \frac{1 + K}{n} + Kn + \exp\left(\frac{1 + K}{2} - \frac{1 + K}{2n^2} - K \ln n \right)$$

as ε tends to 0 on the interval $[1, L]$. On the other hands, we have

$$\frac{dg_0}{dn} = -\left(\frac{1+K}{n^3} - \frac{K}{n}\right) \underbrace{\left[n - \exp\left(\frac{1+K}{2} - \frac{1+K}{2n^2} - K\ln n\right)\right]}_{=:\mathfrak{g}_0(n)}. \tag{3.64}$$

It is easy to see that $\mathfrak{g}_0(n)$ has a unique zero $n = 1$ and that

$$\mathfrak{g}_0(n) > 0 \quad \text{for} \quad n \in (1, \infty). \tag{3.65}$$

Hence, when $K > 0$, $\frac{dg_0}{dn}(n)$ vanishes only at 1 and $n_s^0 := \sqrt{\frac{1+K}{K}}$, and it follows from (3.64)–(3.65) that g_0 is strictly increasing on the interval $(1, n_s^0)$. This implies $g_0(n) < g_0(1)$ on $(1, n_s^0)$. Thus by the uniform convergence of g to g_0 and the construction of n_*^ε, (3.63) holds. Similarly, we can verify that (3.63) holds for $K = 0$.

Step 2: By expanding $g(1) = g(n_*^\varepsilon)$ about $n = 1$, we have

$$\frac{g^{(2)}(1)}{2}(n_*^\varepsilon - 1)^2 + \frac{g^{(3)}(1)}{3!}(n_*^\varepsilon - 1)^3 + \frac{g^{(4)}(n_b^\varepsilon)}{4!}(n_*^\varepsilon - 1)^4 = 0 \tag{3.66}$$

for some $1 < n_b^\varepsilon < n_*^\varepsilon$. Using that $\lim_{\varepsilon \to 0} n_*^\varepsilon = 1$, a direct calculation yields that there are constants $\varepsilon_0, C > 0$ and some functions $\mathfrak{g}_2$, $\mathfrak{g}_3$ of ε such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\begin{cases} g^{(2)}(1) = 2V\gamma\varepsilon + \varepsilon^2 \mathfrak{g}_2(\varepsilon), & \text{(3.67a)} \\ g^{(3)}(1) = -2(1+K) + \varepsilon \mathfrak{g}_3(\varepsilon), & \text{(3.67b)} \\ \dfrac{|g^{(4)}(n_b^\varepsilon)|}{4!} + |\mathfrak{g}_2(\varepsilon)| + |\mathfrak{g}_3(\varepsilon)| < C & \text{(3.67c)} \end{cases}$$

(see Appendix for (3.66) and (3.67)). Dividing (3.66) by $(n_*^\varepsilon - 1)^2$ and using (3.67a), we have

$$(n_*^\varepsilon - 1)\left(-\frac{g^{(3)}(1)}{3!} - \frac{g^{(4)}(n_b^\varepsilon)}{4!}(n_*^\varepsilon - 1)\right) = V\gamma\varepsilon + \frac{\varepsilon^2}{2}\mathfrak{g}_2. \tag{3.68}$$

From (3.63) and (3.67b)–(3.67c), there is sufficiently small $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$-\frac{g^{(3)}(1)}{3!} - \frac{g^{(4)}(n_b^\varepsilon)}{4!}(n_*^\varepsilon - 1) > -\frac{g^{(3)}(1)}{3!} - \frac{1}{8} > \frac{1+K}{4} - \frac{1}{8} \geq \frac{1}{8}. \tag{3.69}$$

Dividing (3.68) by the LHS of (3.69), we get

$$0 < n_*^\varepsilon - 1 < 8V\gamma\varepsilon + 4\varepsilon^2\mathfrak{g}_2. \tag{3.70}$$

Step 3: We divide (3.68) by $-\frac{g^{(3)}(1)}{3!}$ to get

$$n_*^\varepsilon - 1 + \frac{6V\gamma}{g^{(3)}(1)}\varepsilon = -\frac{3\mathfrak{g}_2}{g^{(3)}(1)}\varepsilon^2 - \frac{g^{(4)}(n_b^\varepsilon)}{4g^{(3)}(1)}(n_*^\varepsilon - 1)^2 =: \mathfrak{g}_4. \tag{3.71}$$

By (3.67b)–(3.67c) and (3.70), there is a positive constant C such that

$$|\mathfrak{g}_4| \leq \varepsilon^2 C \tag{3.72}$$

for all $\varepsilon \in (0, \varepsilon_0)$. Subtracting $3\gamma V^{-1}\varepsilon$ from (3.71), one has

$$n_*^\varepsilon - 1 - 3\gamma V^{-1}\varepsilon = \frac{-6V^2 - 3g^{(3)}(1)}{Vg^{(3)}(1)}\gamma\varepsilon + \mathfrak{g}_4. \tag{3.73}$$

Using (3.67b)–(3.67c) and $V = \sqrt{K+1}$, we find that there exists a positive constant C such that

$$\frac{1}{V}\left|\frac{6V^2 + 3g^{(3)}(1)}{g^{(3)}(1)}\right| < C|6V^2 + 3g^{(3)}(1)| = 3C\varepsilon|\mathfrak{g}_3| < \varepsilon C. \tag{3.74}$$

Now (3.72)–(3.74) imply that there exists some positive constant C such that

$$\left|n_*^\varepsilon - 1 - 3\gamma V^{-1}\varepsilon\right| \le \varepsilon^2 C \tag{3.75}$$

for all $\varepsilon \in (0, \varepsilon_0)$.

Step 4: By expanding (3.34a), we get

$$u_*^\varepsilon = (V + \gamma\varepsilon)\left[(n_*^\varepsilon - 1) - \frac{2}{(n_b^\varepsilon)^3}(n_*^\varepsilon - 1)^2\right] \tag{3.76}$$

for some $1 < n_b^\varepsilon < n_*^\varepsilon$. Subtracting $3\gamma\varepsilon$ from (3.76) and then applying (3.75), one has

$$\begin{aligned} |u_*^\varepsilon - 3\gamma\varepsilon| &= \Big|V\left(n_*^\varepsilon - 1 - 3\gamma V^{-1}\varepsilon\right) \\ &\quad - \frac{2V}{(n_b^\varepsilon)^3}(n_*^\varepsilon - 1)^2 + \gamma\varepsilon(n_*^\varepsilon - 1)\left(1 - \frac{2}{(n_b^\varepsilon)^3}(n_*^\varepsilon - 1)\right)\Big| \\ &\le \varepsilon^2 C. \end{aligned} \tag{3.77}$$

In a similar fashion, we obtain from (3.34b) that

$$\phi_*^\varepsilon = (1 + 2V\gamma\varepsilon + \gamma^2\varepsilon^2)(n_*^\varepsilon - 1) + \frac{dh}{dn}(n_b^\varepsilon)(n_*^\varepsilon - 1)^2 \tag{3.78}$$

for some $1 < n_b^\varepsilon < n_*^\varepsilon$. Subtracting $3\gamma V^{-1}\varepsilon$ from (3.78) and using (3.75), we obtain

$$\begin{aligned} \left|\phi_*^\varepsilon - 3\gamma V^{-1}\varepsilon\right| &= \Big|\left(n_*^\varepsilon - 1 - 3\gamma V^{-1}\varepsilon\right) \\ &\quad + \gamma\varepsilon(n_*^\varepsilon - 1)(2V + \gamma\varepsilon) + \frac{dh}{dn}(n_b^\varepsilon)(n_*^\varepsilon - 1)^2\Big| \\ &\le \varepsilon^2 C. \end{aligned} \tag{3.79}$$

Now (3.31) follows from (3.75), (3.77) and (3.79). □

3.5 Asymptotic Behavior of Solitary Waves

In order to prove Theorem 3.2, it is enough to show that (3.29) holds on the half-interval $[0, +\infty)$ due to (3.19).

3.5.1 Uniform Decay Estimate

For notational simplicity, we let

$$\widetilde{N}_\varepsilon(\xi) := \frac{n^\epsilon(\xi) - 1}{\varepsilon} \quad \text{and} \quad \widetilde{E}_\varepsilon(\xi) := \frac{E^\varepsilon(\xi)}{\varepsilon} = \frac{-\phi^{\varepsilon\prime}(\xi)}{\varepsilon} \tag{3.80}$$

throughout this subsection. From Theorem 3.1, we know that $\widetilde{N}_\varepsilon > 0$ for $\xi \in \mathbb{R}$ and $\widetilde{E}_\varepsilon \geq 0$ for $\xi \in [0, +\infty)$. We first prove some lemmas. In Lemma 3.6, it is enough to control only $\widetilde{N}_\varepsilon$ since the RHS of (3.39) is linear in E^ε.

Lemma 3.5. *There exist positive constants ε_0, C, and δ_0 such that the following statements hold.*

1. *For all $\varepsilon \in (0, \varepsilon_0)$,*

$$\widetilde{N}_\varepsilon(0) > 2\gamma V^{-1}, \tag{3.81a}$$

$$4\gamma^2 > \widetilde{E}_\varepsilon'(0) > 2\gamma^2, \tag{3.81b}$$

$$1/2 < \sup_{\xi\in\mathbb{R}} h(n^\varepsilon) < 3/2, \tag{3.81c}$$

$$\sup_{\xi\in\mathbb{R}} \Big(|\widetilde{N}_\varepsilon'(\xi)| + |\widetilde{E}_\varepsilon(\xi)| + |\widetilde{E}_\varepsilon''(\xi)| \Big) \leq C. \tag{3.81d}$$

2. *If $0 < \delta < \delta_0$ and $\widetilde{N}_\varepsilon \leq \delta$ for all $\varepsilon \in (0, \varepsilon_0)$, then there holds*

$$|\widetilde{E}_\varepsilon| > \sqrt{\gamma}\, \widetilde{N}_\varepsilon \tag{3.82}$$

for all $0 < \varepsilon < \varepsilon_0$.

Here δ, δ_0 and C are independent of ξ and ε.

Proof. It is trivial that (3.81a) holds for all small ε by (3.31). Using (3.31), we see from (3.39b) that there exists some function $\tilde{\mathfrak{g}}(\varepsilon, n_*^\varepsilon)$ such that

$$\begin{aligned} \widetilde{E}_\varepsilon'(0) &= \frac{1}{\varepsilon^2}\left(n^\varepsilon(0) - e^{H(n^\varepsilon(0))}\right) \\ &= \frac{1}{\varepsilon^2}\left(-2V\gamma\varepsilon(n_*^\varepsilon - 1) + (1+K)(n_*^\varepsilon - 1)^2\right) + \varepsilon\tilde{\mathfrak{g}}(\varepsilon, n_*^\varepsilon), \end{aligned} \tag{3.83}$$

where $\tilde{\mathfrak{g}}(\varepsilon, n_*^\varepsilon)$ is bounded by some positive constant C uniformly in $\varepsilon \in (0, \varepsilon_0)$, and thus

$$\lim_{\varepsilon\to 0} \widetilde{E}_\varepsilon'(0) = 3\gamma^2$$

(see Appendix for (3.83)). Hence, (3.81b) holds for small ε.

Since $1 < n^\varepsilon(\xi) \leq n_*^\varepsilon$ for $\xi \in \mathbb{R}$, there is a positive constant C such that for all $0 < \varepsilon < \varepsilon_0$,

$$\begin{aligned} \sup_{\xi\in\mathbb{R}} |h(n^\varepsilon) - 1| &\leq \sup_{\xi\in\mathbb{R}} \left|(V+\gamma\varepsilon)^2 - K(n^\varepsilon)^2 - (n^\varepsilon)^3\right| \\ &\leq (V+\gamma\varepsilon)^2 - (K+1) + \sup_{\xi\in\mathbb{R}} \left|K\left(1-(n^\varepsilon)^2\right) + \left(1-(n^\varepsilon)^3\right)\right| \\ &\leq \varepsilon C, \end{aligned} \tag{3.84}$$

where we used $V = \sqrt{K+1}$ and (3.31) in the third inequality. It is obvious that (3.84) implies (3.81c) for all small ε. Now we choose $\varepsilon_0 > 0$ so small that (3.81a)– (3.81c) hold.

To obtain (3.81d), we apply the Taylor expansion to the RHS of (3.43) about $n = 1$ and divide the resulting equation by $2^{-1}\varepsilon^3$. Then we obtain

$$(\widetilde{E}_\varepsilon)^2 = \frac{g^{(2)}(1)}{\varepsilon}(\widetilde{N}_\varepsilon)^2 + \frac{g^{(3)}(n_b^\varepsilon)}{3}(\widetilde{N}_\varepsilon)^3 \tag{3.85}$$

for some $n_b^\varepsilon \in (1, n^\varepsilon)$, where

$$g^{(2)}(1) = \gamma\varepsilon(2V + \gamma\varepsilon)(1 + 2V\gamma\varepsilon + \gamma^2\varepsilon^2) \tag{3.86}$$

(see Appendix for the explicit forms of $g^{(2)}$ and $g^{(3)}$). From (3.31) and that $1 < n^\varepsilon(\xi) \le n_*^\varepsilon$ for $\xi \in \mathbb{R}$, we see that the RHS of (3.85) is uniformly bounded by some positive constant C for all $\varepsilon \in (0, \varepsilon_0)$ and $\xi \in \mathbb{R}$ as long as $1 < n_b^\varepsilon < n^\varepsilon$. Hence, we have

$$\sup_{\xi\in\mathbb{R}} |\widetilde{E}_\varepsilon(\xi)| \le C. \tag{3.87}$$

Dividing (3.39a) by εh, we have from (3.87) and (3.81c) that

$$\sup_{\xi\in\mathbb{R}} |\widetilde{N}_\varepsilon'| = \sup_{\xi\in\mathbb{R}} \frac{|\widetilde{E}_\varepsilon|}{|h(n^\varepsilon)|} \le C. \tag{3.88}$$

Taking the derivative of (3.39b) in ξ, the mean value theorem yields that

$$\begin{aligned} \sup_{\xi\in\mathbb{R}} |\widetilde{E}_\varepsilon''| &= \sup_{\xi\in\mathbb{R}} \frac{1}{\varepsilon}|\widetilde{N}_\varepsilon'||1 - e^{H(n^\varepsilon)}h(n^\varepsilon)| \\ &\le \sup_{\xi\in\mathbb{R}} \frac{1}{\varepsilon}|\widetilde{N}_\varepsilon'| \left(C|n^\varepsilon - 1| + |1 + K - (V+\gamma\varepsilon)^2|\right) \\ &\le C, \end{aligned} \tag{3.89}$$

where we used $n^\varepsilon > 1$ in the second line, and (3.88), (3.31) and $V = \sqrt{1+K}$ in the last line. From (3.87)–(3.89), we obtain (3.81d).

From (3.85)–(3.86), we get

$$|\widetilde{E}_\varepsilon| = \sqrt{\frac{g^{(2)}(1)}{\varepsilon}}\widetilde{N}_\varepsilon\sqrt{1 + \frac{\varepsilon g^{(3)}(n_b^\varepsilon)}{3g^{(2)}(1)}\widetilde{N}_\varepsilon} \ge \sqrt{2\gamma}\widetilde{N}_\varepsilon\sqrt{1 + \frac{\varepsilon g^{(3)}(n_b^\varepsilon)}{3g^{(2)}(1)}\widetilde{N}_\varepsilon} \tag{3.90}$$

since $\widetilde{N}_\varepsilon > 0$ for $\xi \in \mathbb{R}$ and $V \ge 1$. We choose $\delta_0 > 0$ so that

$$\delta_0 \sup_{\xi\in\mathbb{R}} \sup_{0<\varepsilon<\varepsilon_0} \left| \frac{\varepsilon g^{(3)}(n_b^\varepsilon)}{3g^{(2)}(1)} \right| < \frac{1}{2}. \tag{3.91}$$

Then (3.82) follows from (3.90). In (3.91), the supremum exists by (3.86) and (3.31). □

Lemma 3.6. *There exist constants $\varepsilon_0, \delta_1 > 0$ such that the following statement holds: for each $0 < \delta < \delta_1$, there exists $\xi_\delta > 0$ such that for all $\xi \ge \xi_\delta$ and $0 < \varepsilon < \varepsilon_0$,*

$$0 < \widetilde{N}_\varepsilon(\xi) \le \delta$$

Here δ_1, δ, and ξ_δ are independent of ξ and ε.

Proof. From Theorem 3.1, we see that for each ε, $\lim_{\xi\to\infty} \widetilde{N}_\varepsilon(\xi) = 0$ and $\widetilde{N}_\varepsilon(\xi)$ is strictly increasing on $[0,\infty)$. This observation and (3.81a), together with the intermediate value theorem, yields that for each $0 < \delta < \gamma V^{-1}$ and $0 < \varepsilon < \varepsilon_0$, there is a unique $\xi_{\varepsilon,\delta} > 0$ such that

$$\widetilde{N}_\varepsilon(\xi_{\varepsilon,\delta}) = \delta. \tag{3.92}$$

By (3.82) and (3.92), we see that for all $0 < \varepsilon < \varepsilon_0$ and $0 < \delta < \min\{\delta_0, \gamma V^{-1}\}$,

$$\widetilde{E}_\varepsilon(\xi_{\varepsilon,\delta}) > \sqrt{\gamma}\delta \tag{3.93}$$

since $\widetilde{E}_\varepsilon(\xi) \geq 0$ on $[0,\infty)$. Here the mean value theorem yields that

$$\widetilde{E}'_\varepsilon(\xi) = \widetilde{E}'_\varepsilon(0) - \widetilde{E}''_\varepsilon(\xi_b^\varepsilon)\xi$$

for some $\xi_b^\varepsilon \in (0,\xi)$. Thus, using (3.81b) and (3.81d), we have that there is a small $\xi_0 > 0$, independent of ε, such that

$$\widetilde{E}'_\varepsilon(\xi) > \gamma^2 \tag{3.94}$$

holds for all $\varepsilon \in (0,\varepsilon_0)$ and $\xi \in (0,\xi_0)$. Integrating (3.94) over $[0,\xi]$ for $\xi \in (0,\xi_0)$, we have

$$\widetilde{E}_\varepsilon(\xi) > \gamma^2\xi \tag{3.95}$$

since $\widetilde{E}_\varepsilon(0) = 0$. Thus for all $\delta > 0$ with $\gamma^{-\frac{3}{2}}\delta < \xi_0$, we have

$$\widetilde{E}_\varepsilon(\gamma^{-\frac{3}{2}}\delta) > \sqrt{\gamma}\delta. \tag{3.96}$$

For a moment, we assume that there exists a number $\delta_1 \leq \min\{\delta_0, \gamma V^{-1}, \gamma^{\frac{3}{2}}\xi_0\}$ such that for all $0 < \delta < \delta_1$ and $0 < \varepsilon < \varepsilon_0$,

$$\gamma^{-\frac{3}{2}}\delta < \xi_{\varepsilon,\delta} \tag{3.97}$$

holds. Then (3.93) and (3.96)–(3.97) imply that

$$\widetilde{E}_\varepsilon(\xi) > \sqrt{\gamma}\delta \tag{3.98}$$

for all $\xi \in [\gamma^{-\frac{3}{2}}\delta, \xi_{\varepsilon,\delta}]$ and $0 < \varepsilon < \varepsilon_0$ (see Figure 3). Applying (3.81c) and (3.98) to the ODE (3.39a), we have

$$-\widetilde{N}'_\varepsilon(\xi) = \frac{\widetilde{E}_\varepsilon(\xi)}{h(n_\varepsilon)} \geq \frac{2}{3}\sqrt{\gamma}\delta \tag{3.99}$$

for $\xi \in [\gamma^{-\frac{3}{2}}\delta, \xi_{\varepsilon,\delta}]$ and $0 < \varepsilon < \varepsilon_0$. Integrating (3.99) from $\gamma^{-\frac{3}{2}}\delta$ to $\xi_{\varepsilon,\delta}$, we obtain

$$0 < \widetilde{N}_\varepsilon(\xi_{\varepsilon,\delta}) \leq -\frac{2}{3}\sqrt{\gamma}\delta(\xi_{\varepsilon,\delta} - \gamma^{-\frac{3}{2}}\delta) + \sup_{0<\varepsilon<\varepsilon_0} \widetilde{N}_\varepsilon(\gamma^{-\frac{3}{2}}\delta). \tag{3.100}$$

Now it is clear that $\xi_\delta := \sup_{0<\varepsilon<\varepsilon_0} \xi_{\varepsilon,\delta} < \infty$ for each $0 < \delta < \delta_1$. If not, then there is a sequence $\{\varepsilon_k\}$ such that the RHS of (3.100) diverges to $-\infty$ as $k \to \infty$, which is a contradiction. Hence, we have that for each $0 < \delta < \delta_1$,

$$\delta = \widetilde{N}_\varepsilon(\xi_{\varepsilon,\delta}) \geq \widetilde{N}_\varepsilon(\xi_\delta) \geq \widetilde{N}_\varepsilon(\xi)$$

for all $0 < \varepsilon < \varepsilon_0$ and $\xi \geq \xi_\delta$.

To complete the proof, we verify (3.97). It is sufficient to show that there exists $\delta_1 \leq \min\{\delta_0, \gamma V^{-1}, \gamma^{\frac{3}{2}}\xi_0\}$ such that

$$\gamma^{-\frac{3}{2}}\delta < \inf_{0<\varepsilon<\varepsilon_0} \xi_{\varepsilon,\delta} \tag{3.101}$$

for all $0 < \delta < \delta_1$. Here we note that the infimum in (3.101) exists and finite since $0 < \xi_{\varepsilon,\delta} < \infty$.

To verify (3.97), we suppose $\inf_{0<\varepsilon<\varepsilon_0} \xi_{\varepsilon,\delta} = 0$ for some $0 < \delta \leq \gamma V^{-1}$. Then for some sequence $\{\varepsilon_k\}$, we have $\xi_{\varepsilon_k,\delta} \to 0$ as $k \to \infty$. By the mean value theorem, (3.81a) and (3.92) imply that

$$-\widetilde{N}'_{\varepsilon_k}(\bar{\xi}_{\varepsilon_k,\delta}) = \frac{\widetilde{N}_{\varepsilon_k}(0) - \widetilde{N}_{\varepsilon_k}(\xi_{\varepsilon_k,\delta})}{\xi_{\varepsilon_k,\delta}} \geq \frac{\frac{2\gamma}{V} - \delta}{\xi_{\varepsilon_k,\delta}} > 0 \tag{3.102}$$

for some $0 < \bar{\xi}_{\varepsilon_k,\delta} < \xi_{\varepsilon_k,\delta}$. This is a contradiction since the RHS of (3.102) tends to $+\infty$ as $k \to \infty$ while the LHS of (3.102) stays bounded by (3.81d). Thus we obtain $\inf_{0<\varepsilon<\varepsilon_0} \xi_{\varepsilon,\delta} > 0$ for all $0 < \delta \leq \gamma V^{-1}$. By the definition, $\xi_{\varepsilon,\delta}$ decreases in δ for each fixed ε. This implies that

$$\inf_{0<\varepsilon<\varepsilon_0} \xi_{\varepsilon,\delta} \geq \inf_{0<\varepsilon<\varepsilon_0} \xi_{\varepsilon,\frac{\gamma}{V}} > 0 \tag{3.103}$$

for all $0 < \delta \leq \gamma V^{-1}$. We let $\delta_1 := \min\{\delta_0, \gamma V^{-1}, \gamma^{\frac{3}{2}}\xi_0, \gamma^{\frac{3}{2}} \inf_{0<\varepsilon<\varepsilon_0} \xi_{\varepsilon,\frac{\gamma}{V}}\}$. Then (3.103) implies that (3.101) holds for all $0 < \delta < \delta_1$.

□

Now using Lemma 3.5 and Lemma 3.6, we show the uniform exponential decay estimates for the solutions. This will play an important role in remainder analysis.

Proposition 3.7. *Let k be any non-negative interger. Then there exist constants $C_*, \varepsilon_1 > 0$, and $C_k > 0$ such that*

$$\begin{cases} |\widetilde{N}_\varepsilon^{(k)}(\xi)| + |\widetilde{E}_\varepsilon^{(k)}(\xi)| \leq C_k e^{-C_*\xi}, & (3.104a)\\ \dfrac{|u^{\varepsilon(k)}(\xi)|}{\varepsilon} + \dfrac{|\phi^{\varepsilon(k)}(\xi)|}{\varepsilon} \leq C_k e^{-C_*\xi} & (3.104b) \end{cases}$$

for all $\varepsilon \in (0, \varepsilon_1)$ and $\xi \geq 0$. Here C_* and C_k are uniform in ξ and ε. C_* is independent of k.

Proof. Applying Taylor's expansion, (3.39) is written as

$$\begin{pmatrix} n^\varepsilon - 1 \\ E^\varepsilon \end{pmatrix}' = A \begin{pmatrix} n^\varepsilon - 1 \\ E^\varepsilon \end{pmatrix} + \begin{pmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \end{pmatrix}, \tag{3.105}$$

where

$$A := \begin{pmatrix} 0 & (K - J^2)^{-1} \\ \varepsilon^{-1}(1 + K - J^2) & 0 \end{pmatrix}$$

is the Jacobian matrix of (3.39) at $(n^\varepsilon, E^\varepsilon) = (1, 0)$ (see (3.53)). Here, $\mathcal{R}_1$ and $\mathcal{R}_2$ are functions of $(n^\varepsilon, E^\varepsilon)$, and there is a constant $C > 0$ such that

$$\begin{cases} |\mathcal{R}_1| \leq C\left((n^\varepsilon - 1)^2 + (n^\varepsilon - 1)E^\varepsilon\right), & (3.106a)\\ |\mathcal{R}_2| \leq \varepsilon^{-1}C(n^\varepsilon - 1)^2 & (3.106b) \end{cases}$$

for all $\varepsilon \in (0, \varepsilon_0)$ by (3.31) (see Appendix). The eigenvalues of A are

$$\pm \frac{1}{\sqrt{\varepsilon}} \sqrt{\frac{J^2 - 1 - K}{J^2 - K}} = \pm \sqrt{\frac{2V\gamma + \gamma^2 \varepsilon}{1 + 2V\gamma\varepsilon + (\gamma\varepsilon)^2}}, \tag{3.107}$$

where we have used (3.32). Let $\lambda = \lambda(\varepsilon)$ be the positive eigenvalue of A. By (3.107), we can choose sufficiently small $\varepsilon_* = \varepsilon_*(K, \gamma) > 0$ such that

$$\frac{d\lambda}{d\varepsilon}(\varepsilon) = \frac{1}{2\lambda(\varepsilon)} \frac{\gamma^2(1 - 4V^2) + O(\varepsilon)}{(1 + 2V\gamma\varepsilon + \gamma^2\varepsilon^2)^2} < 0$$

for all $\varepsilon \in (0, \varepsilon_*)$ since $V = \sqrt{1+K}$. Hence $\lambda(\varepsilon)$ decreases in ε on $(0, \varepsilon_*)$. Now we fix such ε_*. One may easily check using (3.107) that $(J^2(\varepsilon) - K)\lambda(\varepsilon)$ decreases in ε. Hence we obtain that for all $\varepsilon \in (0, \varepsilon_*)$,

$$\begin{cases} -\lambda < -\lambda(\varepsilon_*), & (3.108a) \\ \sqrt{2V\gamma} < (J^2 - K)\lambda < (J^2(\varepsilon_*) - K)\lambda(\varepsilon_*). & (3.108b) \end{cases}$$

Considering diagonalization of the matrix A, (3.105) can be written as

$$P \begin{pmatrix} n^\varepsilon - 1 \\ E^\varepsilon \end{pmatrix}' = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} P \begin{pmatrix} n^\varepsilon - 1 \\ E^\varepsilon \end{pmatrix} + P \begin{pmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \end{pmatrix}, \tag{3.109}$$

where

$$P := \frac{1}{2} \begin{pmatrix} 1 & -[(J^2 - K)\lambda]^{-1} \\ 1 & [(J^2 - K)\lambda]^{-1} \end{pmatrix}.$$

We multiply the second component of (3.109) by $2\varepsilon^{-1}$ and then use (3.106) to obtain

$$\begin{aligned} \widetilde{N}_\varepsilon' + [(J^2 - K)\lambda]^{-1} \widetilde{E}_\varepsilon' \leq -\lambda \underbrace{\left(\widetilde{N}_\varepsilon + [(J^2 - K)\lambda]^{-1} \widetilde{E}_\varepsilon \right)}_{=:I_1} \\ + \varepsilon C \widetilde{N}_\varepsilon^2 + \varepsilon C \widetilde{N}_\varepsilon \widetilde{E}_\varepsilon + C[(J^2 - K)\lambda]^{-1} \widetilde{N}_\varepsilon^2. \end{aligned} \tag{3.110}$$

Since $\widetilde{N}_\varepsilon \geq 0$ and $\widetilde{E}_\varepsilon \geq 0$ for $\xi \in [0, \infty)$, we have from (3.108) that

$$\begin{aligned} -\lambda I_1 &\leq -\lambda(\varepsilon_*) I_1 \\ &\leq -\frac{\lambda(\varepsilon_*)}{2} I_1 \underbrace{- \frac{\lambda(\varepsilon_*)}{2} \left(\widetilde{N}_\varepsilon + \left[(J^2(\varepsilon_*) - K)\lambda(\varepsilon_*)\right]^{-1} \widetilde{E}_\varepsilon \right)}_{=:I_2} \end{aligned} \tag{3.111}$$

By Lemma 3.6 and (3.108b), we can choose a sufficiently small $\delta \in (0, \delta_1)$ such that

$$I_2 + \varepsilon C \widetilde{N}_\varepsilon^2 + \varepsilon C \widetilde{N}_\varepsilon \widetilde{E}_\varepsilon + C[(J^2 - K)\lambda]^{-1} \widetilde{N}_\varepsilon^2 < \frac{1}{2} I_2 < 0 \tag{3.112}$$

holds for all $\varepsilon \in (0, \varepsilon_0)$ and $\xi \geq \xi_\delta$. Let $\varepsilon_1 := \min\{\varepsilon_*, \varepsilon_0\}$. Combining (3.110)–(3.112), we get

$$\widetilde{N}_\varepsilon' + [(J^2 - K)\lambda]^{-1} \widetilde{E}_\varepsilon' < -\frac{\lambda(\varepsilon_*)}{2} \left(\widetilde{N}_\varepsilon + [(J^2 - K)\lambda]^{-1} \widetilde{E}_\varepsilon \right). \tag{3.113}$$

We multiply (3.113) by $e^{2^{-1}\lambda(\varepsilon_*)\xi}$ and integrate the resultant over $[\xi_\delta, \xi]$ to get

$$\widetilde{N}_\varepsilon(\xi) + \frac{\widetilde{E}_\varepsilon(\xi)}{(J^2-K)\lambda} \leq \left[\widetilde{N}_\varepsilon(\xi_\delta) + \frac{\widetilde{E}_\varepsilon(\xi_\delta)}{(J^2-K)\lambda}\right] e^{-\frac{\lambda(\varepsilon_*)}{2}(\xi-\xi_\delta)}. \tag{3.114}$$

Let $C_* := 2^{-1}\lambda(\varepsilon_*)$. Then (3.104a) for the case $k=0$ easily follows from (3.114) by using the bounds (3.81d), (3.108b), and the estimate (3.31). (3.104b) is obtained from (3.104a) and (3.34). This finishes the proof for the case $k=0$. By the induction argument, one can prove the cases $k \geq 1$ by using the system (3.39) (see (3.140)). □

3.5.2 Proof of Theorem 3.2–Derivation of the Remainder Equations

In what follows, we derive the equations for the remainders $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)$:

$$\begin{cases} u_R^\varepsilon - Vn_R^\varepsilon = \mathcal{M}_1^\varepsilon, & (3.115a) \\ \phi_R^\varepsilon - Vu_R^\varepsilon + Kn_R^\varepsilon = \mathcal{M}_2^\varepsilon, & (3.115b) \\ \phi_R^\varepsilon - n_R^\varepsilon = V\mathcal{M}_1^\varepsilon + \mathcal{M}_2^\varepsilon, & (3.115c) \end{cases}$$

and

$$\phi_R^{\varepsilon\,\prime\prime} - F_\varepsilon \phi_R^\varepsilon = \mathcal{M}_3^\varepsilon, \tag{3.116}$$

where

$$F_\varepsilon(\xi) := 2V\gamma - 2V^2 n_{\mathrm{KdV}} - V^2\frac{\phi_R^\varepsilon}{\varepsilon} \tag{3.117}$$

and $\mathcal{M}_i^\varepsilon$ $(i=1,2,3)$, defined in (3.119)–(3.120) and (3.127), are some functions of $n_{\mathrm{KdV}}, n_R^\varepsilon, u_R^\varepsilon$, and ϕ_R^ε. For notational simplicity, we let $n_{\mathrm{KdV}} = n_K$. Since $n_K(\xi) = n_K(x-\gamma t)$ satisfies (KdV), it also satisfies

$$-\gamma n_K' + Vn_K n_K' + (2V)^{-1} n_K''' = 0. \tag{3.118}$$

Putting (3.28) into (3.33a), a direct calculation yields (3.115a), where

$$\begin{aligned} \mathcal{M}_1^\varepsilon := {} & (\gamma\varepsilon - \varepsilon V n_K) n_R^\varepsilon - (\varepsilon n_K + n_R^\varepsilon) u_R^\varepsilon + \varepsilon^2(\gamma n_K - V n_K^2) \\ & + \varepsilon \underbrace{(Vn_K - Vn_K)}_{=0}. \end{aligned} \tag{3.119}$$

Similarly, we obtain (3.115b) from (3.28) and (3.33b), where

$$\begin{aligned} \mathcal{M}_2^\varepsilon := {} & (\gamma\varepsilon - \varepsilon V n_K) u_R^\varepsilon - \frac{|u_R^\varepsilon|^2}{2} + \frac{K}{2}\left(2\varepsilon n_K n_R^\varepsilon + |n_R^\varepsilon|^2\right) - KO_{n^\varepsilon}(\varepsilon^3) \\ & + \varepsilon^2\left(\gamma V n_K - \frac{V^2 n_K^2}{2} + \frac{K n_K^2}{2}\right) - \varepsilon \underbrace{(n_K - V^2 n_K + K n_K)}_{=0} \end{aligned} \tag{3.120}$$

and $O_{n^\varepsilon}(\varepsilon^3) := \ln n^\varepsilon - (n^\varepsilon - 1) + \frac{1}{2}(n^\varepsilon - 1)^2$. On the other hand, (3.115c) follows from (3.115a)–(3.115b) since $V = \sqrt{1+K}$.

Now we derive (3.116)-(3.117). Plugging (3.28) into (3.17c), we obtain

$$n_R^\varepsilon - \phi_R^\varepsilon = -\varepsilon^2 n_K'' - \varepsilon \phi_R^\varepsilon{}'' + \frac{1}{2}(\varepsilon n_K + \phi_R^\varepsilon)^2 + O_{\phi^\varepsilon}(\varepsilon^3), \tag{3.121}$$

where $O_{\phi^\varepsilon}(\varepsilon^3) := e^{\phi^\varepsilon} - 1 - \phi^\varepsilon - \frac{1}{2}(\phi^\varepsilon)^2$. By adding (3.115b) and (3.121), the term ϕ_R^ε in the LHS of (3.121) is canceled, and one obtains

$$\begin{aligned} -Vu_R^\varepsilon + (1+K)n_R^\varepsilon = \varepsilon^2 \left(-n_K'' + \frac{n_K^2}{2} \right) - \varepsilon \phi_R^\varepsilon{}'' + \frac{1}{2}\left(2\varepsilon n_K \phi_R^\varepsilon + |\phi_R^\varepsilon|^2 \right) \\ + O_{\phi^\varepsilon}(\varepsilon^3) + \mathcal{M}_2^\varepsilon. \end{aligned} \tag{3.122}$$

Multiplying (3.115a) by V, and then adding the resultant to (3.122), the LHS of (3.122) is canceled from $V = \sqrt{1+K}$. Thus we have

$$\begin{aligned} 0 &= V\mathcal{M}_1^\varepsilon + \{\text{the RHS of (3.122)}\} \\ &= V\left[(\gamma\varepsilon - \varepsilon V n_K) n_R^\varepsilon - (\varepsilon n_K + n_R^\varepsilon) u_R^\varepsilon \right] - \varepsilon \phi_R^\varepsilon{}'' + \frac{1}{2}\left(2\varepsilon n_K \phi_R^\varepsilon + |\phi_R^\varepsilon|^2 \right) \\ &\quad + (\gamma\varepsilon - \varepsilon V n_K) u_R^\varepsilon - \frac{|u_R^\varepsilon|^2}{2} + \frac{K}{2}\left(2\varepsilon n_K n_R^\varepsilon + |n_R^\varepsilon|^2 \right) - K O_{n^\varepsilon}(\varepsilon^3) \\ &\quad + O_{\phi^\varepsilon}(\varepsilon^3) + \varepsilon^2 \left(\underline{2V\gamma n_K - V^2 n_K^2 - \frac{V^2 n_K^2}{2} + \frac{1+K}{2} n_K^2 - n_K''} \right). \end{aligned} \tag{3.123}$$

Since $V = \sqrt{1+K}$ and n_K satisfies (3.118), the underlined terms of (3.123) are canceled. Using (3.115a), we substitute u_R^ε of (3.123) with $Vn_R^\varepsilon + \mathcal{M}_1^\varepsilon$. Then it is straightforward to obtain

$$\varepsilon \phi_R^\varepsilon{}'' + \left(\varepsilon \widetilde{F} + \frac{1}{2}(3V^2 - K) n_R^\varepsilon \right) n_R^\varepsilon - \frac{1}{2}\left(2\varepsilon n_K + \phi_R^\varepsilon \right) \phi_R^\varepsilon = \widetilde{\mathcal{M}}_3^\varepsilon, \tag{3.124}$$

where

$$\begin{cases} \widetilde{F}(\xi) := -2V\gamma + (3V^2 - K) n_K, & (3.125\text{a}) \\ \widetilde{\mathcal{M}}_3^\varepsilon := \mathcal{M}_1^\varepsilon \left(\gamma\varepsilon - 2V\varepsilon n_K - 2V n_R^\varepsilon - \frac{1}{2}\mathcal{M}_1^\varepsilon \right) - K O_{n^\varepsilon}(\varepsilon^3) + O_{\phi^\varepsilon}(\varepsilon^3). & (3.125\text{b}) \end{cases}$$

Using (3.115c), we substitute n_R^ε of the LHS of (3.124) by $\phi_R^\varepsilon + V\mathcal{M}_1^\varepsilon + \mathcal{M}_2^\varepsilon$ and then divide the resulting equation by ε. Then we obtain

$$\phi_R^\varepsilon{}'' - \left[2V\gamma - (3V^2 - K - 1) n_K - \frac{1}{2}(3V^2 - K - 1) \frac{\phi_R^\varepsilon}{\varepsilon} \right] \phi_R^\varepsilon = \mathcal{M}_3^\varepsilon, \tag{3.126}$$

where

$$\mathcal{M}_3^\varepsilon := \frac{\widetilde{\mathcal{M}}_3^\varepsilon}{\varepsilon} - (V\mathcal{M}_1^\varepsilon + \mathcal{M}_2^\varepsilon) \left[\widetilde{F} + \frac{(3V^2 - K)}{2\varepsilon} \left[2\phi_R^\varepsilon + (V\mathcal{M}_1^\varepsilon + \mathcal{M}_2^\varepsilon) \right] \right]. \tag{3.127}$$

Applying $V = \sqrt{K+1}$ to (3.126), we arrive at (3.116)–(3.117).

The following lemma directly follows from the definitions of $n_{\text{KdV}}, n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon, F_\varepsilon$, and $\mathcal{M}_i^\varepsilon$ $(i = 1, 2, 3)$, and Proposition 3.3 and Proposition 3.7.

Lemma 3.8. *Let k be any non-negative integer. Then there exist constants ε_1, $\widetilde{C}_* > 0$ (independent of k), and C_k, $\xi_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$,*

$$\begin{cases} |{n_R^\varepsilon}^{(k)}|,\ |{u_R^\varepsilon}^{(k)}|,\ |{\phi_R^\varepsilon}^{(k)}| \leq C_k \varepsilon e^{-\widetilde{C}_* \xi}, \quad (\xi \geq 0), & (3.128a) \\ |M_i^{\varepsilon(k)}| \leq C_k \varepsilon^2 e^{-\widetilde{C}_* \xi}, \quad (\xi \geq 0, \quad \textit{for } i = 1,2,3), & (3.128b) \\ \sup_{\xi \in [0,\infty)} |{F_\varepsilon}^{(k)}| \leq C_k, & (3.128c) \\ F_\varepsilon(\xi) > V\gamma, \quad (\xi \geq \xi_1). & (3.128d) \end{cases}$$

Here $\widetilde{C}_*$ and C_k are uniform in ε and ξ. ξ_1 is uniform in ε.

Now we shall obtain the remainder estimates around $\xi = 0$ using a continuation argument.

Proposition 3.9. *For any fixed $\xi_* > 0$, there is a constant $C_{\xi_*} > 0$ such that for all $0 < \varepsilon < \varepsilon_1$,*

$$\sup_{\xi \in [0,\xi_*]} \left(|{\phi_R^\varepsilon}'(\xi)|^2 + |\phi_R^\varepsilon(\xi)|^2 \right) \leq C_{\xi_*} \varepsilon^4 \tag{3.129}$$

holds. Here C_{ξ_*} is uniform in ε but depends on ξ_*.

Proof. We multiply (3.116) by $2{\phi_R^\varepsilon}'$, and then add $(|\phi_R^\varepsilon|^2)' = 2\phi_R^\varepsilon {\phi_R^\varepsilon}'$ to the resulting equation. Then we have

$$\left(|{\phi_R^\varepsilon}'|^2 + |\phi_R^\varepsilon|^2 \right)' = 2(1 + F_\varepsilon)\phi_R^\varepsilon {\phi_R^\varepsilon}' + 2\mathcal{M}_3^\varepsilon {\phi_R^\varepsilon}'. \tag{3.130}$$

Applying Young's inequality to (3.130), there is a constant $C > 0$ such that

$$\begin{aligned} \left(|{\phi_R^\varepsilon}'|^2 + |\phi_R^\varepsilon|^2 \right)' &\leq |1 + F_\varepsilon| \left(|\phi_R^\varepsilon|^2 + |{\phi_R^\varepsilon}'|^2 \right) + |\mathcal{M}_3^\varepsilon|^2 + |{\phi_R^\varepsilon}'|^2 \\ &\leq C \left(|{\phi_R^\varepsilon}'|^2 + |\phi_R^\varepsilon|^2 \right) + C\varepsilon^4 \end{aligned} \tag{3.131}$$

for $\xi \in [0,\infty)$, where we have used (3.128b)–(3.128c) in the second line. We multiply (3.131) by $e^{-C\xi}$, and then integrate the resultant over $[0,\xi]$. Then we get

$$\begin{aligned} |{\phi_R^\varepsilon}'|^2(\xi) + |\phi_R^\varepsilon|^2(\xi) &\leq \left(|{\phi_R^\varepsilon}'|^2(0) + |\phi_R^\varepsilon|^2(0) \right) e^{C\xi} + \varepsilon^4 (e^{C\xi} - 1) \\ &\leq \varepsilon^4 C^2 e^{C\xi} + \varepsilon^4 (e^{C\xi} - 1), \end{aligned} \tag{3.132}$$

where we have used the estimate (3.31) and that ${\phi_R^\varepsilon}'(0) = 0$ thanks to ${\phi^\varepsilon}'(0) = n_{\mathrm{K}}'(0) = 0$. To finish the proof, for any fixed $\xi_* > 0$ we take the supremum of the LHS of (3.132) over $[0,\xi_*]$. □

Now we prove the uniform decay estimates for the remainders in the far-field region.

Proposition 3.10. *There exist positive constants $\varepsilon_1, \xi_1, C, \alpha$ such that for all $0 < \varepsilon < \varepsilon_1$,*

$$\int_{\xi_1}^{\infty} \left(|{\phi_R^\varepsilon}'(\xi)|^2 + |\phi_R^\varepsilon(\xi)|^2 \right) e^{\alpha\xi} \, d\xi \leq C\varepsilon^4 \tag{3.133}$$

Here α is independent of ξ and ε.

Proof. We multiply (3.116) by $\phi_R^\varepsilon e^{\alpha\xi}$ for $0 < \alpha < \widetilde{C}_*$ and then integrate the resulting equation over $[\xi_1, +\infty)$, where ξ_1 and $\widetilde{C}_*$ are the constants given in Lemma 3.8. Then one obtain

$$\begin{aligned}\int_{\xi_1}^{\infty} \left[F_\varepsilon(\xi)|\phi_R^\varepsilon|^2 + |\phi_R^{\varepsilon\prime}|^2\right] e^{\alpha\xi}\, d\xi = \int_{\xi_1}^{\infty} -\mathcal{M}_3^\varepsilon \phi_R^\varepsilon e^{\alpha\xi} - \alpha \phi_R^{\varepsilon\prime}\phi_R^\varepsilon e^{\alpha\xi}\, d\xi \\ - (\phi_R^{\varepsilon\prime}\phi_R^\varepsilon e^{\alpha\xi})(\xi_1),\end{aligned} \tag{3.134}$$

where we have used the fact that $\lim_{\xi\to\infty} \phi_R^{\varepsilon\prime}\phi_R^\varepsilon e^{\alpha\xi} = 0$ for all $\varepsilon > 0$, which is true by (3.128a) since $0 < \alpha < \widetilde{C}_*$. By (3.128d),

$$\int_{\xi_1}^{\infty} \left[V\gamma|\phi_R^\varepsilon|^2 + |\phi_R^{\varepsilon\prime}|^2\right] e^{\alpha\xi}\, d\xi < \text{the LHS of (3.134)}. \tag{3.135}$$

By applying the Young inequality, and then using Proposition 3.9 and (3.128b), we find that there is a positive constant $C_{\xi_1} > 0$ such that

$$\begin{aligned}\text{the RHS of (3.134)} &\le \frac{V\gamma + \alpha}{2}\int_{\xi_1}^{\infty} |\phi_R^\varepsilon|^2 e^{\alpha\xi}\, d\xi + \frac{1}{2V\gamma}\int_{\xi_1}^{\infty} |\mathcal{M}_3^\varepsilon|^2 e^{\alpha\xi}\, d\xi \\ &\quad + \frac{\alpha}{2}\int_{\xi_1}^{\infty} |\phi_R^{\varepsilon\prime}|^2 e^{\alpha\xi}\, d\xi + |(\phi_R^{\varepsilon\prime}\phi_R^\varepsilon e^{\alpha\xi})|(\xi_1) \\ &\le \frac{V\gamma + \alpha}{2}\int_{\xi_1}^{\infty} |\phi_R^\varepsilon|^2 e^{\alpha\xi}\, d\xi + \frac{\alpha}{2}\int_{\xi_1}^{\infty} |\phi_R^{\varepsilon\prime}|^2 e^{\alpha\xi}\, d\xi \\ &\quad + \varepsilon^4 C_{\xi_1,\alpha}.\end{aligned} \tag{3.136}$$

Now we choose α so that $0 < \alpha < \min\{V\gamma, 2, \widetilde{C}_*\}$. From (3.135)–(3.136), we finish the proof. □

Proof of Theorem 3.2. We first prove that for every non-negative integers k, there is positive constant C_k such that

$$\int_0^{\infty} |\phi_R^{\varepsilon\,(k)}(\xi)|^2 e^{\alpha\xi}\, d\xi \le C_k\varepsilon^4 \tag{3.137}$$

for all $\varepsilon \in (0, \varepsilon_1)$. As the induction hypothesis, we assume that for $k = 0, 1, \cdots, n$, there is some positive constant C_k satisfying (3.137). We take the n-th derivative of (3.116) in ξ and then multiply the resulting equation by $\phi_R^{\varepsilon\,(n+2)} e^{\alpha\xi}$. Then we obtain

$$\begin{aligned}\int_0^{\infty} |\phi_R^{\varepsilon\,(n+2)}|^2 e^{\alpha\xi}\, d\xi &= \int_0^{\infty} \left[(F_\varepsilon\phi_R^\varepsilon)^{(n)} + \mathcal{M}_3^{\varepsilon\,(n)}\right] \phi_R^{\varepsilon\,(n+2)} e^{\alpha\xi}\, d\xi \\ &\le \frac{1}{2}\int_0^{\infty} |\phi_R^{\varepsilon\,(n+2)}|^2 e^{\alpha\xi}\, d\xi + \int_0^{\infty} |(F_\varepsilon\phi_R^\varepsilon)^{(n)}|^2 e^{\alpha\xi}\, d\xi \\ &\quad + \int_0^{\infty} |\mathcal{M}_3^{\varepsilon\,(n)}|^2 e^{\alpha\xi}\, d\xi \\ &\le \frac{1}{2}\int_0^{\infty} |\phi_R^{\varepsilon\,(n+2)}|^2 e^{\alpha\xi}\, d\xi + \sum_{i=0}^{n} C_i \int_0^{\infty} |\phi_R^{\varepsilon\,(i)}|^2 e^{\alpha\xi}\, d\xi \\ &\quad + C_n\varepsilon^4 \\ &\le \frac{1}{2}\int_0^{\infty} |\phi_R^{\varepsilon\,(n+2)}|^2 e^{\alpha\xi}\, d\xi + C_n\varepsilon^4,\end{aligned}$$

where we used the Young inequality in the second line, (3.128b)–(3.128c) in the third line, and the induction hypothesis in the fourth line. Thus (3.137) is true for $k = n+2$. By Proposition 3.9–3.10, we know that (3.137) holds for $k = 0, 1$. This finishes the proof of (3.137).

By the Cauchy-Schwarz inequality and the fundamental theorem of calculus , we find that (3.137) implies that for every k,

$$\begin{aligned} \sup_{\xi\in[0,\infty)} |\phi_R^{\varepsilon\,(k)}(\xi)e^{\frac{\alpha}{2}\xi}|^2 &\le 2\|\phi_R^{\varepsilon\,(k)}\|_{L^2_\alpha}\left(\|\phi_R^{\varepsilon\,(k+1)}\|_{L^2_\alpha} + \frac{\alpha}{2}\|\phi_R^{\varepsilon\,(k)}\|_{L^2_\alpha}\right) \\ &\le C_k\varepsilon^4, \end{aligned} \tag{3.138}$$

where $\|\cdot\|_{L^2_\alpha} := \|\cdot e^{\frac{\alpha}{2}\xi}\|_{L^2([0,\infty))}$. Now the estimates for n_R^ε and u_R^ε follow from (3.115) by applying (3.138) and (3.128b). To finish the proof, we recall that $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)$ is symmetric about $\xi = 0$.

□

3.6 Appendix

Solitary wave solutions to (KdV) We consider the KdV equation

$$n_t + Vnn_x + \frac{1}{2V}n_{xxx} = 0,$$

where $V > 0$ is a constant. By letting $\xi = x - \gamma t$ for a constant $\gamma > 0$, we have

$$-\gamma n_\xi + Vnn_\xi + \frac{1}{2V}n_{\xi\xi\xi} = 0.$$

We impose $n, n_\xi, n_{\xi\xi} \to 0$ as $\xi \to \pm\infty$. By integrating in ξ, we obtain

$$-\gamma n + \frac{V}{2}n^2 + \frac{1}{2V}n_{\xi\xi} = 0,$$

and then we multiply it by n_ξ to get

$$-\gamma nn_\xi + \frac{V}{2}n^2n_\xi + \frac{1}{2V}n_{\xi\xi}n_\xi = 0.$$

Integrating in ξ, we have

$$n_\xi^2 = 2V\gamma n^2 - \frac{2V^2}{3}n^3 = 2Vn^2\left(\gamma - \frac{V}{3}n\right),$$

and hence

$$\frac{dn}{n\sqrt{1-\frac{V}{3\gamma}n}} = \sqrt{2V\gamma}\,d\xi.$$

Now we let $\frac{V}{3\gamma}n = \text{sech}^2 w$. Since $\frac{V}{3\gamma}dn = -2\text{sech}^2 w\,\tanh w\,dw$ and $1 - \text{sech}^2 w = \tanh^2 w$,

$$\sqrt{2V\gamma}\,d\xi = \frac{-\frac{3\gamma}{V}2\text{sech}^2 w\,\tanh w\,dw}{\frac{3\gamma}{V}\text{sech}^2 w\,\tanh w} = -2\,dw.$$

Integrating in ξ, we get

$$w = \frac{-\sqrt{2V\gamma}}{2}\xi + \xi_0$$

for some constant ξ_0. We let $\xi_0 = 0$. Since $\text{sech}^2 w$ is symmetric in w,

$$n = \frac{3\gamma}{V}\text{sech}^2\left(\frac{\sqrt{2V\gamma}}{2}\xi\right) = \frac{3\gamma}{V}\text{sech}^2\left(\frac{\sqrt{2V\gamma}}{2}(x-\gamma t)\right).$$

Solutions of the equation (3.23) It is trivial that $z = 1$ satisfies (3.23). By taking the logarithm of (3.23), we find that (3.23) is equivalent to that

$$0 = K \ln z + \ln K + \ln \left[(z-1)^2 + K^{-1} \right] - \frac{K}{2} \left(z^2 - 1 \right) =: f(z).$$

It is straightforward to obtain that

$$\frac{df}{dz}(z) = \frac{-K(z-1)^2}{z\left[(z-1)^2 + K^{-1}\right]} \left(z - \sqrt{\frac{1+K}{K}} \right) \left(z + \sqrt{\frac{1+K}{K}} \right).$$

Since $\lim_{z\to+\infty} f(z) = -\infty$ and $f(1) = 0$, there exists a unique $\zeta_K > \sqrt{\frac{1+K}{K}} > 1$ such that there holds that $f(\zeta_K) = 0$ and $f(z) > 0$ for all $z \in (1, \zeta_K)$. We note that this result also shows that (3.24) holds.

Derivation of (3.66)–(3.67) **and** (3.85)–(3.86) By taking the derivatives of g in n,

$$\begin{cases} g^{(1)}(n) = -J^2 n^{-2} + K + e^H h, \quad g^{(2)}(n) = 2J^2 n^{-3} + e^H h^2 + e^H h', \\ g^{(3)}(n) = -6J^2 n^{-4} + e^H h^3 + 3e^H hh' + e^H h'', \\ g^{(4)}(n) = 24J^2 n^{-5} + e^H h^4 + 6e^H h^2 h' + 3e^H (h')^2 + 4e^H hh'' + e^H h'''. \end{cases}$$

From (3.35) and (3.37), we have

$$H(1) = 0, \quad h(1) = J^2 - K, \quad h'(1) = -3J^2 + K, \quad h''(1) = 12J^2 - 2K, \tag{3.139}$$

and hence we get $g^{(1)}(1) = 0$, $g^{(2)}(1) = (J^2 - K)(J^2 - 1 - K)$ and

$$g^{(3)}(1) = 6J^2 - 2K + (J^2 - K)^3 + 3(J^2 - K)(-3J^2 + K).$$

Using (3.32), a direct calculation yields (3.67a)–(3.67b). From these results and (3.35), (3.37), and (3.31)–(3.32), we see that (3.67c) holds. (3.85)–(3.86) follow in a similar fashion.

Derivation of (3.83) Let $q(n) := n - e^{H(n)}$. It is straightforward to obtain that

$$q^{(1)}(n) = 1 - e^H h, \quad q^{(2)}(n) = (-h^2 - h')e^H, \quad q^{(3)} = (-h^3 - 3hh' - h'')e^H.$$

Using (3.139) and (3.32), we get

$$\begin{aligned} \frac{1}{\varepsilon^2}(n^\varepsilon - e^{H(n^\varepsilon)}) &= \frac{1}{\varepsilon^2} \Big[(1 + K - J^2)(n^\varepsilon - 1) \\ &\quad - \frac{1}{2}((J^2 - K)^2 - 3J^2 + K)(n^\varepsilon - 1)^2 + g^{(3)}(n_b^\varepsilon)(n^\varepsilon - 1)^3 \Big] \\ &= (-2V\gamma + O(\varepsilon))\, \widetilde{N}_\varepsilon + (1 + K + O(\varepsilon))\, \widetilde{N}_\varepsilon^2 + \varepsilon g^{(3)}(n_b^\varepsilon) \widetilde{N}_\varepsilon^3 \end{aligned} \tag{3.140}$$

for some $1 < n_b^\varepsilon < n^\varepsilon$. By (3.31), we obtain (3.83).

Derivation of (3.106) It is straightforward to check that $\partial_E^2\mathcal{R}_1 = \partial_{nE}\mathcal{R}_2 = \partial_E^2\mathcal{R}_2 = 0$ and

$$\begin{cases} \partial_n^2\mathcal{R}_1 = Eh^{-2}h'' - 2Eh^{-3}(h')^2, \quad \partial_{nE}\mathcal{R}_1 = h'h^{-2}, \\ \partial_n^2\mathcal{R}_2 = -\varepsilon^{-1}e^H(h^2 + h'), \end{cases}$$

Now, (3.106) follows from (3.35), (3.37) and (3.31),

3.7 Small Mass Limit of the two-fluid Euler-Poisson System

We consider the two-fluid 1D Euler-Poisson system (2.16) with the isothermal pressure for ions and electrons. In the non-dimensionalized form (by applying the same scaling as (2.20) for the electron quantities), it is given by

$$\begin{cases} \partial_t n_i + \partial_x(n_i u_i) = 0, \\ \partial_t n_e + \partial_x(n_e u_e) = 0, \\ \partial_t u_i + u_i\partial_x u_i + K\partial_x(\ln n_i) = -\partial_x\phi, \\ \mu\left(\partial_t u_e + u_e\partial_x u_e\right) + \partial_x(\ln n_e) = \partial_x\phi, \\ -\partial_x^2\phi = n_i - n_e, \end{cases} \tag{3.141}$$

with the far-field condition

$$n_i, n_e \to 1, \quad u_i, u_e, \phi \to 0,$$

where $\mu = m_e/m_i$ is a constant representing the electron-ion mass ratio.

We recall that the Euler-Poisson system for ions derived from the two-fluid Euler-Poisson system by letting $m_e = 0$. It would be meaningful to justify the solutions of these two system is close to each other for sufficiently small $\mu > 0$. As far as the author knows, there is no rigorous study on this subject. In this section, we present some formal computation, which shows that solutions to (3.141) converges to the those to (1DEP) as $\mu \to 0$ in terms of the KdV limit.

By introducing the Gardner-Morikawa transformation (GM), (3.141) becomes

$$\begin{cases} \varepsilon\partial_t n_i - V\partial_x n_i + \partial_x(n_i u_i) = 0, \\ \varepsilon\partial_t n_e - V\partial_x n_e + \partial_x(n_e u_e) = 0, \\ \varepsilon\partial_t u_i - V\partial_x u_i + u_i\partial_x u_i + K\dfrac{\partial_x n_i}{n_i} = -\partial_x\phi, \\ \mu\left(\varepsilon\partial_t u_e - V\partial_x u_e + u_e\partial_x u_e\right) + \dfrac{\partial_x n_e}{n_e} = \partial_x\phi, \\ -\varepsilon\partial_x^2\phi = n_i - n_e. \end{cases} \tag{3.142}$$

Assuming that the solutions to (3.142) is represented by the formal expansion as (3.5), one can check that the coefficients of ε^0 is zero, and the coefficients of ε satisfy

$$\begin{cases} -V\partial_x n_i^{(1)} + \partial_x u_i^{(1)} = 0, \\ -V\partial_x n_e^{(1)} + \partial_x u_e^{(1)} = 0, \\ -V\partial_x u_i^{(1)} + K\partial_x n_i^{(1)} = -\partial_x\phi^{(1)}, \\ -\mu V\partial_x u_e^{(1)} + \partial_x n_e^{(1)} = \partial_x\phi^{(1)}, \\ n_i^{(1)} = n_e^{(1)}. \end{cases} \tag{3.143}$$

In order to have the non-trivial first order profiles, V must satisfy

$$(1+\mu)V^2 = 1+K \quad \Leftrightarrow \quad V^2 - K = 1-\mu V^2. \tag{3.144}$$

From (3.143), we have the relation

$$\begin{cases} u_i^{(1)} = V n_i^{(1)}, \\ u_e^{(1)} = V n_e^{(1)}, \\ \phi^{(1)} = V u_i^{(1)} - K n_i^{(1)} = (V^2-K) n_i^{(1)}, \\ \phi^{(1)} = -\mu V u_e^{(1)} + n_e^{(1)} = (1-\mu V^2) n_e^{(1)}, \\ n_i^{(1)} = n_e^{(1)}. \end{cases} \tag{3.145}$$

We note that $V^2 - K = 1-\mu V^2 > 0$ for sufficiently small μ and that (3.145) is not an overdetermined system.

At ε^2 order, we obtain

$$\begin{cases} \partial_t n_i^{(1)} - V\partial_x n_i^{(2)} + \partial_x u_i^{(2)} + \partial_x(n_i^{(1)} u_i^{(1)}) = 0, \\ \partial_t u_i^{(1)} - V\partial_x u_i^{(2)} + u_i^{(1)}\partial_x u_i^{(1)} + K\partial_x n_i^{(2)} - K n_i^{(1)}\partial_x n_i^{(1)} = -\partial_x \phi^{(2)}, \\ \partial_t n_e^{(1)} - V\partial_x n_e^{(2)} + \partial_x u_e^{(2)} + \partial_x(n_e^{(1)} u_e^{(1)}) = 0, \\ \mu\left(\partial_t u_e^{(1)} - V\partial_x u_e^{(2)} + u_e^{(1)}\partial_x u_e^{(1)}\right) + \partial_x n_e^{(2)} - n_e^{(1)}\partial_x n_e^{(1)} = \partial_x\phi^{(2)}, \\ -\partial_x^2\phi^{(1)} = n_i^{(2)} - n_e^{(2)}. \end{cases} \tag{3.146}$$

Multiply the first equation by V and then add it to the second equation. Multiply the third equation by μV and then add it to the fourth equation, Then $u_i^{(2)}$ and $u_e^{(2)}$ terms are canceled and we have

$$\begin{cases} 2V\partial_t n_i^{(1)} + (K-V^2)\partial_x n_i^{(2)} + (3V^2-K) n_i^{(1)}\partial_x n_i^{(1)} = -\partial_x\phi^{(2)}, \\ 2\mu V\partial_t n_i^{(1)} + (1-\mu V^2)\partial_x n_e^{(2)} + (3\mu V^2 - 1) n_i^{(1)}\partial_x n_i^{(1)} = \partial_x \phi^{(2)}, \\ -\partial_x^3\phi^{(1)} = \partial_x n_i^{(2)} - \partial_x n_e^{(2)}, \end{cases} \tag{3.147}$$

where we have used (3.145). Add the first and second equation and then use (3.144) and the third equation. Then, we have

$$2V(1+\mu)\partial_t n_i^{(1)} + (1-\mu V^2)\partial_x^3\phi^{(1)} + (3V^2 - K + 3\mu V^2 - 1) n_i^{(1)}\partial_x n_i^{(1)} = 0. \tag{3.148}$$

Equivalently, using (3.144),

$$\partial_t n_i^{(1)} + \frac{(1-\mu V^2)^2}{2V(1+\mu)}\partial_x^3 n_i^{(1)} + V n_i^{(1)}\partial_x n_i^{(1)} = 0. \tag{3.149}$$

In the moving frame $\xi = x - \gamma t$, by imposing that $n_i^{(1)} \to 0$ as $|\xi| \to \infty$, we obtain

$$-\gamma n_i^{(1)} + \frac{(1-\mu V^2)^2}{2V(1+\mu)}(n_i^{(1)})'' + \frac{V}{2}(n_i^{(1)})^2 = 0.$$

From (3.144) and (3.145),

$$-\frac{\gamma}{1-\mu V^2}\phi^{(1)} + \frac{1-\mu V^2}{2V(1+\mu)}(\phi^{(1)})'' + \frac{V}{2(1-\mu V^2)^2}(\phi^{(1)})^2 = 0.$$

The solution to this equation with $\phi^{(1)}, \phi^{(1)\prime} \to 0$ as $|\xi| \to \infty$ is

$$\phi^{(1)}(\xi) = \frac{3(1-\mu V^2)\gamma}{V}\operatorname{sech}^2\left(\frac{\sqrt{2\gamma V(1+\mu)}}{2(1-\mu V^2)}\xi\right).$$

We note that $1 - \mu V^2 = \dfrac{1-\mu K}{1+\mu} > 0$ if $V = \sqrt{\dfrac{1+K}{1+\mu}}$ and $1 - \mu K > 0$. As $\mu \to 0$, $\phi^{(1)}(\xi)$ converges to (3.3), the rigorous approximation of $\varepsilon^{-1}\phi$ of the one-fluid Euler-Poisson system.

4 Linear Stability of Solitary Waves for the Euler-Poisson System

4.1 Notions of Stability for Solitary Waves

That we can observe some phenomena or objects means that they stay 'stable' for enough time we can detect them. The subject of stability in mathematical aspects would be thus important. It should be pointed out that, however, the terminology 'stability' is somewhat vague; in which sense are solitary waves stable? We need to establsih suitable notions of stability, and it would depend on the properties of solitary waves and the underlying structures of partial differential equations.

We have seen in Section 3 that in a certain scaled frame, the small amplitude EP solitary waves are well approximated by the KdV solitary waves. On the other hand, the stability of solitary waves for the KdV equation has been extensively studied. In what follows, we investigate those notions of stability of traveling solitary waves in the context of our problem.

The result of Section 3 states that the 1D Euler-Poisson system

$$\begin{cases} \partial_t n + \partial_s((1+n)u) = 0, \\ \partial_t u + u\partial_s u + K\dfrac{\partial_s n}{1+n} = -\partial_s \phi, \\ -\partial_s^2 \phi = (1+n) - e^{\phi}, \end{cases} \tag{4.1}$$

with

$$n, u, \phi \to 0 \quad \text{as} \quad s \to \pm\infty, \tag{4.2}$$

has a two-parameter (c, γ) family of traveling solitary wave solutions

$$(n, u, \phi)(s, t) = (n_c, u_c, \phi_c)(s - ct + \gamma)$$

for all $\sqrt{1+K} < c < \zeta_K\sqrt{K}$ (when $K > 0$ for instance) and $\gamma \in \mathbb{R}$.[9,10] We let $\mathbf{u} := (n, u)^T$ and $\mathbf{u}_c := (n_c, u_c)^T$. ϕ or ϕ_c is determined by the Poisson equation of (4.1). In the moving frame $x = s - ct$, $\mathbf{u}_c$ is the stationary solitary wave solution. We point out that the speed c is a parameter and the amplitude of the waves depends on the speed parameter. This is the distinguishing feature of traveling solitary waves.

Orbital Stability For a fixed c, $\mathbf{u}_c(\cdot + \gamma)|_{\gamma\in\mathbb{R}}$ is a one-parameter family of solitary waves. One may consider $\mathbf{u}_c(\cdot + \gamma)$ as a curve (or orbit) parametrized by $\gamma \in \mathbb{R}$ in some function spaces. In this point of view, orbital stability means that the solution of an evolution equation stays

[9] Let $c = \sqrt{1+K} + \varepsilon$ and $n_c(s - ct) := n(\varepsilon^{1/2}(s - (\sqrt{1+K} + \varepsilon)t)) - 1$.

[10] By letting $c = -c'$ and $u = -u'$, we obtain the traveling waves moving to the left direction. This contrasts with that the KdV equation does not have solitary waves traveling to the left direction. This is one reason people say that the KdV equation is uni-directional. Other explaination is that the group velocity of the KdV has one sign. We remark that the KdV equation has the reversibility $t \to -t'$, $x \to -x'$.

close to the orbit $\mathbf{u}_c(\cdot+\gamma)|_{\gamma\in\mathbb{R}}$ for all time $t>0$. More precisely, we say that the one-parameter family of solitary waves $\mathbf{u}_c(\cdot+\gamma)|_{\gamma\in\mathbb{R}}$ is *orbitally stable*, if for a given $\varepsilon>0$ there is some $\delta>0$ such that

$$\|\mathbf{u}_0-\mathbf{u}_c\|\leq\delta\Rightarrow\inf_{\gamma\in\mathbb{R}}\|\mathbf{u}(\cdot,t)-\mathbf{u}_c(\cdot+\gamma)\|\leq\varepsilon\quad\text{for all } t>0.$$

Since each point of the orbit is some spatial translation of $\mathbf{u}_c$, the orbital stability implies that the 'shape' of the solitary wave is stable. We remark that the orbital stability of the *traveling* solitary wave $\mathbf{u}_c(s-ct)$ in the original variable is equivalent to the orbital stability of the *stationary* solitary wave $\mathbf{u}_c(x)$ in the moving frame $x=s-ct$, if one considers translation invariant norms such as L^2-Sobolev norms.

The classical frameworks such as [2] and [3] for the orbital stability of solitary waves invoke the special structures of converved quantities and the second variation of a constrained Hamiltonian. The issue for our case is that whether the Euler-Poisson system has such a 'good' structure. Indeed, the 1D Euler-Poisson system (4.1) possesses a Hamiltonian

$$\begin{aligned}H_E(n,u)&:=H(1+n,u)\\&=\int\frac{(1+n)u^2}{2}+K((1+n)\ln(1+n)-n)-\frac{|\partial_x\phi|^2}{2}+(1+n)\phi-e^{\phi}+1\,dx\end{aligned}$$

and the momentum

$$M_{\mathrm{E}}(n,u):=M(1+n,u)-\int u\,dx=\int nu\,dx.$$

where H and M are defined in (2.6) and (2.5b) respectively. H_E and M_E are conserved quantities. We let $V_{\mathrm{E}}(n,u):=H_{\mathrm{E}}-cM_{\mathrm{E}}$. Some formal calculations of (2.8) and (2.9) yield that

$$\frac{\delta V_E}{\delta n}=\frac{u^2}{2}+K\ln(1+n)+\phi-cu,\qquad\frac{\delta V_E}{\delta u}=(1+n)u-cn.$$

We see that the solution (n_c,u_c) to the EP system is a critical point of the constrained Hamiltonian V_E. The second variation of V_E is given by

$$\begin{bmatrix}\frac{\delta^2V_E}{\delta n^2}&\frac{\delta^2V_E}{\delta n\delta u}\\\frac{\delta^2V_E}{\delta u\delta n}&\frac{\delta^2V_E}{\delta u^2}\end{bmatrix}=\begin{bmatrix}\frac{K}{1+n}&u-c\\u-c&1+n\end{bmatrix}+\begin{bmatrix}\partial_n\phi&0\\0&0\end{bmatrix},$$

where $\partial_n\phi$ denotes a formal variational derivative. Here the first matrix on the RHS a saddle point at $(n,u)=(n_c,u_c)$ for sufficiently small ε. And what $\partial_n\phi$ is?

Asymptotic Stability On the other hand, the notion of orbital stability of solitary waves does not tell us the asymptotic 'location' of the perturbed waves. For the initial data of $\mathbf{u}(s,t)$ sufficiently close to $\mathbf{u}_c(s+\gamma)$, if there exists a fixed $\gamma_+\in\mathbb{R}$ such that

$$\mathbf{u}(s,t)\to\mathbf{u}_c(s-ct+\gamma_+)\quad\text{as}\quad t\to+\infty,$$

then we say that a one-parameter family of waves $\mathbf{u}_c(\cdot+\gamma)|_{\gamma\in\mathbb{R}}$ is *asymptotically (orbitally) stable.* While this notion of stability is typical for the viscous shock waves of viscous conservation laws, it cannot be expected for solitary waves in general. Roughly speaking, this is because the speed of traveling solitary waves is a parameter which is also related to the amplitude of waves. To illustrate, we consider the initial data $\mathbf{u}_0(s) = \mathbf{u}_{c'}(s) \approx \mathbf{u}_c(s)$ for $c' \approx c$ so that we have $\mathbf{u}(s,t) = \mathbf{u}_{c'}(s - c't)$. Since $\mathbf{u}_{c'}(s - c't)$ and $\mathbf{u}_c(s - ct)$ travel with different speeds, $\mathbf{u}(s,t) = \mathbf{u}_{c'}(s - c't)$ cannot approach to any fixed translation of $\mathbf{u}_c(s - ct)$ as $t \to +\infty$. In fact, this is a reason that the orbital stability is an appropriate notion of the stability for solitary waves.

Small perturbations of traveling solitary waves can yield a small change in the speed. Hence, one should allow the parameter c vary. We expects that if the initial data $\mathbf{u}_0(s)$ is sufficiently close to $\mathbf{u}_c(s + \gamma)$, then there is some fixed (c_+, γ_+) sufficiently close to (c, γ) such that

$$\mathbf{u}(s,t) - \mathbf{u}_{c_+}(s - c_+ t + \gamma_+) \to 0 \quad \text{as } t \to +\infty. \tag{4.3}$$

In such a case, we say that *the two-parameter family of solitary waves is asymptotically stable*, and this can be understood as follows: as $t \to +\infty$, $\mathbf{u}(s,t)$ converges to a point of the two-dimensional manifold $\{\mathbf{u}_c(\cdot + \gamma) : c \in \mathbb{R} \text{ and } \gamma \in \mathbb{R}\}$ in some function spaces. We investigate in which norm the convergence (4.3) can be expected.

Small perturbation of the EP solitary waves is governed by the linearized EP system around the solitary waves. We assume that in a moving frame $x = s - ct$, small amplitude waves are governed by

$$\begin{cases} \partial_t n - c\partial_x n + \partial_x u = 0, \\ \partial_t u - c\partial_x u + K\partial_x n = -\partial_x \phi, \\ -\partial_x^2 \phi = n - \phi. \end{cases} \tag{4.4}$$

This assumption would be plausible if small waves are sufficiently far from the peak of large solitary waves. If we assume that the solution of (4.4) is a superposition of wave packets $A(k)e^{ikx + iw(k)t}$, then each packet propagates with the group velocities

$$-\frac{d\omega_\pm(k)}{dk} = -c \mp \frac{1 + K(1+k^2)^2}{\sqrt{\frac{1}{1+k^2} + K(1+k^2)^2}}, \qquad \begin{cases} -\dfrac{d\omega_+(k)}{dk} < -\sqrt{1+K}, \\ -\dfrac{d\omega_+(k)}{dk} < -\varepsilon. \end{cases}$$

This implies that small waves travel to the left in the frame $x = s - ct$.[11] In this regard, we see that L^2-Sobolev norm might be inappropriate for the asymptotic stability. First of all, L^2 norm is invariant in space translation. In fact, L^2-Sobolev norm of the solution $(n, u, \phi, \partial_x \phi)$ to the linearized EP system (4.4) is conserved.[12,13]

Since large solitary waves travel faster than smaller solitary waves, we expect the convergence (4.3) in the moving frame with the speed of the largest solitary wave by introducing exponentially

[11] In the original space variable s, the wave packets propagate in both directions.

[12] Precisely, when $K = 0$, $\|u\|_{H^1}^2 + \|\phi\|_{L^2}^2 + 2\|\partial_x\phi\|_{L^2}^2 + \|\partial_x^2\phi\|_{L^2}^2$ is conserved, and we have $\|n\|_{L^2}^2 = \|\phi\|_{L^2}^2 + 2\|\partial_x\phi\|_{L^2}^2 + \|\partial_x^2\phi\|_{L^2}^2$. Thus, $\|u\|_{H^1}^2 + \|n\|_{L^2}^2$ is also conserved. When $K > 0$, $K\|n\|_{L^2}^2 + \|(u, \phi, \partial_x\phi)\|_{L^2}^2$ is conserved.

[13] Question: do the solutions decay in L^∞ norm? If the data is localized in low frequency, similar behavior to the Airy equation?

weighted norm:

$$\|f(x)\|_{H^s_\eta(\mathbb{R})} := \|e^{\eta x} f(x)\|_{H^s(\mathbb{R})}, \quad (\eta > 0).$$

In the same spirit, this weighted norm was introduced in the context of the asymptotic stability of the KdV solitary waves [27]. In order to see the meaning of convergence in the weight norm, we suppose that a function $f(x,t)$ converges to zero in $H^1_\eta(\mathbb{R})$. For any fixed bounded interval $I \subset \mathbb{R}$, $f(x,t)$ is bounded, and there are two possibilities:

1. $f(x,t)$ escapes the interval I to the left, or
2. $f(x,t)$ decreases to zero without leaving I.

In both cases, $f(x,t)$ uniformly converges to zero on the interval I, and hence, convergence in $H^1_\eta(\mathbb{R})$ norm implies *local uniform convergence*. The first mechanism of stability is called *convective* stability, and the second one is called *absolute* stability. As we observed, the stability mechanism for the solitary waves is convective in the moving frame of the largest solitary wave.

Linear Asymptotic Stability As a first step toward the non-linear stability, we will study the linear version of the asymptotic stability (4.3) in Section 4. In the moving frame $x = s - ct$, we may write (4.1) as (see (4.11))

$$\partial_t \mathbf{u} = \mathcal{F}_c(\mathbf{u}_c). \tag{4.5}$$

Here we note that the nonlinear operator $\mathcal{F}_c$ involves a parameter c and that the two-parameter family of solitary waves $\mathbf{u}_c(\cdot + \gamma)$ is a stationary solution to (4.5). Let $\widetilde{\mathbf{u}} := \mathbf{u} - \mathbf{u}_c$ and consider the linearized EP system around $\mathbf{u}_c$:

$$\partial_t \widetilde{\mathbf{u}} = \mathcal{L}\widetilde{\mathbf{u}}. \tag{4.6}$$

Due to the translation invariance and the speed parameter, $\mathcal{L}$ has a zero eigenvalue with algebraic multiplicity at least two. More precisely, we have

$$\mathcal{L}\partial_x \mathbf{u}_c = 0, \quad \mathcal{L}\partial_c \mathbf{u}_c = -\partial_x \mathbf{u}_c.$$

Indeed, by taking ∂_γ and ∂_c of $0 = \mathcal{F}_c\left(\mathbf{u}_c(\cdot + \gamma)\right)$, we observe that

$$\begin{aligned} 0 = \partial_\gamma \left[\mathcal{F}_c\left(\mathbf{u}_c(\cdot+\gamma)\right)\right] \quad &\Rightarrow \quad 0 = (\nabla_{\mathbf{u}} \mathcal{F}_c)\partial_x \mathbf{u}_c(\cdot+\gamma) \\ &\Rightarrow \quad 0 = \mathcal{L}\partial_x \mathbf{u}_c(\cdot+\gamma), \end{aligned}$$

$$\begin{aligned} 0 = \partial_c \left[\mathcal{F}_c\left(\mathbf{u}_c(\cdot+\gamma)\right)\right] \quad &\Rightarrow \quad 0 = (\partial_c \mathcal{F}_c)\mathbf{u}_c(\cdot+\gamma) + (\nabla_{\mathbf{u}} \mathcal{F}_c)\partial_c \mathbf{u}_c(\cdot+\gamma) \\ &\Rightarrow \quad 0 = \partial_x \mathbf{u}_c(\cdot+\gamma) + \mathcal{L}\partial_c \mathbf{u}_c(\cdot+\gamma). \end{aligned}$$

Thus,

$$\widetilde{\mathbf{u}}(x,t) = \partial_x \mathbf{u}_c + (\partial_c \mathbf{u}_c - t\partial_x \mathbf{u}_c)$$

is a non-decaying solution to (4.6). We will study the linear asymptotic stability modulo the span of these two non-decaying modes, $\mathrm{span}\{\partial_x \mathbf{u}_c, \partial_c \mathbf{u}_c\}$.

The linear asymptotic stability can be interpreted as follows: the solution to the linearized equation around $\mathbf{u}_c(\cdot+\gamma)$ asymptotically converges to a point of the tangent plane of the two-parameter family of solitary waves at $\mathbf{u}_c(\cdot+\gamma)$. By differentiating $\mathbf{u}_c(s-ct+\gamma)$ in c and γ, we have the formal expansion of $\mathbf{u}_{c_+}(\cdot+\gamma_+)$ around (c,γ):

$$\begin{aligned}\mathbf{u}_{c_+}(s-c_+t+\gamma_+) \approx \mathbf{u}_c(s-ct+\gamma) &+ (\gamma_+-\gamma)\left[\partial_x\mathbf{u}_c(s-ct+\gamma)\right] \\ &+ (c_+-c)\left[\partial_c\mathbf{u}_c(s-ct+\gamma) - t\partial_x\mathbf{u}_c(s-ct+\gamma)\right].\end{aligned} \tag{4.7}$$

The RHS of (4.7) can be interpreted as the tangent plane of the two-dimensional manifold $\mathbf{u}_{c_+}(s-c_+t+\gamma_+)$ at $\mathbf{u}_c(s-ct+\gamma)$ with a basis

$$\{\partial_x\mathbf{u}_c(\cdot+\gamma), \partial_c\mathbf{u}_c(\cdot+\gamma) - t\partial_x\mathbf{u}_c(\cdot+\gamma)\}.$$

The position $(c_+-c, \gamma_+-\gamma)$ at which the solution of the linear equation converges as $t\to+\infty$ is determined by the initial condition of the linear equation.

4.2 Main Results

The previous section concerns the existence of traveling solitary wave solutions to (4.1)–(4.2) and the asymptotic behavior of the solutions in a certain stretched moving frame as $\varepsilon\to 0$. We summarize the result.

Theorem 4.1. *Let* $\mathsf{V}=\sqrt{1+K}$ *and* $c=\sqrt{1+K}+\varepsilon$ *for* $K\geq 0$. *Let* k *be any non-negative integer. Then the following hold:*

1. *(a) For all sufficiently small* $\varepsilon>0$, *(4.1)–(4.2) admits a non-trivial (smooth) traveling solitary wave solution* $(n_c,u_c,\phi_c)(x)$, *where* $x=s-ct$.

   (b) There exist positive constant ε_0, C, *and* C_k *such that for all* $\varepsilon\in[0,\varepsilon_0]$,

$$|\partial_x^k n_c(x)| + |\partial_x^k u_c(x)| + |\partial_x^k \phi_c(x)| \leq C_k\varepsilon^{k/2+1}e^{-C\varepsilon^{1/2}|x|}. \tag{4.8}$$

2. *Let* $(n_*,u_*,\phi_*)(\xi) := \varepsilon^{-1}(n_c,u_c,\phi_c)(x)$, *where* $\xi:=\varepsilon^{1/2}x$. *There exist positive constants* ε_0, C *and* C_k *such that for all* $\varepsilon\in(0,\varepsilon_0]$,

$$|\partial_\xi^k(n_*-\Psi_K)| + |\partial_\xi^k(u_*-\mathsf{V}\Psi_K)| + |\partial_\xi^k(\phi_*-\Psi_K)| \leq C_k\varepsilon e^{-C|\xi|}, \tag{4.9}$$

   where

$$\Psi_K(\xi) := \frac{3}{\mathsf{V}}\operatorname{sech}^2\left(\sqrt{\frac{\mathsf{V}}{2}}\xi\right) \tag{4.10}$$

   is a solution to

$$-\partial_\xi\Psi_K + \mathsf{V}\Psi_K\partial_\xi\Psi_K + \frac{1}{2\mathsf{V}}\partial_\xi^2\Psi_K = 0.$$

   For $\varepsilon=0$, $(n_*,u_*,\phi_*)=(\Psi_K,\mathsf{V}\Psi_K,\Psi_K)$.

In the moving frame $x = s - ct$ and $t = t$, (4.1) becomes

$$\begin{cases} \partial_t n - c\,\partial_x n + \partial_x((1+n)u) = 0, \\ \partial_t u - c\,\partial_x u + u\partial_x u + K\dfrac{\partial_x n}{1+n} = -\partial_x \phi, \\ -\partial_x^2 \phi = (1+n) - e^{\phi}. \end{cases} \tag{4.11}$$

From Theorem 4.1, the solitary wave solutions $(n_c, u_c, \phi_c)(x)$ satisfy

$$\begin{cases} -\,c\,\partial_x n_c + \partial_x((1+n_c)u_c) = 0, & (4.12a) \\ -\,c\,\partial_x u_c + u_c\partial_x u_c + K\dfrac{\partial_x n_c}{1+n_c} = -\partial_x \phi_c, & (4.12b) \\ -\,\partial_x^2 \phi_c = (1+n_c) - e^{\phi_c}. & (4.12c) \end{cases}$$

The linearization of the Euler-Poisson system (4.11) around (n_c, u_c, ϕ_c) is given by

$$\begin{cases} \partial_t \begin{pmatrix} n \\ u \end{pmatrix} + L_1\partial_x \begin{pmatrix} n \\ u \end{pmatrix} + L_2 \begin{pmatrix} n \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ -\partial_x \phi \end{pmatrix}, & (4.13a) \\ -\,\partial_x^2 \phi = n - e^{\phi_c}\phi, & (4.13b) \end{cases}$$

where $L_1 = L_1(x, \varepsilon)$ and $L_2 = L_2(x, \varepsilon)$ are the matrices defined by

$$L_1 := \begin{pmatrix} -c + u_c & 1 + n_c \\ \dfrac{K}{1+n_c} & -c + u_c \end{pmatrix}, \quad L_2 := \begin{pmatrix} \partial_x u_c & \partial_x n_c \\ -\dfrac{K\partial_x n_c}{(1+n_c)^2} & \partial_x u_c \end{pmatrix}. \tag{4.14}$$

For given $n \in L^2(\mathbb{R})$, a unique solution $\phi =: (-\partial_x^2 + e^{\phi_c})^{-1}(n)$ to the linear Poisson equation (4.13b) exists in $H^2(\mathbb{R})$ since ϕ_c is small.

The solution of the linearized EP system (4.13) will be represented in terms of the C_0-semigroup. The spectral information of the generator gives the asymptotic behavior of the semigroup (see Section 5 for the related spectral and semigroup theory). By substituting the Ansatz $(n, u, \phi) = e^{\lambda t}(n, u, \phi)(x)$ into (4.13), we consider the eigenvalue problem for the Euler-Poisson system,

$$(\lambda - \mathcal{L})(n, u)^T = (0, 0)^T, \tag{4.15}$$

where we let

$$\mathcal{L} \begin{pmatrix} n \\ u \end{pmatrix} := -\left[(L_1\partial_x + L_2) \begin{pmatrix} n \\ u \end{pmatrix} + \begin{pmatrix} 0 \\ \partial_x(-\partial_x^2 + e^{\phi_c})^{-1}(n) \end{pmatrix} \right]. \tag{4.16}$$

Due to the translation invariance and that the speed c is a parameter, $\lambda = 0$ is an eigenvalue of $\mathcal{L}$ in L^2 with algebraic multiplicity at least two. Indeed, by differentiating (4.12) in x and c, we obtain that

$$\mathcal{L}\partial_x(n_c, u_c)^T = (0, 0)^T, \quad \mathcal{L}\partial_c(n_c, u_c)^T = -\partial_x(n_c, u_c)^T. \tag{4.17}$$

Since $(n_c, u_c, \phi_c)^T$ and $\partial_x(n_c, u_c, \phi_c)^T$ exponentially decay to zero as $|x| \to +\infty$, the essential spectrum of $\mathcal{L}$ in L^2 coincides with the imaginary axis of the complex plane. Moreover, we will show that there is no eigenvalue of $\mathcal{L}$ with $\operatorname{Re}\lambda > 0$. In other words, *the two-parameter family of solitary waves for the Euler-Poisson system is spectrally stable in* L^2 (Theorem 4.21).

To study the linear asymptotic stability, we define

$$\|f(x)\|_{H^s_\eta(\mathbb{R})} := \|e^{\eta x} f(x)\|_{H^s(\mathbb{R})}, \quad (\eta > 0), \tag{4.18}$$

where $H^s(\mathbb{R})$ is the usual L^2 Sobolev norm. While the norm (4.18) moves the essential spectrum, it does not change the location of the zero eigenvalue of $\mathcal{L}$. The essential spectrum of $\mathcal{L}$ in L^2_η space consists of the images of two parametrized curves strictly lying on the open left-half plane of the complex plane (Proposition 4.6). $\lambda = 0$ is the only eigenvalue of $\mathcal{L}$ with multiplicity two on some open set containing the closed right-half plane (Theorem 4.22). The corresponding eigenvector and the generalized eigenvector are given by $\partial_x(n_c, u_c)^T$ and $\partial_c(n_c, u_c)^T$ respectively. This idea−separating the essential spectrum and the embedded eigenvalue−was first introduced by [33] in the study of stability of traveling waves of parabolic system, and it is successfully adopted to the stability of the KdV solitary waves in [27].

In order to prove the linear asymptotic stability, we will show in Section 4.6 that

(i) $\mathcal{L}$ generates a C_0-semigroup,

(ii) $\lambda = 0$ is an isolated eigenvalue of $\mathcal{L}$ with algebraic multiplicity two,

(iii) $(\lambda - \mathcal{L})^{-1}$ is uniformly bounded on $\operatorname{Re}\lambda > 0$, outside any small neighbourhood of the origin.

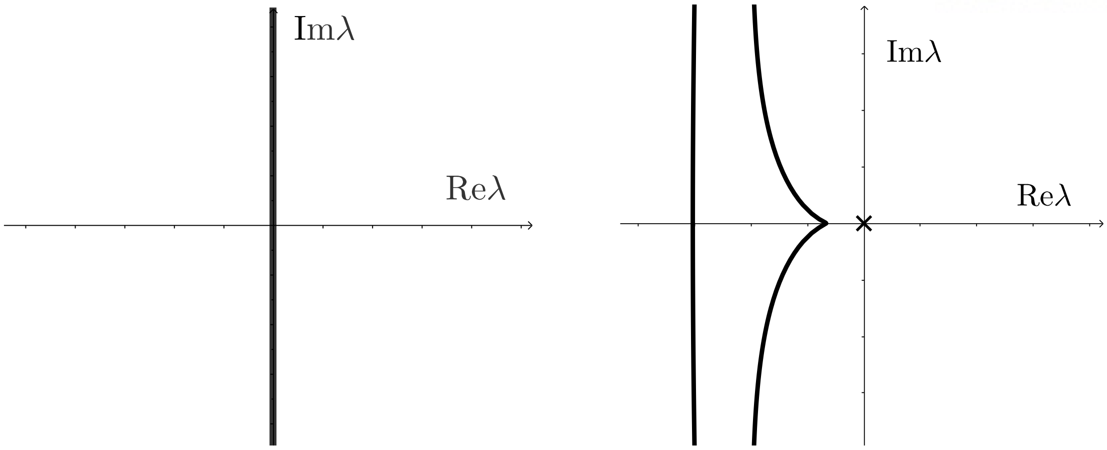
Applying the result of Prüss [29] (see Theorem 5.29), we then obtain our main result, Theorem 4.2. For a Hilbert space $\mathcal{H}$, we denote $\mathcal{H} \times \mathcal{H}$ by $(\mathcal{H})^2$.

Theorem 4.2 (Linear convective stability of solitary waves)**.** *Consider the operator* $\mathcal{L} : (L^2_\eta)^2 \to (L^2_\eta)^2$ *with dense domain* $(H^1_\eta)^2$. *For* $0 < c_0 < \sqrt{\frac{2\mathrm{V}}{3}}$ *and* $\varepsilon > 0$, *let* $\eta = c_0\varepsilon^{1/2}$. *Then there exist* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$, *the following holds: for given* $(n_0, u_0)^T \in (L^2_\eta)^2$ *with* $P(n_0, u_0)^T = 0$, *where* P *is the spectral projection onto the generalized eigenspace of* $\mathcal{L}$, *we have*

$$\|(n(t), u(t))\|_{(L^2_\eta)^2} \leq Ce^{-\varepsilon' t}\|(n_0, u_0)\|_{(L^2_\eta)^2} \tag{4.19}$$

for some $\varepsilon' > 0$.

To study the eigenvalue problem, we will apply the Evans function, which is particularly useful for detecting eigenvalues and their algebraic multiplicity. Calculating the Evans function is not simple in general. On the other hand, the Evans function for the KdV equation is explicitly known [27]. Our strategy is to show that in a special scaling, the Evans function for the EP system converges to that for the KdV equation. The work of [28] concerns a similar issue for some Boussinesq systems. We refer to Section 5 for the general description on the linear stability of nonlinear waves and some prerequisites such as spectral theory, semigroup theory, and the Evans function.



(a) Spectrum of in unweighted spaces (b) Spectrum in exponentially weighted spaces

Figure 5: The bold curves indicate the essential spectrums of $\mathcal{L}$. The zero eigenvalue of $\mathcal{L}$ is isolated in $e^{\eta x}$-weighted L^2 spaces for sufficiently small $\eta > 0$.

Eigenvalue problem of the Euler-Poisson system and the KdV equation We formally observe how the eigenvalue problem (4.15) is related to the eigenvalue problem of the KdV equation. By intoducing the KdV scaling

$$\xi = \varepsilon^{1/2} x, \quad \lambda = \varepsilon^{3/2}\Lambda, \tag{4.20}$$

(4.15) becomes

$$\begin{cases} \varepsilon\Lambda n - c\partial_\xi n + \partial_\xi u + \partial_\xi(\varepsilon n_* u + \varepsilon u_* n) = 0, & (4.21a) \\ \varepsilon\Lambda u - c\partial_\xi u + K\partial_\xi\left(\dfrac{n}{1+\varepsilon n_*}\right) + \partial_\xi(\varepsilon u_* u) = -\partial_\xi\phi, & (4.21b) \\ -\varepsilon\partial_\xi^2\phi = n - e^{\varepsilon\phi_*}\phi. & (4.21c) \end{cases}$$

By integrating (4.21a)–(4.21b) in ξ, we formally obtain that (recall that $c = \sqrt{1+K+\varepsilon}$)

$$\begin{cases} -\sqrt{1+K}\, n + u = O(\varepsilon), & (4.22a) \\ Kn - \sqrt{1+K}\, u + \phi = O(\varepsilon), & (4.22b) \\ n - \phi = O(\varepsilon). & (4.22c) \end{cases}$$

Taking derivative of (4.21c) in ξ, and then subtracting the resulting equation from (4.21b), $-\partial_\xi\phi$ term in the RHS of (4.21b) is canceled. Then, by applying the Taylor expansion, we obtain

$$\varepsilon\Lambda u - (\sqrt{1+K} + \varepsilon)\partial_\xi u + (K+1)\partial_\xi n - K\partial_\xi(\varepsilon n_* n) + \partial_\xi(\varepsilon u_* u) + \varepsilon\partial_\xi^3\phi - \partial_\xi(\varepsilon\phi_*\phi) = O(\varepsilon^2). \tag{4.23}$$

Multiplying (4.21a) by $\mathsf{V} = \sqrt{1+K}$ and then adding the resulting equation to (4.23), we see that $-\sqrt{1+K}\,\partial_\xi u$ and $(1+K)\partial_\xi n$ terms in (4.23) are canceled, and we have

$$\begin{aligned} &\mathsf{V}\Lambda n - \mathsf{V}\partial_\xi n + \mathsf{V}\partial_\xi(n_* u + u_* n) \\ &+ \Lambda u - \partial_\xi u - K\partial_\xi(n_* n) + \partial_\xi(u_* u) + \partial_\xi^3\phi - \partial_\xi(\phi_*\phi) = O(\varepsilon). \end{aligned} \tag{4.24}$$

Using the relation (4.22) and Theorem 4.1, we obtain from (4.24) that

$$\Lambda n - \partial_\xi n + \mathsf{V}\partial_\xi(\Psi_K n) + \frac{1}{2\mathsf{V}}\partial_\xi^3 n = O(\varepsilon). \tag{4.25}$$

4.3 Reformulation of the Eigenvalue Problem

Since the operator $\mathcal{L}$ involves the nonlocal operator $(-\partial_x^2 + e^{\phi_c})^{-1}(n)$, it is convenient to rewrite $\lambda - \mathcal{L}$ as the associated linear ordinary differential operator. For notational simplicity, we let

$$J = J(x,\varepsilon) := (c-u_c)^2 - K. \tag{4.26}$$

Since $\sup_{x\in\mathbb{R}}|u_c(x)| = O(\varepsilon)$, $J(x,\varepsilon)$ converges to 1 as $\varepsilon \to 0$ uniformly in $x \in \mathbb{R}$.[14] Hence the matrix L_1 is invertible, and we have from (4.15) and (4.16) that

$$\begin{aligned} \begin{pmatrix} n_x \\ u_x \end{pmatrix} &= L_1^{-1}\left[-(\lambda I_2 + L_2)\begin{pmatrix} n \\ u \end{pmatrix} + \begin{pmatrix} 0 \\ -\phi_x \end{pmatrix}\right] \\ &= \frac{1}{J}\begin{pmatrix} -c+u_c & -(1+n_c) \\ \dfrac{-K}{1+n_c} & -c+u_c \end{pmatrix}\begin{pmatrix} -(u_c)_x n - (n_c)_x u - \lambda n \\ \dfrac{K(n_c)_x n}{(1+n_c)^2} - (u_c)_x u - \lambda u - \phi_x \end{pmatrix}. \end{aligned} \tag{4.27}$$

We rewrite the Poisson equation as

$$\phi_x =: \psi, \quad \psi_x = e^{\phi_c}\phi - n. \tag{4.28}$$

By letting $\mathbf{y} := (n,u,\phi,\psi)^T$, the eigenvalue problem (4.15) is written as

$$\mathcal{A}(\lambda)\mathbf{y} := \left(\frac{d}{dx} - A(x,\lambda,\varepsilon)\right)\mathbf{y} = (0,0,0,0)^T, \tag{4.29}$$

where

$$A = A(x,\lambda,\varepsilon) := \begin{pmatrix} L_1^{-1} & 0_2 \\ 0_2 & I_2 \end{pmatrix}\left[\begin{array}{c|cc} -\lambda I_2 - L_2 & \begin{matrix} 0 & 0 \\ 0 & -1 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \\ -1 & 0 \end{matrix} & \begin{matrix} 0 & 1 \\ e^{\phi_c} & 0 \end{matrix} \end{array}\right]. \tag{4.30}$$

[14]Indeed, J is always positive as long as the solitary wave exists. This follows from (4.81d) and the analysis of the function h in the previous work. This fact can be used for the large amplitude solitary wave.

Here A has the form of $A(x,\lambda,\varepsilon) := A_1(x,\varepsilon) + \lambda A_2(x,\varepsilon)$, where

$$A_1 := \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{-L_1^{-1}L_2} & \multicolumn{2}{c}{L_1^{-1}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}} \\ \hline 0 & 0 & 0 & 1 \\ -1 & 0 & e^{\phi_c} & 0 \end{array}\right] \tag{4.31a}$$

$$= \begin{pmatrix} \dfrac{(u_c)_x(c-u_c)}{J} - \dfrac{K(n_c)_x}{J(1+n_c)} & \dfrac{(n_c)_x(c-u_c)}{J} + \dfrac{(u_c)_x(1+n_c)}{J} & 0 & \dfrac{1+n_c}{J} \\ \dfrac{K(u_c)_x}{J(1+n_c)} - \dfrac{K(c-u_c)(n_c)_x}{J(1+n_c)^2} & \dfrac{K(n_c)_x}{J(1+n_c)} + \dfrac{(c-u_c)(u_c)_x}{J} & 0 & \dfrac{c-u_c}{J} \\ 0 & 0 & 0 & 1 \\ -1 & 0 & e^{\phi_c} & 0 \end{pmatrix}, \tag{4.31b}$$

$$A_2 := \left[\begin{array}{c|c} -L_1^{-1} & \mathbf{0}_2 \\ \hline \mathbf{0}_2 & \mathbf{0}_2 \end{array}\right] = \frac{1}{J}\left[\begin{array}{cc|c} c-u_c & 1+n_c & \\ \dfrac{K}{1+n_c} & c-u_c & \mathbf{0}_2 \\ \hline \multicolumn{2}{c|}{\mathbf{0}_2} & \mathbf{0}_2 \end{array}\right]. \tag{4.31c}$$

Proposition 4.3. *Consider the operators* $\lambda - \mathcal{L} : (H^1)^2 \subset (L^2)^2 \to (L^2)^2$ *and* $\mathcal{A}(\lambda) : (H^1)^4 \subset (L^2)^4 \to (L^2)^4$*. Then the following hold.*

(a) $\lambda - \mathcal{L}$ *is not Fredholm with index* 0 *if and only if* $\mathcal{A}(\lambda)$ *is not Fredholm with index* 0*. In this case,* $\lambda \in \sigma_{\mathrm{ess}}(\mathcal{L})$*.*

(b) $\lambda - \mathcal{L}$ *is Fredholm with index* 0 *and* $\mathcal{N}(\lambda - \mathcal{L}) = \{\mathbf{0}\}$ *if and only if* $\mathcal{A}(\lambda)$ *is Fredholm with index* 0 *and* $\mathcal{N}(\mathcal{A}(\lambda)) = \{\mathbf{0}\}$*. In this case,* $\lambda \in \rho(\mathcal{L})$*.*

(c) $\lambda - \mathcal{L}$ *is Fredholm with index* 0 *and* $\mathcal{N}(\lambda - \mathcal{L}) \neq \{\mathbf{0}\}$ *if and only if* $\mathcal{A}(\lambda)$ *is Fredholm with index* 0 *and* $\mathcal{N}(\mathcal{A}(\lambda)) \neq \{\mathbf{0}\}$*. In this case,* $\lambda \in \sigma_{\mathrm{pt}}(\mathcal{L})$*.*

These statements also hold for the operators $\lambda - \mathcal{L} : (H^1_\eta)^2 \subset (L^2_\eta)^2 \to (L^2_\eta)^2$ *and* $\mathcal{A}(\lambda) : (H^1_\eta)^4 \subset (L^2_\eta)^4 \to (L^2_\eta)^4$*.*

The proof of Proposition 4.3 is given in Section 4.7. Due to the characterization of the Fredholm properties of $\mathcal{A}(\lambda)$ in terms of exponential dichotomies ([23],[24]) and the roughness of exponential dichotomies ([7]), there is no $\lambda \in \mathbb{C}$ such that $\mathcal{A}(\lambda)$ (hence $\lambda - \mathcal{L}$) is Fredholm with non-zero index since $\lim_{x\to+\infty} A(x;) = \lim_{x\to-\infty} A(x;)$. Hence $\sigma_{\mathrm{ess}}(\mathcal{L})$ in L^2 space consists of $\lambda \in \mathbb{C}$ such that $\mathcal{A}(\lambda)$ is not Fredholm, and it is characterized by λ for which the asymptotic matrix $A^\infty(\lambda,\varepsilon) := \lim_{x\to\pm\infty} A(x,\lambda,\varepsilon)$ has an eigenvalue μ with $\mathrm{Re}\,\mu = 0$. In L^2_η space, $\sigma_{\mathrm{ess}}(\mathcal{L})$ is characterized by λ for which the matrix $A^\infty(\lambda,\varepsilon) + \eta I$ has an eigenvalue μ' with $\mathrm{Re}\,\mu' = 0$ since studying the spectrum of the operator ∂_x in L^2_η space is equivalent to studying the spectrum of the operator $\partial_x - \eta$ in L^2 space.

4.4 The Evans Function

Before we define the Evans function for the Euler-Poisson system, we briefly summurize the definition of the Evans function and its properties following [26]. In order to locate λ for which the operator $\mathcal{A}(\lambda)$ has a non-trivial kernel, we consider the first-order linear ODE system

$$\frac{d\mathbf{y}}{dx} = A(x, \lambda, \varepsilon)\mathbf{y} \tag{4.32}$$

with the matrix $A(x, \lambda, \varepsilon)$ defined in (4.30). We observe that $A(x, \lambda, \varepsilon)$ converges to the same asymptotic matrix $A^\infty(\lambda, \varepsilon)$ as $x \to \pm\infty$. To define the Evans function for (4.32) on a simply connected domain $\Omega^\varepsilon \subset \mathbb{C}$, we need to verify that for fixed parameter ε,

H1 $A(x, \lambda, \varepsilon)$ is continuous in $(x, \lambda) \in \mathbb{R} \times \Omega^\varepsilon$ and is analytic in λ for fixed x.

H2 $A(x, \lambda, \varepsilon)$ converges to $A^\infty(\lambda, \varepsilon)$ as $|x| \to \infty$, uniformly for λ on any compact subset of Ω^ε.

H3 The integral $\displaystyle\int_{-\infty}^{\infty} |A(x, \lambda, \varepsilon) - A^\infty(\lambda, \varepsilon)|\, dx$ converges for all λ, uniformly on compact subsets of Ω^ε.

H4 For every $\lambda \in \Omega^\varepsilon$, the matrix eigenvalues $\mu_j = \mu_j(\lambda, \varepsilon)$ of $A^\infty = A^\infty(\lambda, \varepsilon)$ can be labelled so that

$$\operatorname{Re}\mu_1 < \mu_* := \min\{\operatorname{Re}\mu_j : j = 2, 3, 4\}.$$

Under the assumptions **H1**–**H4**, (4.32) has a unique solution $\mathbf{y}^+ = \mathbf{y}^+(x, \lambda, \varepsilon)$ satisfying

$$\lim_{x\to+\infty} e^{-\mu_1 x}\mathbf{y}^+ = \mathbf{v}_1, \tag{4.33}$$

where $\mathbf{v}_1 = \mathbf{v}_1(\lambda, \varepsilon)$ is a right eigenvector of A^∞ associated with μ_1. The transposed ODE system

$$\frac{d\mathbf{z}}{dx} = -\mathbf{z}A(x, \lambda, \varepsilon), \tag{4.34}$$

where $\mathbf{z}$ is considered as a row vector, has a unique solution $\mathbf{z}^- = \mathbf{z}^-(x, \lambda, \varepsilon)$ satisfying

$$\lim_{x\to-\infty} e^{\mu_1 x}\mathbf{z}^- = \mathbf{w}_1, \tag{4.35}$$

where $\mathbf{w}_1 = \mathbf{w}_1(\lambda, \varepsilon)$ is the left eigenvector of $-A^\infty$ associated with $-\mu_1$ such that $\mathbf{w}_1\mathbf{v}_1 = 1$. Here the solutions $\mathbf{y}^+$ and $\mathbf{z}^-$ can be constructed so that they are analytic in $\lambda \in \Omega^\varepsilon$ for fixed $x \in \mathbb{R}$. The Evans function $D(\lambda, \varepsilon)$ for (4.32) is then defined by

$$D(\lambda, \varepsilon) := \mathbf{z}^-(x, \lambda, \varepsilon)\mathbf{y}^+(x, \lambda, \varepsilon),$$

which is analytic in λ and independent of x, and it is characterized by

$$\lim_{x\to-\infty} e^{-\mu_1 x}\mathbf{y}^+ = D(\lambda, \varepsilon)\mathbf{v}_1. \tag{4.36}$$

The following is a summary of Proposition 5.39.

Proposition 4.4. *Suppose **H1**–**H4** hold on Ω^ε for each $\varepsilon > 0$. Let $\lambda \in \Omega^\varepsilon$ and $\mathbf{y}^+(x, \lambda, \varepsilon)$ be the solution of* (4.32) *satisfying* (4.33)*. Then, the following statements hold true.*

1. *The following are equivalent:*

   (a) $D(\lambda, \varepsilon) = 0$;

   (b) $\mathbf{y}^+(x, \lambda, \varepsilon) = o(e^{\mu_1 x})$ *as* $x \to -\infty$;

   (c) $\mathbf{y}^+(x, \lambda, \varepsilon) = O(e^{(\mu_* - \theta)x})$ *as* $x \to -\infty$ *for any* $0 < \theta < \mu_* - \operatorname{Re} \mu_1$.

2. *For any solution* $\mathbf{y}(x, \lambda, \varepsilon)$ *of* (4.32)*, the following are equivalent:*

   (a) $\mathbf{y}(x, \lambda, \varepsilon) = O(e^{\mu_1 x})$ *as* $x \to +\infty$;

   (b) $\mathbf{y}(x, \lambda, \varepsilon) = \alpha \mathbf{y}^+(x, \lambda, \varepsilon)$ *for some constant* $\alpha \in \mathbb{C}$;

   (c) $\mathbf{y}(x, \lambda, \varepsilon) = o(e^{(\mu_* - \theta)x})$ *as* $x \to +\infty$ *for any* $0 < \theta < \mu_* - \operatorname{Re} \mu_1$.

We will see that $\operatorname{Re} \mu_1 < 0 < \mu_*$ holds on the natural domain $\operatorname{Re} \lambda > 0$. In this case, Proposition 4.4 implies that $\mathbf{y}^+(x;)$ is an L^2 solution of (4.32) if $D(\lambda, \varepsilon) = 0$. Conversely, if (4.32) has an L^2 solution for some λ, then $D(\lambda, \varepsilon) = 0$. In a similar fashion, we have that (4.32) has a solution in L^2_η if and only if $D(\lambda, \varepsilon) = 0$, provided that

$$\operatorname{Re} \mu_1 + \eta < 0 < \mu_* + \eta. \tag{4.37}$$

We will see that (4.37) holds on some right-half plane containing the imaginary axis.

Remark 4. If λ is an L^2 eigenvalue, then so are $\overline{\lambda}$ and $-\lambda$. Indeed, if $\mathbf{y}(x, \lambda)$ is a solution to (4.32), then we have

$$\frac{d}{dx}\overline{\mathbf{y}(x, \lambda)} = (A_1 + \overline{\lambda} A_2)\overline{\mathbf{y}(x, \lambda)}, \quad \frac{d}{dx}\widetilde{\mathbf{y}} = (A_1 - \lambda A_2)\widetilde{\mathbf{y}}, \tag{4.38}$$

where $\widetilde{\mathbf{y}} := (y_1(-x, \lambda), y_2(-x, \lambda), y_3(-x, \lambda), -y_4(-x, \lambda))^T$, using the symmetry $(n_c, u_c, \phi_c)(x) = (n_c, u_c, \phi_c)(-x)$. We remark that on the domain $\operatorname{Re} \lambda \leq 0$, where $\operatorname{Re} \mu_1 < \mu_* \leq 0$ holds, the zeros of the Evans function is not related to the L^2 eigenvalues in principle. For $\operatorname{Re} \lambda = 0$, for instance, we have $\operatorname{Re} \mu_1 < 0 = \mu_*$, and hence $\mathbf{y}^+$ (an analytic continuation of $\mathbf{y}^+$ defined on the natural domain $\operatorname{Re} \lambda > 0$) may oscillate without decaying as $x \to -\infty$. The eigenfunction $\widetilde{\mathbf{y}}$ corresponding to the eigenvalue $-\lambda$ is not an analytic continuation of $\mathbf{y}^+$. The zeros of the Evans function on $\operatorname{Re} \lambda \leq 0$ correspond to the so-called resonance poles ([26],[28],[35]), and their locations are related to the possible decay rate of (4.19). We show that on $\operatorname{Re} \lambda > -\kappa(\varepsilon)$ there is no resonance pole for $\mathcal{L}$ with respect to the L^2 norm.

4.4.1 The Evans Function for the Euler-Poisson System

We consider the ODE system (4.32) associated with the eigenvalue problem (4.15). By Theorem 4.1, the coefficient matrix $A(x, \lambda, \varepsilon)$ converges to the asymptotic matrix

$$A^\infty(\lambda, \varepsilon) := \begin{pmatrix} \frac{c\lambda}{c^2 - K} & \frac{\lambda}{c^2 - K} & 0 & \frac{1}{c^2 - K} \\ \frac{K\lambda}{c^2 - K} & \frac{c\lambda}{c^2 - K} & 0 & \frac{c}{c^2 - K} \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix} \tag{4.39}$$

as $|x| \to \infty$ exponentially fast. The matrix eigenvalues μ of A^∞ are the zeros of the characteristic polynomial

$$\begin{aligned} d(\mu) = d(\mu;\lambda,\varepsilon) &:= \det\left(\mu I - A^\infty(\lambda,\varepsilon)\right) \\ &= (c^2-K)^{-1}\left((\mu^2-1)\left[(\lambda-c\mu)^2 - K\mu^2\right] + \mu^2\right). \end{aligned} \tag{4.40}$$

For all $\varepsilon \geq 0$ and $\lambda \in \mathbb{C}$, $d(\mu)$ has four zeros μ_j counted with multiplicities.

The right and left eigenvectors (denoted by $\mathbf{v}_j$ and $\mathbf{w}_j$) of A^∞ corresponding to a non-zero simple eigenvalue μ_j satisfying the normalization $\mathbf{w}_j\mathbf{v}_j = 1$ are chosen as follows:

$$\begin{cases} \mathbf{v}_j := \left(1,\ \dfrac{c\mu_j - \lambda}{\mu_j},\ \dfrac{1}{1-\mu_j^2},\ \dfrac{\mu_j}{1-\mu_j^2}\right)^T, & \text{(4.41a)} \\ \mathbf{w}_j := \dfrac{\boldsymbol{\pi}_j}{\boldsymbol{\pi}_j\mathbf{v}_j}, & \text{(4.41b)} \end{cases}$$

where

$$\begin{cases} \boldsymbol{\pi}_j := \left([c\lambda - \mu_j(c^2-K)]\dfrac{1-\mu_j^2}{\mu_j},\ -\lambda\dfrac{1-\mu_j^2}{\mu_j},\ 1,\ \mu_j\right), & \text{(4.42a)} \\ \boldsymbol{\pi}_j\mathbf{v}_j = \dfrac{(1-\mu_j^2)(\lambda^2 - \mu_j^2(c^2-K))}{\mu_j^2} + \dfrac{1+\mu_j^2}{1-\mu_j^2} & \text{(4.42b)} \\ \qquad = \dfrac{\mu_j^2\left[(1-\mu_j^2)^2\frac{\lambda^2}{\mu_j^4} - \frac{2\varepsilon\sqrt{1+K}}{\mu_j^2} - \frac{\varepsilon^2}{\mu_j^2} + 1 + (2-\mu_j^2)(c^2-K)\right]}{1-\mu_j^2}. & \text{(4.42c)} \end{cases}$$

For (4.42c), we have used $c = \sqrt{1+K} + \varepsilon$. We note that $\boldsymbol{\pi}_j\mathbf{v}_j \neq 0$ when μ_j is semi-simple.

Zeros of Characteristic Polynomial $d(\mu)$ To define the Evans function for the Euler-Poisson system, we verify the central assumption **H4** by investigating some properties of the zeros of the characteristic polynomial (4.40).

Since $d(\pm 1) \neq 0$ and $c^2 - K \neq 0$ for all $\varepsilon \geq 0$ and $\lambda \in \mathbb{C}$, $d(\mu) = 0$ is equivalent to that μ satisfies one of the equations

$$\begin{cases} d_+(\mu) = d_+(\mu;\varepsilon) := \mu\left(c + \sqrt{\dfrac{1}{1-\mu^2} + K}\right) = \lambda, & \text{(4.43a)} \\ d_-(\mu) = d_-(\mu;\varepsilon) := \mu\left(c - \sqrt{\dfrac{1}{1-\mu^2} + K}\right) = \lambda. & \text{(4.43b)} \end{cases}$$

Plugging $\lambda = i\omega$ and $\mu = ik$ for $\omega, k \in \mathbb{R}$ into $\lambda = d_\pm(\mu)$, we obtain

$$\omega = \omega_\pm(k;\varepsilon) := ck \pm k\sqrt{\frac{1}{1+k^2} + K}.$$

We observe the behavior of $\omega_\pm$, which is important for the study on the solutions of (4.43). We have

$$\begin{cases} \dfrac{\mathrm{d}\omega_\pm}{\mathrm{d}k} = c \pm \dfrac{1 + K(1+k^2)^2}{\sqrt{\frac{1}{1+k^2} + K}\,(1+k^2)^2}, \\ \dfrac{\mathrm{d}^2\omega_-}{\mathrm{d}k^2} = -\dfrac{k(K(k^4 - 2k^2 - 3) - 3)}{(1+k^2)^3(K(1+k^2)+1)\sqrt{K + \frac{1}{1+k^2}}}. \end{cases}$$

$\frac{d\omega_\pm}{dk}$ are symmetric about $k = 0$. It is clear that $\frac{d\omega_+}{dk} \geq \sqrt{1+K}$ for all $k \in \mathbb{R}$ and $\varepsilon \geq 0$. On the other hand,

$$\frac{d\omega_-}{dk}(0) = \varepsilon, \quad \lim_{k\to\pm\infty} \frac{d\omega_-}{dk} = \sqrt{1+K} + \varepsilon - \sqrt{K}.$$

Solving the numerator of $\frac{d^2\omega_-}{dk^2}$, one can check that $\frac{d\omega_-}{dk}$ increases on $(0, \overline{k}_+)$ and decreases on $(\overline{k}_+, \infty)$ where[15]

$$\overline{k}_+ := \sqrt{\frac{K + \sqrt{4K^2 + 3K}}{K}}.$$

Proposition 4.5. *The zeros μ_j of $d(\mu)$ can be labelled so that the following splitting properties hold.*

1. *For $\varepsilon > 0$,*

$$\operatorname{Re}\mu_1 < 0 = \operatorname{Re}\mu_2 = \operatorname{Re}\mu_3 < \operatorname{Re}\mu_4, \qquad \textit{when } \operatorname{Re}\lambda = 0, \tag{4.45a}$$

$$\operatorname{Re}\mu_1 < 0 < \operatorname{Re}\mu_j, \quad (j = 2,3,4), \qquad \textit{when } \operatorname{Re}\lambda > 0. \tag{4.45b}$$

2. *For $\varepsilon = 0$,*

$$\operatorname{Re}\mu_1 < 0 = \operatorname{Re}\mu_2 = \operatorname{Re}\mu_3 < \operatorname{Re}\mu_4, \qquad \textit{when } \operatorname{Re}\lambda = 0 \textit{ and } \lambda \neq 0, \tag{4.46a}$$

$$\operatorname{Re}\mu_1 < 0 < \operatorname{Re}\mu_j, \quad (j = 2,3,4), \qquad \textit{when } \operatorname{Re}\lambda > 0. \tag{4.46b}$$

Remark 5. When $\operatorname{Re}\lambda < 0$, one may check that $\operatorname{Re}\mu_j < 0 < \operatorname{Re}\mu_4$, $j = 1,2,3$, for all $\varepsilon \geq 0$. For $\varepsilon = 0$ and $\lambda = 0$, we have $\mu_j = 0$ for all $j = 1,2,3,4$.

Proof. We first consider the case $\varepsilon > 0$. We note that $d_-(\mu_j) = 0$ for

$$\mu_1 = -\sqrt{1 - \frac{1}{c^2 - K}}, \quad \mu_4 = \sqrt{1 - \frac{1}{c^2 - K}}.$$

We recall that the functions $k \in \mathbb{R} \mapsto -id_\pm(ik) = \omega_\pm(k) \in \mathbb{R}$ are one-to-one and onto. Moreover, $d\omega_\pm/dk$ are strictly positive for $\varepsilon > 0$. Hence, for $\operatorname{Re}\lambda = 0$, there are exactly two solutions μ_2 and μ_3 with $\operatorname{Re}\mu_2 = \operatorname{Re}\mu_3 = 0$ satisfying $d_+(\mu_2) = \lambda$ and $d_-(\mu_3) = \lambda$. This proves (4.45a). Since $d_\pm(ik) \in i\mathbb{R}$ for all $k \in \mathbb{R}$, any solutions μ of $d_\pm(\mu;\varepsilon) = \lambda$ with $\operatorname{Re}\lambda \neq 0$ cannot lie in the imaginary axis . This implies that as long as $\operatorname{Re}\lambda > 0$, the number of solutions μ of (4.43) lying on the left half-plane or the right half-plane does not change. Expanding $d_\pm$ in μ around the origin of $\mathbb{C}$, we have

$$d_\pm(\mu;\varepsilon) = \partial_\mu d_\pm(0;\varepsilon)\mu \pm \frac{\mu^3}{2\sqrt{1+K}} + O(\mu^5), \tag{4.47}$$

[15]The behavior of ω is more complicated than the case $K = 0$. See Figure 6. When $K = 0$, it seems that the convergence of the Evans function can be shown in larger domain as the Boussinesq system following [28]. For this case, [17] only concerns near $\lambda = 0$.

where $\partial_\mu d_\pm(0;\varepsilon) = c \pm \sqrt{1+K} > 0$ for $\varepsilon > 0$. Here, the higher order terms does not involve ε. Thus, if we slightly move λ to the left from the origin along the real axis, then the real parts of the solutions μ_2 and μ_3 become positive. This implies that (4.45b) holds.

Next we consider the case $\varepsilon = 0$. We note that $\partial_\mu d_-(0;0) = 0$. Hence the solutions of $\lambda = d_-(\mu)$ are approximated by three solutions of $\lambda = -\dfrac{\mu^3}{2\sqrt{1+K}}$ as long as $|\lambda|$ is small. We have that for $n = 1, 2, 3$,

$$\mu = (2\sqrt{1+K}|\lambda|)^{1/3} e^{i(\pm\frac{\pi}{2}+\frac{2\pi n}{3})} \qquad \text{when } \lambda = e^{\pm i\pi/2}|\lambda|, \tag{4.48a}$$

$$\mu = (2\sqrt{1+K}|\lambda|)^{1/3} e^{i(\frac{\pi}{3}+\frac{2\pi n}{3})} \qquad \text{when } \lambda = |\lambda|, \tag{4.48b}$$

$$\mu = (2\sqrt{1+K}|\lambda|)^{1/3} e^{i\frac{2\pi n}{3}} \qquad \text{when } \lambda = -|\lambda|. \tag{4.48c}$$

From (4.48) and the previous argument, we easily deduce that (4.46) holds. □

Now we find the domain where the relation (4.37) holds. By setting $\mu = ik - \eta$, we have

$$d_\pm(\mu) = (ik-\eta)\left(c \pm \sqrt{\frac{1}{1-(ik-\eta)^2} + K}\right), \quad k \in \mathbb{R}. \tag{4.49}$$

Since $c = \sqrt{1+K} + \varepsilon$, we see that at $k = 0$,

$$\begin{aligned} d_-(-\eta) &= -\eta\left(1 + 2\sqrt{1+K}\varepsilon + \varepsilon^2 - \frac{1}{1-\eta^2}\right)\left(c + \sqrt{\frac{1}{1-\eta^2}+K}\right)^{-1} \\ &= -\frac{\eta\left(-\eta^2 + \varepsilon(2\sqrt{1+K}+\varepsilon)(1-\eta^2)\right)}{1-\eta^2}\left(c + \sqrt{\frac{1}{1-\eta^2}+K}\right)^{-1}. \end{aligned} \tag{4.50}$$

For $0 < c_0 < \sqrt{\frac{2\mathrm{V}}{3}}$, let $\eta = c_0\varepsilon^{1/2}$. Then, we have $d_-(-\eta) < 0$ for all sufficiently small $\varepsilon > 0$,[16] and hence the domain

$$\Omega^\varepsilon := \{\lambda : \operatorname{Re}\lambda > d_-(-\eta)\}$$

contains the closed right-half plane $\{\lambda : \operatorname{Re}\lambda \geq 0\}$.

Proposition 4.6. *For $0 < c_0 < \sqrt{\frac{2\mathrm{V}}{3}}$ and $\varepsilon > 0$, let $\eta = c_0\varepsilon^{1/2}$. Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the following hold.*

1. *The curves $\{d_\pm(\mu) : \mu = ik - \eta,\ k \in \mathbb{R}\}$ lie on $\mathbb{C} \setminus \Omega^\varepsilon = \{\lambda : \operatorname{Re}\lambda \leq d_-(-\eta) < 0\}$.*

2. *For $\lambda \in \Omega^\varepsilon$, the zeros of $d(\mu)$ can be labeled so that they satisfy*

$$\operatorname{Re}\mu_1 < -\eta < \operatorname{Re}\mu_j, \quad (j = 2, 3, 4). \tag{4.51}$$

Proof. It is obvious that $d_+(-\eta) < d_-(-\eta)$. On the other hand, for $\mu = ik - \eta$, by the Cauchy-Riemann equation,

$$\left.\frac{\partial(\operatorname{Re} d_\pm)}{\partial\eta}\right|_{\eta=0} = \left.\frac{\partial(\operatorname{Re} d_\pm)}{\partial(-\operatorname{Re}\mu)}\right|_{\operatorname{Re}\mu=0} = -\left.\frac{\partial(\operatorname{Im} d_\pm)}{\partial(\operatorname{Im}\mu)}\right|_{\operatorname{Re}\mu=0} = -\frac{\mathrm{d}\omega_\pm}{\mathrm{d}k}.$$

[16] The exponent 1/2 of ε is sharp. Consider $\eta = c_0\varepsilon^{1/4}$ for instance.

Now the first assertion easily follows from the previous discussion on the behavior of $\omega_\pm$. To prove (4.51), we let $\mu = \mu' - \eta$. Then, the zeros μ_j' of $d(\mu' - \eta)$ with $\lambda = 0$ are

$$\mu_1' = \eta - \sqrt{\frac{c^2 - 1 - K}{c^2 - K}}, \quad \mu_2' = \eta + \sqrt{\frac{c^2 - 1 - K}{c^2 - K}}, \quad \mu_3' = \mu_4' = \eta.$$

It is easy to see that $\mu_1' < 0 < \mu_j'$ $(j = 2, 3, 4)$ for all sufficiently small $\varepsilon > 0$. By the first assertion, the same argument as the proof of Proposition 4.5 yields that

$$\operatorname{Re} \mu_1' < 0 < \operatorname{Re} \mu_j' \quad (j = 2, 3, 4)$$

as long as $\lambda \in \Omega^\varepsilon$. Now the proof is finished by adding $-\eta$. □

From Proposition 4.5–4.6, we see that for each $\varepsilon > 0$, **H4** holds for $\lambda \in \Omega^\varepsilon$. When $\varepsilon = 0$, since $\mu(\lambda)$ is continuous in λ, it is clear that there is an open set including $\Omega_0 := \{\lambda : \operatorname{Re} \lambda \geq 0, \lambda \neq 0\}$ such that **H4** holds on the open set. **H1**–**H3** are clear from (4.8), (4.31) and (4.39). Hence, we have the following proposition.

Proposition 4.7. *There exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in [0, \varepsilon_0)$, the Evans function $D(\lambda, \varepsilon)$ for the system* (4.32) *associated with the eigenvalue problem* (4.15) *is defined on the domain Ω^ε. $D(\lambda, \varepsilon)$ is analytic in $\lambda \in \Omega^\varepsilon$ for each fixed $\varepsilon \in [0, \varepsilon_0)$.*

By Proposition 4.3 and the remark below it, Proposition 4.5–4.6 also lead the following.

Proposition 4.8. *1. For the operator $\mathcal{L} : (L^2)^2 \to (L^2)^2$, $\sigma_{\text{ess}}(\mathcal{L}) = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda = 0\}$.*

2. For $0 < c_0 < \sqrt{\frac{2\mathsf{V}}{3}}$ and $\varepsilon > 0$, let $\eta = c_0 \varepsilon^{1/2}$. For the operator $\mathcal{L} : (L^2_\eta)^2 \to (L^2_\eta)^2$,

$$\sigma_{\text{ess}}(\mathcal{L}) = \{d_\pm(\mu) : \mu = ik - \eta, \, k \in \mathbb{R}\} \subseteq \{\lambda : \operatorname{Re} \lambda \leq d_-(-\eta) < 0\}$$

for all sufficiently small $\varepsilon > 0$.

Zeros of the Evans function $D(\lambda, \varepsilon)$ and non-trivial solutions in unweighted and weighted L^2 spaces

Proposition 4.9. *For $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > 0$, the system* (4.32) *associated with the eigenvalue problem* (4.15) *has a nontrivial solution in L^2 if and only if $D(\lambda, \varepsilon) = 0$.*[17]

Proof. If $D(\lambda, \varepsilon) = 0$, then $\mathbf{y}^+$ satisfying (4.33) is an L^2 solution to (4.32) by (4.45b) and Proposition 4.4. If $\mathbf{y}$ is a non-trivial solution of (4.32) in L^2, then $\mathbf{y}$ is bounded in x since $\mathbf{y} \in H^1$. Again, from (4.45b) and Proposition 4.4, we see that $\mathbf{y} = \alpha \mathbf{y}^+$ for some constant $\alpha \in \mathbb{C}$. Since $\mathbf{y}^+$ is bounded, we see that $\mathbf{y}^+ = o(e^{\mu_1 x})$ as $x \to -\infty$, equivalently, $D(\lambda, \varepsilon) = 0$. □

[17] In principle, this statement is not true when $\operatorname{Re} \lambda = 0$ because of the matrix eigenvalues with $\operatorname{Re} \mu_i = 0$. For gKdV, gBBM, and some Boussinesq equation, this proposition can be extended to $\operatorname{Re} \lambda \geq 0$ using the symmetry that if $y(x)$ is a solution, then $\overline{y(-x)}$ is also a solution, which is valid for $\operatorname{Re} \lambda = 0$. (See [26].) EP also has this property.

Proposition 4.10. *For* $0 < c_0 < \sqrt{\dfrac{2\mathsf{V}}{3}}$ *and* $\varepsilon > 0$*, let* $\eta = c_0\varepsilon^{1/2}$*. Then there exists* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$*, the following holds: for* $\lambda \in \Omega^\varepsilon$*, the system* (4.32) *associated with the eigenvalue problem* (4.15) *has a nontrivial solution in* L^2_η *if and only if* $D(\lambda, \varepsilon) = 0$*.*

Proof. Since $e^{\eta x}\mathbf{y}^+ = O(e^{(\eta+\mu_1)x})$ as $x \to +\infty$, $e^{\eta x}\mathbf{y}^+$ exponentially decays to zero as $x \to +\infty$ by (4.51). Suppose that $D(\lambda, \varepsilon) = 0$. By Proposition 4.4, $e^{\eta x}\mathbf{y}^+ = O(e^{\eta x}e^{(\mu_*-\theta)x})$ as $x \to -\infty$. By (4.51), we see that $e^{\eta x}\mathbf{y}^+$ exponentially decays to zero as $x \to -\infty$. Hence, $\mathbf{y}^+$ is a solution to (4.32) in L^2_η.

Suppose that $\mathbf{y}$ is a nontrivial solution to (4.32) such that $e^{\eta x}\mathbf{y} \in L^2$. Since $A(x, \lambda, \varepsilon)$ is uniformly bounded in x, $e^{\eta x}\partial_x\mathbf{y}$ is also a L^2 function. Hence, $e^{\eta x}\mathbf{y}$ is uniformly bounded in x since $e^{\eta x}\mathbf{y} \in H^1$, and there holds that

$$\mathbf{y} = O(e^{-\eta x}) \quad \text{as} \quad |x| \to \infty. \tag{4.52}$$

Multiplying (4.52) by $e^{(-\mu_* x+\theta|x|)}$, we have

$$e^{(-\mu_* x+\theta|x|)}\mathbf{y} = O(e^{(-\eta x-\mu_* x+\theta|x|)}) \quad \text{as} \quad x \to +\infty.$$

By (4.51), $-\eta - \mu_* + \theta < 0$ for sufficiently small $\theta > 0$, hence we have $\mathbf{y} = o(e^{\mu_* x-\theta|x|})$ as $x \to +\infty$. By Proposition 4.4, this implies that $\mathbf{y}$ is a constant multiple of $\mathbf{y}^+$, and thus from (4.52) we have

$$\mathbf{y}^+ = O(e^{-\eta x}) \quad \text{as} \quad x \to -\infty.$$

Together with (4.51), this yields that $\mathbf{y}^+ = o(e^{\mu_1 x})$ as $x \to -\infty$, which is equivalent to $D(\lambda, \varepsilon) = 0$ by Proposition 4.4 (or (4.36)). □

4.4.2 The Evans Function for the KdV Equation

In the KdV scaing, we formally obtained (by letting $\varepsilon \to 0$ in (4.25) and $n = p_2$)

$$\Lambda p_2 - \partial_\xi p_2 + \mathsf{V}\partial_\xi(p_2\Psi_K) + (2\mathsf{V})^{-1}\partial_\xi^3 p_2 = 0. \tag{4.53}$$

By introducing the change of variables

$$\Lambda = (2\mathsf{V})^{1/2}\tilde{\Lambda}, \quad \tilde{\xi} = (2\mathsf{V})^{1/2}\xi, \quad \tilde{p}(\tilde{\xi}; \tilde{\Lambda}) = p_2(\xi; \Lambda), \tag{4.54}$$

and using (4.10), (4.53) becomes

$$\tilde{\Lambda}\tilde{p} - \partial_{\tilde{\xi}}\tilde{p} + \partial_{\tilde{\xi}}(\tilde{\Psi}_K\tilde{p}) + \partial_{\tilde{\xi}}^3\tilde{p} = 0, \quad \text{where } \tilde{\Psi}_K(\tilde{\xi}) = 3\,\mathrm{sech}^2(\frac{1}{2}\tilde{\xi}). \tag{4.55}$$

The eigenvalue problem (4.55) is studied in [27]. We briefly summarize some results of [27], and then apply those directly to the eigenvalue problem (4.53). The characteristic polynomial associated with (4.55) is

$$\tilde{d}_{KdV}(\tilde{\kappa}) = \tilde{d}_{KdV}(\tilde{\kappa}; \tilde{\Lambda}) := \tilde{\Lambda} - \tilde{\kappa} + \tilde{\kappa}^3. \tag{4.56}$$

It is shown that for $\tilde{\Lambda} \in \tilde{\Omega}_{KdV} := \mathbb{C} \setminus (-\infty, -\frac{2}{3\sqrt{3}}]$, the zeros $\tilde{\kappa}_j$ of $\tilde{d}_{KdV}(\tilde{\kappa})$ can be labelled so that

$$\operatorname{Re}\tilde{\kappa}_1 < \operatorname{Re}\tilde{\kappa}_j \quad (j = 2, 3), \tag{4.57}$$

and in particular, $\operatorname{Re}\tilde{\kappa}_1 < 0$. It turns out that the Evans function $\tilde{D}_{KdV}(\tilde{\Lambda})$ for the KdV equation (4.55) is defined on the domain $\tilde{\Omega}_{KdV}$, and $\tilde{D}_{KdV}(\tilde{\Lambda})$ is characterized with the property that

$$\tilde{p}^+(\tilde{\xi}; \tilde{\Lambda}) \sim \tilde{D}_{KdV}(\tilde{\Lambda}) e^{\tilde{\kappa}_1 \tilde{\xi}} \quad \text{as} \quad \tilde{\xi} \to -\infty, \tag{4.58}$$

where $\tilde{p}^+$ is a unique solution to (4.55) satisfying

$$\tilde{p}^+(\tilde{\xi}; \tilde{\Lambda}) \sim e^{\tilde{\kappa}_1 \tilde{\xi}} \quad \text{as} \quad \tilde{\xi} \to +\infty. \tag{4.59}$$

$\tilde{D}_{KdV}(\tilde{\Lambda})$ can be constructed so that $\tilde{D}_{KdV}(\tilde{\Lambda}) \to 1$ as $|\tilde{\Lambda}| \to \infty$. In [27], the Evans function for (4.55) is explicitly given by

$$\tilde{D}_{KdV}(\tilde{\Lambda}) = \left(\frac{\tilde{\kappa}_1 + 1}{\tilde{\kappa}_1 - 1} \right)^2. \tag{4.60}$$

From (4.56), one can check that $\tilde{D}_{KdV}(\tilde{\Lambda})$ vanishes only at $\tilde{\Lambda} = 0$ and that the multiplicity of $\tilde{\Lambda} = 0$ is two as a zero of $\tilde{D}_{KdV}(\tilde{\Lambda})$ by taking derivatives in $\tilde{\Lambda}$.

Now we apply the above results to construct the Evans function for the KdV equation (4.53). Since Ψ_K and $\partial_\xi \Psi_K$ decay to zero exponentially fast, the characteristic polynomial associated with (4.53) is

$$d_{KdV}(\kappa) = d_{KdV}(\kappa; \Lambda) := 2\mathsf{V}(\Lambda - \kappa + (2\mathsf{V})^{-1}\kappa^3). \tag{4.61}$$

We note that the zeros κ_j of $d_{KdV}(\kappa)$ are related to $\tilde{\kappa}_j$, and we have

$$\kappa_j = (2\mathsf{V})^{1/2} \tilde{\kappa}_j. \tag{4.62}$$

Hence, from the relation (4.54) and (4.62), it follows that for $\Lambda \in \Omega_{KdV} := \mathbb{C} \setminus (-\infty, -\frac{2\sqrt{2\mathsf{V}}}{3\sqrt{3}}]$, we can label κ_j so that

$$\operatorname{Re}\kappa_1 < \operatorname{Re}\kappa_j \quad (j = 2, 3), \quad \operatorname{Re}\kappa_1 < 0. \tag{4.63}$$

The Evans function for the equation (4.53) is defined on Ω_{KdV}, the following is a simple corollary of the result in [27]. (See also [28].)

Corollary 4.11. *1. The Evans function $D_{KdV}(\Lambda)$ for the KdV equation* (4.53) *is defined on the domain $\Omega_{KdV} := \mathbb{C} \setminus (-\infty, -\frac{2\sqrt{2\mathsf{V}}}{3\sqrt{3}}]$.*

2. $D_{KdV}(\Lambda)$ satisfies that

$$p_2^+(\xi; \Lambda) \sim D_{KdV}(\Lambda) e^{\kappa_1 \xi} \quad \text{as} \quad \xi \to -\infty, \tag{4.64}$$

where κ_1 is a unique zero of $d_{KdV}(\kappa)$ satisfying (4.63)*, and p_2 is a unique solution to* (4.53) *satisfying*

$$p_2^+(\xi; \Lambda) \sim e^{\kappa_1 \xi} \quad \text{as} \quad \xi \to +\infty. \tag{4.65}$$

3. $D_{KdV}(\Lambda) = \left(\dfrac{\kappa_1 + \sqrt{2\mathsf{V}}}{\kappa_1 - \sqrt{2\mathsf{V}}}\right)^2$, *and* $\Lambda = 0$ *is the only zero of* $D_{KdV}(\Lambda)$ *with multiplicity two.*

4. $D_{KdV}(\Lambda) \to 1$ *as* $|\Lambda| \to \infty$ *with* $\Lambda \in \Omega_{KdV}$.

Proof. The first two statements can be checked following [26]. To prove the last two assertions, it is enough to check that $D_{KdV}(\Lambda) = \tilde{D}_{KdV}(\tilde{\Lambda})$. From the relation (4.54) and (4.62), we have

$$\tilde{\kappa}_1 \tilde{\xi} = \kappa_1 \xi. \tag{4.66}$$

From the change of variable (4.54), $\tilde{p}^+((2\mathsf{V})^{1/2}\xi; (2\mathsf{V})^{-1/2}\Lambda)$ is a solution of (4.53) since $\tilde{p}^+(\tilde{\xi}; \tilde{\Lambda})$ is a solution of (4.55). From (4.59), (4.65) and (4.66), we have

$$\tilde{p}^+((2\mathsf{V})^{1/2}\xi; (2\mathsf{V})^{-1/2}\Lambda) \sim e^{\kappa_1 \xi} \quad \text{as} \quad \xi \to +\infty.$$

Hence, we must have $\tilde{p}^+(\tilde{\xi}; \tilde{\Lambda}) = p_2^+(\xi; \Lambda)$, and we conclude that $D_{KdV}(\Lambda) = \tilde{D}_{KdV}(\tilde{\Lambda})$ from (4.58), (4.64) and (4.66). ☐

4.4.3 The Evans Function for the Euler-Poisson System in the KdV Scaling

Motivated by the formal derivation of the linearized KdV equation, we take the transformation

$$\xi = \varepsilon^{1/2} x, \quad \lambda = \varepsilon^{3/2}\Lambda, \tag{4.67a}$$

$$\begin{cases} n(x) = p_2(\xi), \quad u(x) = \varepsilon p_1(\xi) + \mathsf{V} p_2(\xi), \\ \phi(x) = p_2(\xi) + \varepsilon p_4(\xi), \quad \psi(x) = \varepsilon^{1/2} p_3(\xi). \end{cases} \tag{4.67b}$$

Let $\mathbf{p} := (p_1, p_2, p_3, p_4)^T$. Then, we have $S\mathbf{p} = \mathbf{y}$, where S given by

$$S := \begin{pmatrix} 0 & 1 & 0 & 0 \\ \varepsilon & \mathsf{V} & 0 & 0 \\ 0 & 1 & 0 & \varepsilon \\ 0 & 0 & \varepsilon^{1/2} & 0 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} -\mathsf{V}\varepsilon^{-1} & \varepsilon^{-1} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon^{-1/2} \\ -\varepsilon^{-1} & 0 & \varepsilon^{-1} & 0 \end{pmatrix}, \tag{4.68}$$

is a matrix for the transformation (4.67b), and (4.32) becomes

$$\frac{d\mathbf{p}}{d\xi} = A_*(\xi, \Lambda, \varepsilon)\mathbf{p}, \tag{4.69}$$

where

$$A_*(\xi, \Lambda, \varepsilon) := \frac{1}{\sqrt{\varepsilon}} S^{-1} A\left(\frac{\xi}{\sqrt{\varepsilon}}, \varepsilon^{3/2}\Lambda, \varepsilon\right) S. \tag{4.70}$$

(See (4.68) and (4.135) for the more specific form of $A_*(\xi, \Lambda, \varepsilon)$.) Expanding $A_*(\xi, \Lambda, \varepsilon)$, we obtain

$$A_*(\xi, \Lambda, \varepsilon) = A_*(\xi, \Lambda, 0) + \widetilde{A}_*(\xi, \Lambda, \varepsilon), \tag{4.71}$$

where

$$A_*(\xi,\Lambda,0) := \begin{pmatrix} 0 & -2\mathsf{V}\partial_\xi\Psi_K - \Lambda & 1-2\mathsf{V}\Psi_K & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \Psi_K & 0 & 1 \\ 0 & -(2\mathsf{V}^2+1)\partial_\xi\Psi_K - 2\mathsf{V}\Lambda & 2\mathsf{V}-(2\mathsf{V}^2+1)\Psi_K & 0 \end{pmatrix}. \tag{4.72}$$

Here, there exists $\varepsilon_0 > 0$ and a positive function $C(\Lambda)$ of Λ (independent of ε and bounded on any compact set of $\mathbb{C}$), such that $|\widetilde{A}_*(\xi,\Lambda,\varepsilon)| \le \varepsilon C(\Lambda)e^{-\widetilde{C}|\xi|}$ for all $\varepsilon \in [0,\varepsilon_0]$, $\xi \in \mathbb{R}$ and $\Lambda \in \mathbb{C}$. When $\varepsilon = 0$, we see that the last three equation of (4.69) implies the KdV equation (4.53) (recall that $n = p_2$).

For all $\varepsilon \ge 0$, the matrix eigenvalues ν of the asymptotic matrix $A_*^\infty(\Lambda,\varepsilon)$ of $A_*(\xi,\Lambda,\varepsilon)$ in ξ is the zeros of the characteristic polynomial

$$\begin{aligned} d_*(\nu) = d_*(\nu;\Lambda,\varepsilon) &:= \det\left(A_*^\infty(\Lambda,\varepsilon) - \nu I\right) = \frac{1}{\varepsilon^2}\det\left(A^\infty(\varepsilon^{3/2}\Lambda,\varepsilon) - \sqrt{\varepsilon}\,\nu I\right) \\ &= \frac{d(\sqrt{\varepsilon}\,\nu;\varepsilon^{3/2}\Lambda,\varepsilon)}{\varepsilon^2} \\ &= \frac{\nu\, d_{\mathrm{KdV}}(\nu) - \varepsilon(\Lambda-\nu)^2 + \varepsilon\nu^2\left[\varepsilon\Lambda^2 - 2c\Lambda\nu + (2\mathsf{V}+\varepsilon)\nu^2\right]}{c^2-K}, \end{aligned} \tag{4.73}$$

where $d_{\mathrm{KdV}}(\nu)$ is defined in (4.61).

Zeros of the characteristic polynomial $d_*(\nu)$ To define the Evans function for the Euler-Poisson system in the KdV scaling, we first verify **H4**. From the scaling (4.67a) and the second line of (4.73), we see that for $\varepsilon > 0$, the zeros ν_j of $d_*(\nu)$ are related to the zeros μ_j of $d(\mu)$ by

$$\nu_j = \varepsilon^{-1/2}\mu_j. \tag{4.74}$$

When $\varepsilon = 0$, the zeros of $d_*(\nu)$ are comprised of 0 and the three zeros κ_j of $d_{KdV}(\kappa)$. Together with (4.51) and (4.63), these observations imply the following proposition.

Proposition 4.12. *The zeros ν_j of $d_*(\nu)$ can be labelled so that they satisfy*

$$\operatorname{Re}\nu_1 < \operatorname{Re}\nu_j \quad (j = 2,3,4) \tag{4.75}$$

for all Λ such that $\varepsilon^{3/2}\Lambda = \lambda \in \Omega^\varepsilon$ when $\varepsilon > 0$ and for all $\Lambda \in \Omega_{KdV}$ when $\varepsilon = 0$.

For fixed $0 < c_0 < \sqrt{\frac{2\mathsf{V}}{3}}$, we define

$$\Omega_*^\varepsilon := \{\Lambda : \operatorname{Re}\Lambda > \varepsilon^{-3/2}d_-(c_0\varepsilon^{1/2})\} = \varepsilon^{-3/2}\Omega^\varepsilon \qquad \text{for } \varepsilon > 0, \tag{4.76a}$$

$$\Omega_*^0 := \{\Lambda : \operatorname{Re}\Lambda > -c_0\left(1 - \frac{c_0^2}{2\mathsf{V}}\right)\} \qquad \text{for } \varepsilon = 0. \tag{4.76b}$$

From (4.50), we see that $\lim_{\varepsilon\to 0}\varepsilon^{-3/2}d_-(c_0\varepsilon^{1/2}) = -c_0\left(1 - \frac{c_0^2}{2\mathsf{V}}\right)$. Hence, the domain Ω_*^ε approaches to Ω_*^0 as $\varepsilon \to 0$. On the other hand, for all $0 < c_0 < \sqrt{\frac{2\mathsf{V}}{3}}$

$$-\frac{2\sqrt{2\mathsf{V}}}{3\sqrt{3}} < -c_0\left(1 - \frac{c_0^2}{2\mathsf{V}}\right) < 0$$

holds, which implies that $\{\Lambda : \operatorname{Re}\Lambda \geq 0\} \subset \Omega_*^0 \subset \Omega_{KdV}$.

The right eigenvector $\mathbf{v}_j^*$ of the asymptotic matrix A_*^∞ corresponding to a non-zero ν_j is given by

$$\begin{aligned}\mathbf{v}_j^* := S^{-1}\mathbf{v}_j &= \left(-\frac{\mathsf{V}}{\varepsilon}+\frac{c}{\varepsilon}-\frac{\lambda}{\varepsilon\mu_j}, 1, \frac{1}{\sqrt{\varepsilon}}\frac{\mu_j}{1-\mu_j^2}, -\frac{1}{\varepsilon}+\frac{1}{\varepsilon(1-\mu_j^2)}\right)^T \\ &= \left(1-\frac{\Lambda}{\nu_j},\ 1,\ \frac{\nu_j}{1-\varepsilon\nu_j^2},\ \frac{\nu_j^2}{1-\varepsilon\nu_j^2}\right),\end{aligned} \tag{4.77}$$

where we have used (4.67a), (4.74), and the definition of c.

Proposition 4.13. *There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in [0, \varepsilon_0)$, the Evans function $D_*(\Lambda, \varepsilon)$ for (4.69)–(4.70) is defined on the domain Ω_*^ε so that*

$$\lim_{\xi \to -\infty} e^{-\nu_1 \xi}\mathbf{p}^+(\xi, \Lambda) = D_*(\Lambda, \varepsilon)\mathbf{v}_1^*, \tag{4.78}$$

where $\mathbf{p}^+(\xi, \Lambda)$ is a unique solution to (4.69) satisfying

$$\lim_{\xi \to +\infty} e^{-\nu_1 \xi}\mathbf{p}^+(\xi, \Lambda) = \mathbf{v}_1^*. \tag{4.79}$$

Moreover, $D_*(\Lambda, \varepsilon)$ is analytic in $\Lambda \in \Omega_*^\varepsilon$.

Proof. **H4** is verified from the above discussion and (4.75). **H1–H3** clear from Theorem 4.1 and (4.70). □

4.4.4 Relation Among $D(\lambda, \varepsilon)$, $D_*(\Lambda, \varepsilon)$, $D_{KdV}(\Lambda)$

Proposition 4.14. *There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon)$, $D_*(\Lambda, \varepsilon) = D(\lambda, \varepsilon)$ for $\Lambda = \varepsilon^{-3/2}\lambda \in \Omega_*^\varepsilon$. When $\varepsilon = 0$, we have $D_*(\Lambda, 0) = D_{KdV}(\Lambda)$ for $\Lambda \in \Omega_*^0$.*

Proof. Considering the transform (4.67) and the relation (4.74), we observe that

$$S\mathbf{p}^+(\sqrt{\varepsilon}x, \varepsilon^{-3/2}\lambda)$$

is a solution to (4.32), and moreover $S\mathbf{p}^+ = O(e^{\mu_1 x})$ as $x \to +\infty$ from (4.79). Hence, by the second assertion of Proposition 4.4, $S\mathbf{p}^+$ is a constant multiple of $\mathbf{y}^+$. In particular, we have $p_2^+ = Cy_1^+$ for some constant C. On the other hand, we note that the first component of $\mathbf{v}_1$ and the second component of $\mathbf{v}_1^*$ are 1. Thus, from (4.33) and (4.79), we conclude that $p_2^+(\xi, \Lambda) = y_1^+(x, \lambda)$. Then, it is clear from (4.36) and (4.78) that $D(\lambda, \varepsilon) = D_*(\Lambda, \varepsilon)$. Similarly, we obtain $D_*(\Lambda, 0) = D_{KdV}(\Lambda)$. □

Regarding continuity, the following proposition can be proved by the fixed point argument following [28], Section 8.

Proposition 4.15. *There exists $\varepsilon_0 > 0$ such that $D(\lambda, \varepsilon)$ and $D_*(\Lambda, \varepsilon)$ are jointly continuous on the sets $\{(\lambda, \varepsilon) : \lambda \in \Omega^\varepsilon,\ \varepsilon \in [0, \varepsilon_0]\}$ and $\{(\Lambda, \varepsilon) : \Lambda \in \Omega_*^\varepsilon\ \varepsilon \in [0, \varepsilon_0]\}$, respectively.*

4.4.5 The Order of $D(\lambda, \varepsilon)$

We show that $\lambda = 0$ is a zero of the Evans function $D(\lambda, \varepsilon)$ with the order at least two. Once we know that $D(\lambda, \varepsilon) = 0$, the formula for the derivatives of the Evans function (see Theorem 5.45) becomes much simpler. In this case, we have

$$\partial_\lambda D(\lambda, \varepsilon) = -\int_{-\infty}^{\infty} \mathbf{z}^-(x)\partial_\lambda A(x, \lambda, \varepsilon)\mathbf{y}^+(x)\, dx \tag{4.80}$$

in the sense of an improper integral, where $\mathbf{z}^-$ is the solution to the transposed ODE system of (4.29) satisfying (4.34) and (4.35).

We will use the following identities for solitary waves solutions:

$$(1 + n_c)(c - u_c) = c, \tag{4.81a}$$

$$(c - u_c)(n_c)_x = (u_c)_x(1 + n_c), \tag{4.81b}$$

$$\frac{c(u_c)_x - K(n_c)_x}{1 + n_c} = (\phi_c)_x, \tag{4.81c}$$

$$(\phi_c)_x = \frac{J}{1 + n_c}(n_c)_x, \tag{4.81d}$$

$$(n_c)_x J = c(u_c)_x - K(n_c)_x. \tag{4.81e}$$

The identity (4.81a) is obtained by integrating the first equation of (4.12) in x. Differentiating (4.81a) in x, we have (4.81b). (4.81c) follows from the second equation of (4.12) and (4.81a). (4.81d) follows from (4.81a)–(4.81c). (4.81e) follows from (4.81c)–(4.81d).

Lemma 4.16. *When $\lambda = 0$, for each $\varepsilon \in (0, \varepsilon_0]$ there exists a constant $\alpha_1 \neq 0$ such that $\mathbf{y}^+(x, 0, \varepsilon) = \alpha_1 \mathbf{y}_1(x)$, where*

$$\mathbf{y}_1(x) := (\partial_x n_c, \partial_x u_c, \partial_x \phi_c, \partial_x^2 \phi_c)^T.$$

Proof. By differentiating (4.12) in x, we see that $\mathbf{y}_1$ satisfies the ODE system (4.29) associated with the eigenvalue problem (4.15) with $\lambda = 0$. From Proposition 4.5, we have $\operatorname{Re} \mu_1 < 0 = \mu_*$ when $\lambda = 0$. Since $\mathbf{y}_1$ exponentially decays to zero as $|x| \to \infty$, we have that for sufficiently small $\theta > 0$,

$$\lim_{x \to +\infty} e^{\theta x}\mathbf{y}_1(x) = 0.$$

By Proposition 4.4, this implies that $\mathbf{y}^+ = \alpha_1 \mathbf{y}_1$ for some constant $\alpha_1 \neq 0$. □

The following lemma is obtained by using the solitary wave identities (4.81).

Lemma 4.17. *When $\lambda = 0$, for each $\varepsilon \in (0, \varepsilon_0]$ there exists a constant $\alpha_2 \neq 0$ such that $\mathbf{z}^-(x, 0, \varepsilon) = \alpha_2 \mathbf{z}_1(x)$, where*[18]

$$\mathbf{z}_1(x) := \left(-\frac{cu_c - Kn_c}{1 + n_c}, \frac{(1 + n_c)}{c}(cu_c - Kn_c - n_c J), e^{\phi_c} - 1, -\partial_x \phi_c\right).$$

[18] $n_c u_c = \frac{1 + n_c}{c}(cu_c - Kn_c - n_c J)$ using (4.81a).

We first show that the above two lemmas, together with the derivative formula (4.80), imply the following proposition.

Proposition 4.18. *There exists* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0]$,

$$D(0,\varepsilon) = \partial_\lambda D(0,\varepsilon) = 0.$$

Proof. We denote ∂_x by $'$ for simplicity. From Lemma 4.16, we deduce that $\mathbf{y}^+(x) \to 0$ as $x \to -\infty$. Since $\operatorname{Re}\mu_1 < 0$ when $\lambda = 0$, we have $D(0,\varepsilon) = 0$ by Proposition 4.4. From the formula (4.80), Lemma 4.16 and Lemma 4.17, we obtain

$$\partial_\lambda D(0,\varepsilon) = -\alpha_1\alpha_2 \int_{-\infty}^{\infty} \mathbf{z}_1 A_2 \mathbf{y}_1 \, dx$$

since $\partial_\lambda A(x,\lambda,\varepsilon) = A_2(x,\varepsilon)$. Using the solitary wave identities (4.81), we have

$$\begin{aligned}
\mathbf{z}_1 A_2 \mathbf{y}_1 &= \frac{1}{J}\left[z_1\left((c-u_c)n_c' + u_c'(1+n_c)\right) + z_2\left(\frac{Kn_c'}{1+n_c} + (c-u_c)u_c'\right)\right] \\
&= \frac{1}{J}\left[2z_1 u_c'(1+n_c) + z_2\left(\frac{Kn_c'}{1+n_c} + (c-u_c)u_c'\right)\right] \\
&= \frac{1}{J}\left[-2(cu_c - Kn_c)u_c' + \frac{1}{c}\left(cu_c - Kn_c - n_cJ\right)\left(Kn_c' + cu_c'\right)\right] \\
&= \frac{(cu_c - Kn_c)}{J}\left(-2u_c' + \frac{Kn_c' + cu_c'}{c}\right) - \frac{n_c}{c}(Kn_c' + cu_c') \\
&= \frac{(cu_c - Kn_c)}{cJ}\left(-cu_c' + Kn_c'\right) - \frac{n_c}{c}(Kn_c' + cu_c') \\
&= -\frac{(cu_c - Kn_c)}{c}n_c' - \frac{n_c}{c}(Kn_c' + cu_c') \\
&= -(n_c u_c)',
\end{aligned}$$

where we used (4.81b) in the second line, (4.81a) in the third line and (4.81e) in the sixth line. Since $n_c u_c$ tends to 0 as $|x| \to \infty$, we obtain $\partial_\lambda D(0,\varepsilon) = 0$.

□

Proof of Lemma 4.17. We denote ∂_x by $'$ for simplicity. Using the identities (4.81a), (4.81b) and (4.81e), $A(x,0,\varepsilon) = A_1(x,\varepsilon)$ is simplified as follows:

$$A_1(x,\varepsilon) = \begin{pmatrix} \dfrac{n_c'}{1+n_c} & \dfrac{2u_c'(1+n_c)}{J} & 0 & \dfrac{1+n_c}{J} \\ 0 & \dfrac{Kn_c' + cu_c'}{J(1+n_c)} & 0 & \dfrac{c-u_c}{J} \\ 0 & 0 & 0 & 1 \\ -1 & 0 & e^{\phi_c} & 0 \end{pmatrix}. \tag{4.82}$$

We show that $\mathbf{z}_1 := (z_1, z_2, z_3, z_4)$ satisfies $\mathbf{z}_1' = -\mathbf{z}_1 A_1(x,\varepsilon)$,

$$\begin{cases} z_1' = \dfrac{-n_c'}{1+n_c} z_1 + z_4, & (4.83a) \\ z_2' = -\dfrac{2u_c'(1+n_c)}{J} z_1 - \dfrac{Kn_c' + cu_c'}{J(1+n_c)} z_2, & (4.83b) \\ z_3' = -z_4 e^{\phi_c}, & (4.83c) \\ z_4' = -z_1 \left(\dfrac{1+n_c}{J}\right) - z_2 \left(\dfrac{c-u_c}{J}\right) - z_3. & (4.83d) \end{cases}$$

It is trivial that $(z_3, z_4) = (e^{\phi_c} - 1, -\phi_c')$ satisfies (4.83c). Using (4.81c), we see that (z_1, z_4) satisfies (4.83a) since

$$\begin{aligned} \frac{-n_c'}{1+n_c} z_1 + z_4 &= \frac{n_c'(cu_c - Kn_c)}{(1+n_c)^2} - \phi_c' \\ &= \frac{n_c'(cu_c - Kn_c)}{(1+n_c)^2} - \frac{(1+n_c)(cu_c' - Kn_c')}{(1+n_c)^2} \\ &= -\left(\frac{cu_c - Kn_c}{1+n_c}\right)' = z_1'. \end{aligned}$$

Using (4.81a) and the Poisson equation of (4.12), we obtain that

$$\begin{aligned} -z_1\left(\frac{1+n_c}{J}\right) - z_2\left(\frac{c-u_c}{J}\right) &= \frac{(cu_c - Kn_c)}{J} - \frac{(1+n_c)}{c}(cu_c - Kn_c - n_cJ)\left(\frac{c-u_c}{J}\right) \\ &= \frac{(cu_c - Kn_c)}{J} - \frac{cu_c - Kn_c - n_cJ}{J} \\ &= n_c \\ &= -\phi_c'' + e^{\phi_c} - 1 \\ &= z_4' + z_3. \end{aligned}$$

Thus $\mathbf{z}_1$ satisfies (4.83d). Lastly, we show that (z_1, z_2) satisfies (4.83b). We have

$$\begin{aligned} &- \frac{2u_c'(1+n_c)}{J} z_1 - \frac{(Kn_c + cu_c)'}{J(1+n_c)} z_2 \\ &= \frac{1}{cJ}\left[2cu_c'(cu_c - Kn_c) - (Kn_c + cu_c)'(cu_c - Kn_c - n_cJ)\right] \\ &= \frac{1}{cJ}\left[(cu_c - Kn_c)'(cu_c - Kn_c) + (Kn_c + cu_c)' n_c J\right]. \end{aligned}$$

On the other hand,

$$\begin{aligned} z_2' &= \frac{n_c'}{c}(cu_c - Kn_c - n_cJ) + \frac{(1+n_c)}{c}\left(cu_c' - Kn_c' - n_c'J - n_cJ'\right) \\ &= \frac{1}{cJ}\left[Jn_c'(cu_c - Kn_c - n_cJ) + (1+n_c)J\left(cu_c' - Kn_c' - n_c'J - n_cJ'\right)\right] \\ &= \frac{1}{cJ}\left[(cu_c - Kn_c)'(cu_c - Kn_c - n_cJ) + J(1+n_c)\left(-n_cJ'\right)\right] \\ &= \frac{1}{cJ}\left[(cu_c - Kn_c)'(cu_c - Kn_c - n_cJ) + J(1+n_c)2n_c(c-u_c)u_c'\right] \\ &= \frac{1}{cJ}\left[(cu_c - Kn_c)'(cu_c - Kn_c - n_cJ) + 2cu_c'n_cJ\right] \\ &= \frac{1}{cJ}\left[(cu_c - Kn_c)'(cu_c - Kn_c) + (Kn_c + cu_c)'n_cJ\right], \end{aligned}$$

where we used (4.81e) in the third line, (4.81a) in the fifth line.

Since $\mathbf{z}_1$ exponentially decays to zero as $x \to -\infty$ and $-\operatorname{Re}\mu_1 > 0 = -\mu_*$, we see that there is a constant $\alpha_2 \neq 0$ such that $\mathbf{z}^-(x,0,\varepsilon) = \alpha_2\mathbf{z}_1$. □

4.4.6 Absence of Nonzero Eigenvalues

For a fixed $a > 0$ sufficiently small so that $-c_0\left(1-\frac{c_0^2}{2\mathsf{V}}\right)+a<0$, we define the region

$$\mathfrak{D} := \{\Lambda : \operatorname{Re}\Lambda \geq -c_0\left(1-\frac{c_0^2}{2\mathsf{V}}\right)+a\}. \tag{4.84}$$

Since Ω_*^ε approaches to Ω_*^0 as $\varepsilon \to 0$, there exists $\varepsilon_0 > 0$ such that $\mathfrak{D} \subset \Omega_*^\varepsilon$ for all $\varepsilon \in [0,\varepsilon_0]$. (See (4.76) and the observation below it.)

Theorem 4.19. *On the region $\mathfrak{D}$ defined in* (4.84), *we have*

$$\sup_{\Lambda\in\mathfrak{D}} |D_*(\Lambda,\varepsilon) - D_{KdV}(\Lambda)| \to 0 \quad \text{as } \varepsilon \to 0.$$

Corollary 4.20. *There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0,\varepsilon_0]$, $\Lambda = 0$ is the only zero of $D_*(\Lambda,\varepsilon)$ on the region $\mathfrak{D}$. Moreover, the multiplicity of $\Lambda = 0$ is exactly two.*

Proof of Corollary 4.20. For $\delta > 0$, let Γ_δ be the boundary of the region $\mathfrak{D} \cap \{\Lambda : |\Lambda| \leq \delta^{-1}\}$. Since $D_{KdV} \to 1$ as $|\Lambda| \to \infty$ and $\Lambda = 0$ is the only zero of D_{KdV} on $\Omega_{KdV} \supset \mathfrak{D}$ (see Proposition 4.11), we may choose small $\gamma > 0$ such that

$$\inf_{\Lambda\in\Gamma_\delta} |D_{KdV}(\Lambda)| > \gamma.$$

From Theorem 4.19, there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0]$,

$$|D_{KdV}(\Lambda)| > \gamma > |D_*(\Lambda,\varepsilon) - D_{KdV}(\Lambda)| \quad \text{on } \Gamma_\delta. \tag{4.85}$$

Now the proof is finished by applying Rouché's theorem together with the facts that the multiplicity of the only zero $\Lambda = 0$ of $D_{KdV}(\Lambda)$ is exactly two and that $\Lambda = 0$ is a zero of $D_*(\Lambda,\varepsilon)$ of the multiplicity at least two. □

From Proposition 4.8, Proposition 4.9, Proposition 4.10, the result that $\Lambda = \lambda = 0$ is the only zero of the Evans function $D_*(\Lambda,\varepsilon)$ (and hence $D(\lambda,\varepsilon)$) yields the following results.

Theorem 4.21 (Spectrum of $\mathcal{L}$ in L^2)**.** *Consider the operator $\mathcal{L} : (L^2)^2 \to (L^2)^2$ with dense domain $(H^1)^2$. Then, for all sufficiently small $\varepsilon > 0$, we have*

$$\sigma_{\text{ess}}(\mathcal{L}) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda = 0\}, \quad \sigma_{\text{pt}}(\mathcal{L}) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \neq 0\} = \emptyset.$$

Theorem 4.22 (Spectrum of $\mathcal{L}$ in L^2_η)**.** *Consider the operator $\mathcal{L} : (L^2_\eta)^2 \to (L^2_\eta)^2$ with dense domain $(H^1_\eta)^2$. For $0 < c_0 < \sqrt{\frac{2\mathsf{V}}{3}}$ and $\varepsilon > 0$, let $\eta = c_0\varepsilon^{1/2}$. There exist a constant $\varepsilon_0 > 0$ and a real-valued function $\kappa(\varepsilon)$ such that for all $\varepsilon \in (0,\varepsilon_0)$,*

$$\sigma_{\text{ess}}(\mathcal{L}) \subset \{\lambda : \operatorname{Re}\lambda \leq -\kappa(\varepsilon) < 0\}, \quad \sigma_{\text{pt}}(\mathcal{L}) \cap \{\lambda : \operatorname{Re}\lambda > -\kappa(\varepsilon)\} = \{0\}.$$

In particular, $\sigma_{\text{ess}}(\mathcal{L})$ is parametrized by two disjoint curves $\{d_\pm(\mu) : \mu = ik - \eta,\ k \in \mathbb{R}\}$.

To prove Theorem 4.19, we divide $\mathfrak{D}$ into four regions as follows: for $\delta > 0$ will be chosen later, we set

$$\begin{aligned}
\mathfrak{D}_1 &:= \mathfrak{D} \cap \{\Lambda : |\Lambda| \le \delta^{-1}\},\\
\mathfrak{D}_2 &:= \mathfrak{D} \cap \{\lambda : \varepsilon^{3/2}\delta^{-1} \le |\lambda| \le \delta\} = \mathfrak{D} \cap \{\Lambda : \delta^{-1} \le |\Lambda| \le \varepsilon^{-3/2}\delta\}\\
\mathfrak{D}_3 &:= \mathfrak{D} \cap \{\lambda : \delta \le |\lambda| \le \delta^{-1}\},\\
\mathfrak{D}_4 &:= \mathfrak{D} \cap \{\lambda : \delta^{-1} \le |\lambda|\}.
\end{aligned}$$

Lemma 4.23. *For any fixed constant $\delta > 0$,*

$$\sup_{\Lambda \in \mathfrak{D}_1} |D_*(\Lambda, \varepsilon) - D_{KdV}(\Lambda)| \to 0, \quad \text{as} \quad \varepsilon \to 0. \tag{4.87}$$

Proof. $D_*(\Lambda, \varepsilon)$ is uniformly continuous on a fixed compact set $\{(\Lambda, \varepsilon) : \varepsilon \in [0, \varepsilon_0], \Lambda \in \mathfrak{D}_1\}$ since it is jointly continuous on the set. Hence, $\sup_{\Lambda \in \mathfrak{D}_1} |D_*(\Lambda, \varepsilon) - D_{KdV}(\Lambda)|$ is continuous on $[0, \varepsilon_0]$. Now (4.87) follows from that $D_*(\Lambda, 0) = D_{KdV}(\Lambda)$. See Prop 4.15 and Prop 4.14. □

Lemma 4.24. *There exist constants $C_2, \delta_2, \varepsilon_2 > 0$ such that for all $\varepsilon \in [0, \varepsilon_2]$ and $\delta \in (0, \delta_2]$,*

$$\sup_{\lambda \in \mathfrak{D}_2} |D(\lambda, \varepsilon) - 1| < C_2 \delta^{1/3}. \tag{4.88}$$

Here C_2 is independent of ε and δ.

Lemma 4.25. *For any fixed constant $\delta > 0$,*

$$\sup_{\lambda \in \mathfrak{D}_3} |D(\lambda, \varepsilon) - 1| \to 0, \quad \text{as} \quad \varepsilon \to 0. \tag{4.89}$$

Proof. This is true since $D(\lambda, \varepsilon)$ is jointly continuous and $D(\lambda, 0) = 1$. □

Lemma 4.26. *There exist constants $C_4, \delta_4, \varepsilon_4 > 0$ such that for all $\varepsilon \in [0, \varepsilon_4]$ and $\delta \in (0, \delta_4]$,*

$$\sup_{\lambda \in \mathfrak{D}_4} |D(\lambda, \varepsilon) - 1| < C_4 \varepsilon^{1/2}. \tag{4.90}$$

Here C_4 is independent of ε and δ.

Proof of Theorem 4.19. Let $\gamma > 0$ is given. From the property of D_{KdV}, (4.88), and (4.90), there exist constants $\delta_\gamma, \varepsilon_\gamma > 0$ such that for all $\varepsilon \in (0, \varepsilon_\gamma)$ and $\delta \in (0, \delta_\gamma]$, there hold that

$$\sup_{|\Lambda| \ge \delta_\gamma^{-1}} |D_{KdV}(\Lambda) - 1| < \frac{\gamma}{2}, \tag{4.91}$$

$$\sup_{\lambda \in \mathfrak{D}_2} |D(\lambda, \varepsilon) - 1| < \frac{\gamma}{2}, \quad \sup_{\lambda \in \mathfrak{D}_4} |D(\lambda, \varepsilon) - 1| < \frac{\gamma}{2}. \tag{4.92}$$

We fix $\delta = \delta_\gamma > 0$. Then, from (4.89), there is $\varepsilon_3 > 0$ such that for all $\varepsilon \in (0, \varepsilon_3]$,

$$\sup_{\lambda \in \mathfrak{D}_3} |D(\lambda, \varepsilon) - 1| < \frac{\gamma}{2}. \tag{4.93}$$

Since $D(\lambda,\varepsilon) = D_*(\Lambda,\varepsilon)$, it follows from (4.91),(4.92),(4.93) that for $j = 2,3,4$,

$$\sup_{\lambda\in\mathfrak{D}_j} |D_*(\Lambda,\varepsilon) - D_{KdV}(\Lambda)| < \gamma. \tag{4.94}$$

From (4.87), there exists $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0,\varepsilon_1]$,

$$\sup_{\Lambda\in\mathfrak{D}_1} |D_*(\Lambda,\varepsilon) - D_{KdV}(\Lambda)| < \gamma. \tag{4.95}$$

From (4.94) and (4.95), we conclude that there is $\varepsilon_0 := \min\{\varepsilon_\gamma, \varepsilon_1, \varepsilon_3\}$ such that for all $\varepsilon \in (0,\varepsilon_0]$,

$$\sup_{\lambda\in\mathfrak{D}} |D_*(\Lambda,\varepsilon) - D_{KdV}(\Lambda)| < \gamma.$$

This finishes the proof. □

Remark 6 (The reason for dividing into $\mathfrak{D}_2$ and $\mathfrak{D}_3$). There is no way to extend $D(\lambda,\varepsilon)$ to a (uniformly or jointly) continuous function defined on a compact set including $(\lambda,\varepsilon) = (0,0)$. If there were such an extension, its value at $(0,0)$ must be 1 since $D(\lambda,0) = 1$ for all $\lambda \in \Omega^0$. However, we will see that $D(0,\varepsilon) = 0$ for $\varepsilon \in (0,\varepsilon_0]$.

4.5 Asymptotic Behavior of $D(\lambda,\varepsilon)$

4.5.1 Proof of Lemma 4.24

We first observe the behavior of the roots of the characteristic polynomial $d(\mu)$ on $\mathfrak{D}_2$, which can be obtained by a perturbation argument. The proof is given in Appendix.

Proposition 4.27. *1. There exists a constant $\varepsilon_0 > 0$ such that as long as λ and ε satisfy $\varepsilon^{3/2}\delta^{-1} < |\lambda| < \delta$ and $0 \le \varepsilon \le \varepsilon_0$, where $\delta > 0$ is small, there holds that*

$$\mu_j = (-2\sqrt{1+K}\,\lambda)^{1/3} e^{2\pi i j/3}(1+\widetilde{\beta}_j) \quad (j = 1,2,3),$$

where $\widetilde{\beta}_j$ are functions of λ and ε, and $\widetilde{\beta}_j \to 0$ uniformly in ε as $\delta \to 0$.

2. For $|\lambda| < \delta$ sufficiently small,

$$\mu_4 = \frac{\lambda}{c+\sqrt{1+K}} + \widetilde{\beta}_4,$$

where $\widetilde{\beta}_4$ is a function of μ_4 and ε such that $\widetilde{\beta}_4 = O(\lambda^3)$ uniformly in $0 \le \varepsilon < 1$.

To prove Lemma 4.24, we also apply the following proposition ([26],[28], see Proposition 5.48 for the proof). We note that Proposition 4.27 implies in particular that $\mu_j(\lambda)$ are all distinct for $\lambda \ne 0$ with $\varepsilon^{-3/2}\lambda = \Lambda \in \mathfrak{D}_2$.

Proposition 4.28. *We assume that for a matrix* $A(x,\lambda)$ *with* $\lim_{x\to\pm\infty} A(x,\lambda) = A_\infty(\lambda)$*, the system* (5.39) *satisfies the hypotheses* ***H1–H4***. *We further assume that* $A_\infty(\lambda)$ *is diagonalizable such that for the matrices* W *and* V *defined by*

$$W := \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_n \end{bmatrix}, \quad V := \begin{bmatrix} \mathbf{v}_1, \cdots, \mathbf{v}_n \end{bmatrix},$$

where $\mathbf{w}_i$ *and* $\mathbf{v}_i$ *are the left and right eigenvectors of* $A_\infty(\lambda)$ *associated with* μ_i*, we have*

$$WA_\infty(\lambda)V = \mathrm{diag}\{\mu_j\}, \quad WV = I.$$

Let $R(x,\lambda) := A(x,\lambda) - A_\infty(\lambda)$*. Then, there exists* $0 < \delta_0 < 1$ *such that if* $\int_{-\infty}^{\infty} |WR(x,\lambda)V|\,dx \leq \delta_0$*, then*

$$|D(\lambda) - 1| \leq C \int_{-\infty}^{\infty} |WR(x,\lambda)V|\,dx. \tag{4.96}$$

Proof of Lemma 4.24. Let $R(x,\lambda,\varepsilon) = A(x,\lambda,\varepsilon) - A^\infty(\lambda,\varepsilon)$. It is straightforward to check that

$$|R_{jk}(x,\lambda,\varepsilon)| \leq C\varepsilon e^{-C\varepsilon^{1/2}|x|} E_{jk}, \tag{4.97}$$

where

$$E = \begin{pmatrix} \varepsilon^{1/2} + |\lambda| & \varepsilon^{1/2} + |\lambda| & 0 & 1 \\ \varepsilon^{1/2} + |\lambda| & \varepsilon^{1/2} + |\lambda| & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{4.98}$$

Let

$$V = \begin{bmatrix} \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \end{bmatrix}, \quad W = \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_4 \end{bmatrix}.$$

Let v_{jl} and w_{jl} be the l-th component of $\mathbf{v}_j$ and $\mathbf{w}_j$, respectively. Then,

$$(WRV)_{jk} = \sum_{l=1,2} w_{jl}(R_{l1}v_{k1} + R_{l2}v_{k2} + R_{l4}v_{k4}) + w_{j4}R_{43}v_{k3}. \tag{4.99}$$

Applying (4.41), (4.42), (4.97) and (4.98), one can obtain from (4.99) that

$$|(WRV)_{jk}| \leq C\varepsilon e^{-C\varepsilon^{1/2}|x|} G_{jk} \tag{4.100}$$

where

$$\begin{aligned} G_{jk} := \frac{|1-\mu_j^2|}{|\mu_j|^2|\widetilde{G}_j|} \Bigg\{ & \frac{|\mu_j|}{|1-\mu_k^2|} + \frac{|1-\mu_j^2|}{|\mu_j|} \left(|c\lambda - \mu_j(c^2-K)| + |\lambda|\right) \\ & \times \left[(\varepsilon^{1/2} + |\lambda|)\left(1 + \frac{|c\mu_k - \lambda|}{|\mu_k|}\right) + \frac{|\mu_k|}{|1-\mu_k^2|} \right] \Bigg\} \end{aligned} \tag{4.101}$$

and

$$\widetilde{G}_j = (1-\mu_j^2)^2\frac{\lambda^2}{\mu_j^4} - \frac{2\varepsilon\sqrt{1+K}}{\mu_j^2} - \frac{\varepsilon^2}{\mu_j^2} + 1 + (2-\mu_j^2)(c^2-K).$$

Using Proposition 4.27, it is strightforward to see that as long as $\varepsilon^{3/2}\delta^{-1} < |\lambda| < \delta$,

$$|\widetilde{G}_j| = 1 + o(1) \quad \text{uniformly in} \quad \varepsilon \quad \text{as} \quad \delta \to 0,$$

for $j = 1, 2, 3$, and $|\mu_4|^2|\widetilde{G}_4|$ is bounded from above and below by some positive constant C uniformly in δ and ε. From this, we obtain that $\varepsilon^{1/2}|G_{jk}| \le C\delta^{1/3}$, $(j, k = 1, 2, 3, 4)$, for some positive constant C independent of δ and ε. On the other hand, we have

$$\varepsilon^{1/2}\int_{-\infty}^{\infty} e^{-C\varepsilon^{1/2}|x|}\,dx < C'.$$

This finishes the proof.

□

4.5.2 Proof of Lemma 4.26

To prove Lemma 4.26, we need the asymptotic behavior of the roots of characteristic polynomial $d(\mu)$ for large $|\lambda|$.

Proposition 4.29. *The roots of $d(\mu)$ can be labelled so that they satisfy*[19]

$$\mu_1 = -1 + O(|\lambda|^{-2}), \quad \mu_4 = 1 + O(|\lambda|^{-2}),$$

$$\mu_2 = \frac{c\lambda - \sqrt{-c^2+K+K\lambda^2}}{c^2-K} + O(|\lambda|^{-3}), \quad \mu_3 = \frac{c\lambda + \sqrt{-c^2+K+K\lambda^2}}{c^2-K} + O(|\lambda|^{-3})$$

as $|\lambda| \to \infty$. Here, the big-O terms are uniform in $\varepsilon \in [0, \varepsilon_0]$.

The proof of Proposition 4.29 is given in Appendix, and it is based on a perturbation argument using Rouché's theorem in a similar fashion to Lemma 1.20 of [26].

From Proposition 4.29, on $\mathfrak{D}_4$, we have[20] for $j = 1, 4$,

$$\mu_j = (-1)^j + O(|\lambda|^{-2}), \qquad 1-\mu_j^2 = \lambda^{-2}\left(1+O(|\lambda|^{-1})\right), \tag{4.102a}$$

$$c\mu_j - \lambda = -\lambda\left(1+O(|\lambda|^{-1})\right), \qquad c\lambda - \mu_j(c^2-K) = c\lambda\left(1+O(|\lambda|^{-1})\right)), \tag{4.102b}$$

[19] In the case $K > 0$, the order of the error terms for μ_2 and μ_3 are different from the case $K = 0$. See the proof.

[20] (4.102b) directly follows using the first equation of (4.102a). To show that $1-\mu_j^2 = \lambda^{-2}\left(1+O(|\lambda|^{-1})\right)$, we use $d(\mu_j) = 0$, that is, $\dfrac{1}{1-\mu_j^2} = \dfrac{(\lambda - c\mu_j)^2 - K\mu_j^2}{\mu_j^2}$ rather than use $\mu_j = \pm 1 + O(|\lambda|^{-2})$ directly. Direct calculation using (4.103a) leads (4.103b)–(4.103d).

and for $j = 2, 3$,

$$\mu_j = \frac{\lambda}{c + (-1)^j \sqrt{K}} \left(1 + O(|\lambda|^{-2})\right), \tag{4.103a}$$

$$1 - \mu_j^2 = \frac{-\lambda^2}{(c + (-1)^j \sqrt{K})^2} \left(1 + O(|\lambda|^{-2})\right), \tag{4.103b}$$

$$c\mu_j - \lambda = \frac{(-1)^{1+j}\sqrt{K}}{c + (-1)^j \sqrt{K}} \lambda \left(1 + O(|\lambda|^{-2})\right), \tag{4.103c}$$

$$c\lambda - \mu_j(c^2 - K) = (-1)^j \sqrt{K} \lambda \left(1 + O(|\lambda|^{-2})\right). \tag{4.103d}$$

Additionally, we have

$$\lambda \pm \sqrt{c^2 - K}\, \mu_j = \lambda \left(1 + O(|\lambda|^{-1})\right) \qquad \text{for } j = 1, 4, \tag{4.104a}$$

$$\lambda \pm \sqrt{c^2 - K}\, \mu_j = \lambda \left(1 + \frac{\pm\sqrt{c^2 - K}}{c + (-1)^j \sqrt{K}} + O(|\lambda|^{-2})\right) \qquad \text{for } j = 2, 3. \tag{4.104b}$$

Using the estimates (4.102), (4.103) and (4.104), we have from (4.42b) that

$$\text{for } j = 1, 4, \qquad \boldsymbol{\pi}_j \mathbf{v}_j = \frac{(1 - \mu_j^2)(\lambda^2 - \mu_j^2(c^2 - K))}{\mu_j^2} + \frac{1 + \mu_j^2}{1 - \mu_j^2} = 2\lambda^2 \left(1 + O(|\lambda|^{-1})\right), \tag{4.105a}$$

$$\text{for } j = 2, 3, \qquad \boldsymbol{\pi}_j \mathbf{v}_j = \lambda^2 \left(\frac{(1 - \mu_j^2)(\lambda^2 - \mu_j^2(c^2 - K))}{\mu_j^2 \lambda^2} + \frac{1}{\lambda^2} \frac{1 + \mu_j^2}{1 - \mu_j^2}\right) = -\lambda^2 \left(\frac{2(-1)^j \sqrt{K}}{(c + (-1)^j \sqrt{K})} + O(|\lambda|^{-1})\right). \tag{4.105b}$$

We first observe that on the domain $\mathfrak{D}_4$, Proposition 4.28 is not directly applied as the analysis on the domain $\mathfrak{D}_2$. Applying (4.102),(4.103),(4.104) and (4.105) to (4.101), we have for large $|\lambda|$,

$$|G_{jk}| \le C, \quad (j, k = 1, 4), \qquad |G_{jk}| \le C|\lambda|^{-2}, \quad (j = 1, 4, k = 2, 3),$$

$$|G_{jk}| \le C|\lambda|, \quad (j, k = 2, 3), \qquad |G_{jk}| \le C|\lambda|^2, \quad (j = 2, 3, k = 1, 4).$$

The bound $|G_{jk}| \le C|\lambda|, \quad (j, k = 2, 3)$ is due to $|\lambda|$ term in $(\varepsilon^{1/2} + |\lambda|)$. The bound $|G_{jk}| \le C|\lambda|^2, \quad (j = 2, 3, k = 1, 4)$ results from the growth rate of $|1 - \mu_k^2|^{-1}$ and the boundedness of $|\mu_k|^{-1}$ for $k = 1, 4$ as $|\lambda| \to +\infty$. Hence, we need a more delicate approach to obtain the uniform bounds for $|G_{jk}|$ on the domain $\mathfrak{D}_4$.

To accomplish this, we write

$$A(x, \lambda, \varepsilon) - A^{\infty}(\lambda, \varepsilon) = \lambda R^{(1)}(x, \varepsilon) + R^{(2)}(x, \varepsilon),$$

where (see (4.31))

$$
\begin{aligned}
R^{(1)} = R^{(1)}(x,\varepsilon) &:= A_2(x,\varepsilon) - \lim_{|x|\to\infty} A_2(x,\varepsilon) \\
&= \frac{1}{J}\left[\begin{array}{cc|c} c-u_c & 1+n_c & \multirow{2}{*}{$\mathbf{0}_2$} \\ \frac{K}{1+n_c} & c-u_c & \\ \hline \multicolumn{2}{c|}{\mathbf{0}_2} & \mathbf{0}_2 \end{array}\right] - \frac{1}{c^2-K}\left[\begin{array}{cc|c} c & 1 & \multirow{2}{*}{$\mathbf{0}_2$} \\ K & c & \\ \hline \multicolumn{2}{c|}{\mathbf{0}_2} & \mathbf{0}_2 \end{array}\right].
\end{aligned}
$$

Then using (4.8), a direct computation yields that

$$|R^{(2)}_{jk}(x)| \le C\varepsilon e^{-C\varepsilon^{1/2}|x|} E^{(2)}_{jk}, \tag{4.107}$$

where (compare (4.108) with (4.98))

$$E^{(2)} := \begin{pmatrix} \varepsilon^{1/2} & \varepsilon^{1/2} & 0 & 1 \\ \varepsilon^{1/2} & \varepsilon^{1/2} & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{4.108}$$

We let V_0 be the matrix whose j-th column is $(c+(-1)^j\sqrt{K})|1-\mu_j^2|^{\frac{1}{2}}\,\mathbf{v}_j$ and W_0 be the matrix whose j-th row is $(c+(-1)^j\sqrt{K})^{-1}|1-\mu_j^2|^{-\frac{1}{2}}\mathbf{w}_j$, that is

$$
\begin{aligned}
W_0 &:= \begin{bmatrix} (c+(-1)^1\sqrt{K})^{-1}|1-\mu_1^2|^{-\frac{1}{2}}\mathbf{w}_1 \\ \vdots \\ (c+(-1)^4\sqrt{K})^{-1}|1-\mu_1^2|^{-\frac{1}{2}}\mathbf{w}_4 \end{bmatrix}, \\
V_0 &:= \left[(c+(-1)^1\sqrt{K})|1-\mu_1^2|^{\frac{1}{2}}\,\mathbf{v}_1,\ \cdots,\ (c+(-1)^4\sqrt{K})|1-\mu_4^2|^{\frac{1}{2}}\,\mathbf{v}_4\right].
\end{aligned} \tag{4.109}
$$

We note that μ_j are all distinct on $\mathfrak{D}_4$ by Proposition 4.29, and it is clear that

$$W_0V_0 = I, \quad W_0A^\infty V_0 = \operatorname{diag}(\mu_1,\mu_2,\mu_3,\mu_4) \tag{4.110}$$

and

$$W_0\left(A - A^\infty\right)V_0 = \lambda W_0 R^{(1)} V_0 + W_0 R^{(2)} V_0. \tag{4.111}$$

Estimate of $W_0R^{(2)}V_0$ From the definitions of V_0 and W_0, (4.109), we have

$$
\begin{aligned}
(W_0R^{(2)}V_0)_{jk} &= \frac{c+(-1)^k\sqrt{K}}{c+(-1)^j\sqrt{K}}\sqrt{\frac{|1-\mu_k^2|}{|1-\mu_j^2|}} \\
&\quad\times\left(\sum_{l=1,2} w_{jl}\left(R^{(2)}_{l1}v_{k1} + R^{(2)}_{l2}v_{k2} + R^{(2)}_{l4}v_{k4}\right) + w_{j4}R^{(2)}_{43}v_{k3}\right).
\end{aligned}
$$

From (4.41), (4.42), (4.107) and (4.108), we obtain

$$|(W_0R^{(2)}V_0)_{jk}| \le C\varepsilon e^{-C\varepsilon^{1/2}|x|} G^{(2)}_{jk}, \tag{4.112}$$

where (compare $G^{(2)}_{jk}$ with G_{jk})[21]

$$G^{(2)}_{jk} := \sqrt{\frac{|1-\mu_k^2|}{|1-\mu_j^2|}}\frac{1}{|\boldsymbol{\pi}_j \cdot \mathbf{v}_j|}\left\{\frac{|\mu_j|}{|1-\mu_k^2|} + \frac{|1-\mu_j^2|}{|\mu_j|}\left(|c\lambda - \mu_j(c^2-K)| + |\lambda|\right)\right.$$
$$\left. \times\left[\varepsilon^{1/2}\left(1 + \frac{|c\mu_k - \lambda|}{|\mu_k|}\right) + \frac{|\mu_k|}{|1-\mu_k^2|}\right]\right\}.$$

By a direct calculation using (4.102), (4.103), (4.104) and (4.105), we have

$$|G^{(2)}_{jk}| \le C \quad \text{for } j,k = 1,2,3,4, \tag{4.113}$$

uniformly in ε and $|\lambda| \ge \delta^{-1}$ for sufficiently small δ. From (4.112) and (4.113), we obtain the bound

$$|W_0 R^{(2)}(x) V_0| \le C\varepsilon e^{-C\varepsilon^{1/2}|x|}, \tag{4.114}$$

where the constant C is uniform in ε and λ with $|\lambda| \ge \delta^{-1}$ for sufficiently small δ.

Estimate of $\lambda W_0 R^{(1)} V_0$ Now we estimate $\lambda W_0 R^{(1)} V_0$ part. We have

$$\begin{aligned}(\lambda W_0 R^{(1)} V_0)_{jk} &= \frac{c + (-1)^k\sqrt{K}}{c+(-1)^j\sqrt{K}}\sqrt{\frac{|1-\mu_k^2|}{|1-\mu_j^2|}}\left(\sum_{l=1,2}\lambda w_{jl}\left[R^{(1)}_{l1}v_{k1} + R^{(1)}_{l2}v_{k2}\right]\right)\\
&= \frac{c + (-1)^k\sqrt{K}}{c+(-1)^j\sqrt{K}}\sqrt{\frac{|1-\mu_k^2|}{|1-\mu_j^2|}}\frac{\lambda}{\boldsymbol{\pi}_j\cdot\mathbf{v}_j}\left(\frac{1-\mu_j^2}{\mu_j}\right)\\
&\quad\times\left[(c\lambda - \mu_j(c^2-K))\left(R^{(1)}_{11} + \frac{c\mu_k-\lambda}{\mu_k}R^{(1)}_{12}\right) - \lambda\left(R^{(1)}_{21} + \frac{c\mu_k - \lambda}{\mu_k}R^{(1)}_{22}\right)\right].\end{aligned}$$

Using (4.102),(4.103),(4.104) and (4.105), a direct calculation yields a decomposition

$$\lambda W_0 R^{(1)} V_0 = \frac{\lambda}{2\sqrt{K}}S_1 + \widetilde{R}^{(1)}, \tag{4.115}$$

where $S_1 = S_1(x;\varepsilon)$ is a symmetric matrix defined by

$$S_1 := \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 2\sqrt{K}R^{(1)}_{11} - KR^{(1)}_{12} - R^{(1)}_{21} & KR^{(1)}_{12} - R^{(1)}_{21} & 0\\ 0 & KR^{(1)}_{12} - R^{(1)}_{21} & 2\sqrt{K}R^{(1)}_{11} + KR^{(1)}_{12} + R^{(1)}_{21} & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \tag{4.116}$$

[21]The order of $\sqrt{\frac{|1-\mu_k^2|}{|1-\mu_j^2|}}$ is $|\lambda|^2$ for $j = 1,4, k = 2,3$ and $|\lambda|^{-2}$ for $j = 2,3, k = 1,4$. Moreover, there is no $\varepsilon^{1/2} + |\lambda|$ term, which caused a problem for getting a uniform bound of G_{jk} for $j,k = 2,3$. Multiplying $\mathbf{v}_j$ by $1-\mu_j^2$ does not work since in that case, the factor $\sqrt{\frac{1-\mu_k^2}{1-\mu_j^2}}$ is of the order λ^4 for $j = 1,4, k = 2,3$. One might think that choosing $\mathbf{v}_j$ by $\sqrt{1-\mu_j^2}\mathbf{v}_j$ from the beginning of the construction of the Evans function. However, $\sqrt{1-\mu_j^2}$ is not analytic on the right half plane for any choice of branch cut. See Ahlfors p.148.

and $\widetilde{R}^{(1)}$ is a matrix whose (j,k) entries are functions of $(\lambda, n_c, u_c, \varepsilon)$ such that

$$|(\widetilde{R}^{(1)})_{jk}| \le C\varepsilon e^{-C\varepsilon^{1/2}|x|} \tag{4.117}$$

holds for all $\varepsilon \in [0,\varepsilon_0]$ and $|\lambda| \ge \delta^{-1}$ for sufficiently small δ. (Here, the constant C is uniform in δ, ε and x.) (Recall that $o(1)$ is indeed $O(|\lambda|^{-1})$). The matrix S_1 is positive semi-definite (or non-negative).[22]

Lemma 4.30. *There exists $\varepsilon_0 > 0$ and $C > 0$, independent of ε and x, such that for all $\varepsilon \in [0,\varepsilon_0]$, the symmetric matrix $S_1(x;\varepsilon)$ defined in (4.116) is positive semi-definite(or non-negative) and there holds that $|S_1(x;\varepsilon)| \le C\varepsilon$.*

The proof of Lemma 4.30 is straightforward using the solitary wave identities. Since it is not short, we verify it in Appendix. Now we are ready to prove Lemma 4.26.

Proof of Lemma 4.26. From (4.114), (4.115), (4.116), (4.117), (4.111), (4.110), we have

$$\begin{aligned} W_0(A-\mu_1 I)V_0 &= W_0(A^\infty - \mu_1 I)V_0 + W_0(A-A^\infty)V_0 \\ &= \widetilde{B} + \widetilde{F}, \end{aligned}$$

where

$$\widetilde{B}(x,\lambda,\varepsilon) := \mathrm{diag}(0, \mu_2-\mu_1, \mu_3-\mu_1, \mu_4-\mu_1) + \frac{\lambda}{2\sqrt{K}} S_1 \tag{4.118}$$

and

$$|\widetilde{F}(x,\lambda,\varepsilon)| \le C\varepsilon e^{-C\varepsilon^{1/2}|x|}.$$

We note that for all $\varepsilon \in (0,\varepsilon_0]$,

$$\int_{-\infty}^{\infty} |\widetilde{F}(x,\lambda,\varepsilon)|\,dx \le C\varepsilon^{1/2}, \tag{4.119}$$

where the constant C is uniform in ε and $|\lambda| \ge \delta^{-1}$.

We let

$$\widetilde{\mathbf{e}}_1 := \Big((c-\sqrt{K})^{-1}|1-\mu_1^2|^{-\frac{1}{2}}, 0,0,0\Big)^T, \quad \widetilde{\mathbf{e}}_1^* := \Big((c-\sqrt{K})|1-\mu_1^2|^{\frac{1}{2}}, 0,0,0\Big).$$

Changing variables $\widetilde{\mathbf{y}}(x) = e^{-\mu_1 x} W_0 \mathbf{y}(x) - \widetilde{\mathbf{e}}_1$, we have

$$\frac{d\widetilde{\mathbf{y}}}{dx} = \widetilde{B}(x;)\widetilde{\mathbf{y}} + \widetilde{F}(x;)(\widetilde{\mathbf{e}}_1 + \widetilde{\mathbf{y}}). \tag{4.120}$$

With a particular choice of $\mathbf{y}^+$, we know that $\widetilde{\mathbf{y}}^+(x) := e^{-\mu_1 x}W_0\mathbf{y}^+(x) - \widetilde{\mathbf{e}}_1$ is a solution of (4.120) satisfying $\lim_{x\to+\infty} \widetilde{\mathbf{y}}^+(x) = 0$ from the definition of W_0 and (4.33).

Let $\Phi(x;s)$ be the fundamental matrix of the simpler system

$$\frac{d\boldsymbol{a}}{dx} = \widetilde{B}(x;)\boldsymbol{a} \tag{4.121}$$

[22] This is why we multiplied the eigenvector $\mathbf{v}_j$ by $c + (-1)^j\sqrt{K}$. Indeed, symmetricity is not necessary, but it gives an easy way to verify that the matrix is non-negative.

satisfying $\Phi(s;s) = I$. In Lemma 4.31, we will show that $|\Phi(x;s)| \le 1$ for $x \le s$. Using this fact and (4.119), one may apply an iteration argument to show that there is a solution $\widetilde{\mathbf{y}}_\sharp^+$ of (4.120) satisfying $\lim_{x\to+\infty} \widetilde{\mathbf{y}}_\sharp^+(x) = 0$ as a fixed point of the bounded linear operator $\widetilde{\mathcal{T}}$ on $C_b([0,\infty))$ defined by

$$(\widetilde{\mathcal{T}}\,\widetilde{\mathbf{y}})(x) := -\int_x^\infty \Phi(x;s)\big[\widetilde{F}(s)(\widetilde{\mathbf{e}}_1 + \widetilde{\mathbf{y}}(s))\big]\,ds.$$

Since $\widetilde{\mathbf{y}}_\sharp^+$ and $\widetilde{\mathbf{y}}^+$ tends to 0 as $x \to +\infty$, we have $\widetilde{\mathbf{y}}_\sharp^+ = \widetilde{\mathbf{y}}^+$ by the one-to-one correspondence between bounded solutions of (4.121) and (4.120).[23] Hence, from the definition of $\widetilde{\mathcal{T}}$, we obtain

$$\sup_{x\in[0,\infty)} |\widetilde{\mathbf{y}}^+(x)| \le C\varepsilon^{1/2}.$$

In a similar fashion, one can obtain that

$$\sup_{x\in(-\infty,0]} |\widetilde{\mathbf{z}}^-(x)| \le C\varepsilon^{1/2},$$

where $\widetilde{\mathbf{z}}^-(x) := \mathbf{z}^-(x)e^{\mu_1 x}V_0 - \widetilde{\mathbf{e}}_1^*$. Since $D(\lambda,\varepsilon) = \mathbf{z}^-\mathbf{y}^+ = (\widetilde{\mathbf{z}}^- + \widetilde{\mathbf{e}}_1^*)(\widetilde{\mathbf{y}}^+ + \widetilde{\mathbf{e}}_1)$, we arrive at

$$|D(\lambda,\varepsilon) - 1| \le C\varepsilon^{1/2}.$$

□

Lemma 4.31. *For $\widetilde{B}(x;)$ given by (4.118) with S_1 is defined in (4.116), let $\Phi(x;x_0) \in \mathbb{C}^{4\times 4}$ be the fundamental matrix of*

$$\frac{d\boldsymbol{a}}{dx} = \widetilde{B}(x;)\boldsymbol{a}$$

satisfying $\Phi(x_0;x_0) = I$. Then, if $\varepsilon > 0$ and $\delta > 0$ are sufficiently small, we have that for all $\lambda \in \mathfrak{D}_4$,

$$|\Phi(x;x_0)\mathbf{q}| \le |\mathbf{q}| \quad \text{for all } x \le x_0 \text{ and } \mathbf{q} \in \mathbb{C}^4. \tag{4.122}$$

Proof. Let $\boldsymbol{a} := (a_1,a_2,a_3,a_4)^T(x)$ and $\mathbf{q} := (q_1,q_2,q_3,q_4)^T$. From the structure of $\widetilde{B}$, we observe that $\frac{da_1}{dx} = 0$ and $\frac{da_4}{dx} = (\mu_4 - \mu_1)a_4(x)$. Since $a_1(x) = q_1$ and $a_4(x) = e^{(\mu_4-\mu_1)(x-x_0)}q_4$, we have

$$|a_1|(x) \le |q_1| \quad \text{and} \quad |a_4|(x) \le |q_4| \quad \text{for } x \le x_0,$$

where we have used that $\mu_4 - \mu_1 = 2 + O(|\lambda|^{-2})$ (see (4.102)).

Let $\widetilde{\boldsymbol{a}} := (a_2,a_3)^T(x)$. Then, $\widetilde{\boldsymbol{a}}$ satisfies

$$\frac{d\widetilde{\boldsymbol{a}}}{dx} = \begin{pmatrix} \mu_2 - \mu_1 & 0 \\ 0 & \mu_3 - \mu_1 \end{pmatrix} + \frac{\lambda}{2\sqrt{K}}\widetilde{S}_1,$$

where

$$\widetilde{S}_1 := \begin{pmatrix} 2\sqrt{K}R_{11}^{(1)} - KR_{12}^{(1)} - R_{21}^{(1)} & KR_{12}^{(1)} - R_{21}^{(1)} \\ KR_{12}^{(1)} - R_{21}^{(1)} & 2\sqrt{K}R_{11}^{(1)} + KR_{12}^{(1)} + R_{21}^{(1)} \end{pmatrix}, \tag{4.123}$$

[23] See Coppel, [6]. Or, one may directly use Proposition 4.4 by considering the asymptotic behavior of $\mathbf{y}_\sharp^+$ and $\mathbf{y}^+$ as $x \to +\infty$, where $\mathbf{y}_\sharp^+$, defined by $\widetilde{\mathbf{y}}_\sharp^+ = e^{-\mu_1 x}W_0\mathbf{y}_\sharp^+ - \widetilde{\mathbf{e}}_1$, is a solution of the ODE (4.32). Indeed, we have $e^{-\mu_1 x}\mathbf{y}_\sharp^+(x) \to V_0\widetilde{\mathbf{e}}_1 = \mathbf{v}_1$ and $e^{-\mu_1 x}\mathbf{y}^+(x) \to \mathbf{v}_1$ as $x \to +\infty$.

and we have

$$\frac{1}{2}\frac{d}{dx}\langle \widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{a}} \rangle \geq \min_{j=2,3} \mathrm{Re}(\mu_j - \mu_1)\langle \widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{a}} \rangle + \frac{\mathrm{Re}\,\lambda}{2\sqrt{K}}\left\langle \widetilde{S}_1 \widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{a}} \right\rangle.$$

where $\langle , \rangle$ denotes the standard inner product on $\mathbb{C}^2$. From (4.102) and (4.103),

$$\min_{j=2,3} \mathrm{Re}\,(\mu_j - \mu_1) = \begin{cases} \dfrac{\mathrm{Re}\,\lambda}{c+\sqrt{K}} + 1 + O(|\lambda|^{-1}) & \text{if } \mathrm{Re}\,\lambda \geq 0, \\ \dfrac{\mathrm{Re}\,\lambda}{c-\sqrt{K}} + 1 + O(|\lambda|^{-1}) & \text{if } \mathrm{Re}\,\lambda < 0, \end{cases}$$

Thus, since $\widetilde{S}_1$ is non-negative, we have

$$\frac{1}{2}\frac{d}{dx}\langle \widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{a}} \rangle > \begin{cases} 0 & \text{if } \mathrm{Re}\,\lambda \geq 0, \\ \dfrac{1}{2}\langle \widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{a}} \rangle & \text{if } \varepsilon^{3/2} c_0^* \leq \mathrm{Re}\,\lambda < 0 \end{cases}$$

where $c_0^* := -c_0\left(1 - \dfrac{c_0^2}{2\mathrm{V}}\right) + a < 0$. Integrating over $[x, x_0]$, we obtain

$$|(a_2, a_3)|^2(x) \leq |(a_2, a_3)|^2(x_0) = |(q_2, q_3)|^2$$

for $x \leq x_0$, which finishes the proof. □

4.6 Linear Convective Stability

The linearized system (4.13) can be rewritten as

$$\partial_t \mathbf{u} = \mathcal{L}\mathbf{u} \tag{4.124}$$

where $\mathcal{L}$ is the operator defined in (4.16) and $\mathbf{u} = (n, u)^T$. We consider the abstract Cauchy problem (4.124) in $(L^2_\eta)^2$. We recall that under the change of variable $\mathbf{u}^\eta := e^{\eta x}\mathbf{u}$, studying the spectrum and resolvent operator of $\mathcal{L}$ in L^2_η space is equivalent to studying those of $\mathcal{L}_\eta := e^{\eta x}\mathcal{L}e^{-\eta x}$ in L^2 space. Specifically,

$$\mathcal{L}^\eta \mathbf{u}^\eta := -(L_1(\partial_x - \eta) + L_2)\mathbf{u}^\eta - \begin{pmatrix} 0 \\ (\partial_x - \eta)(-(\partial_x - \eta)^2 + e^{\phi_c})^{-1}(\mathbf{u}_1^\eta) \end{pmatrix}. \tag{4.125}$$

$\mathcal{L}$ generates C_0-semigroup in (L^2_η). It is enough to check that $-L_1\partial_x$ generates C_0-semigroup in $(L^2)^2$ space since the remaining terms are bounded operators (see Proposition 5.27). If we denote the C_0-semigroup with generator $\mathcal{L}^\eta$ by $e^{\mathcal{L}^\eta t}$, then $e^{\mathcal{L}t} := e^{-\eta x}e^{\mathcal{L}^\eta t}e^{\eta x}$ is a C_0-semigroup generated by $\mathcal{L}$ in $(L^2_\eta)^2$ spaces.

We note that L_1 is symmetrizable. We let

$$L_0^{1/2} := \begin{pmatrix} \dfrac{\sqrt{K}}{\sqrt{1+n_c}} & 0 \\ 0 & \sqrt{1+n_c} \end{pmatrix}$$

and

$$S_0 := L_0^{1/2}(-L_1)L_0^{-1/2} = \begin{pmatrix} c+u_c & -\sqrt{K} \\ -\sqrt{K} & c-u_c \end{pmatrix}, \quad \widetilde{L} := S_0\partial_x - L_0^{1/2}L_1(\partial_x L_0^{-1/2}).$$

If $S_0\partial_x$ generates a C_0-semigroup on $(L^2)^2$, then $\widetilde{L}$ also generates a C_0-semigroup, $e^{\widetilde{L}t}$, as a bounded perturbation of $S_0\partial_x$. Then,

$$e^{-L_1\partial_x t} := L_0^{-1/2} e^{\widetilde{L}t} L_0^{1/2}$$

is a C_0-semigroup with generator $-L_1\partial_x$. Indeed, we have for all $\mathbf{u}^\eta \in (H^1)^2$,

$$\begin{aligned} L_0^{-1/2}\widetilde{L}L_0^{1/2}\mathbf{u}^\eta &= L_0^{-1/2}\left(L_0^{1/2}(-L_1)L_0^{-1/2}\partial_x - L_0^{1/2}L_1(\partial_x L_0^{1/2})\right)L_0^{1/2}\mathbf{u}^\eta \\ &= -L_1L_0^{-1/2}\partial_x(L_0^{1/2}\mathbf{u}^\eta) - L_1(\partial_x L_0^{-1/2})L_0^{1/2}\mathbf{u}^\eta \\ &= -L_1\partial_x\left(L_0^{-1/2}L_0^{1/2}\mathbf{u}^\eta\right) = -L_1\partial_x\mathbf{u}^\eta. \end{aligned}$$

To show that $S_0\partial_x$ generates a C_0-semigroup, we first consider the shifted operator $S_0\partial_x - I$. Since S_0 is a real-valued symmetric matrix, and the derivative of u_c is small, integration by parts yields that

$$\mathrm{Re}\langle (S_0\partial_x - I)\widetilde{\mathbf{u}}^\eta, \widetilde{\mathbf{u}}^\eta\rangle = -\frac{1}{2}\mathrm{Re}\langle(\partial_x S_0)\widetilde{\mathbf{u}}^\eta, \widetilde{\mathbf{u}}^\eta\rangle - \|\widetilde{\mathbf{u}}^\eta\|_{L^2}^2 < 0,$$

that is, $S_0\partial_x - I$ is dissipative by (5.16). By the Lumer-Phillips Generation Theorem (Theorem 5.25), $S_0\partial_x - I$ generates a C_0-contraction semigroup, and hence $S_0\partial_x$ also generates a C_0-semigroup as a bounded perturbation of $S_0\partial_x - I$.

The algebraic multiplicity of $\lambda = 0$ as an eigenvalue of $\mathcal{L}$ in (L^2_η) is two. Since $0 = D(0,\varepsilon) = \partial_\lambda D(0,\varepsilon) \neq \partial_\lambda^2 D(0,\varepsilon)$, we have from (5.46) and Proposition 5.46 that

$$\partial_\lambda^j \mathbf{y}^+(x,0,\varepsilon) = \begin{cases} O(e^{\mu_1 x + \theta x}) & \text{as } x \to +\infty, \quad \text{for } j = 0,1,2,\cdots, \\ O(e^{\mu_* x - \theta x}) & \text{as } x \to -\infty, \quad \text{for } j = 0,1. \end{cases} \tag{4.126}$$

Let us omit the ε-dependence. Differentiating (4.32) in λ, we see that $\mathbf{y}^+(x,\lambda)$ satisfies (recall the form of $A(x,\lambda)$)

$$\left(\frac{d}{dx} - A_1(x)\right)(\partial_\lambda^{j+1}\mathbf{y}^+)|_{\lambda=0} = (j+1)A_2(x)(\partial_\lambda^j\mathbf{y}^+)|_{\lambda=0} \tag{4.127}$$

for $j = 0,1,\cdots$. From (4.51) and (4.126), $e^{\eta x}\partial_\lambda^j\mathbf{y}^+|_{\lambda=0}$ exponentially decay as $|x| \to \infty$ for $j = 0,1$. We show the following two statements:

1. every non-trivial L^2_η solution of (4.32) with $\lambda = 0$ is a constant multiple of $\mathbf{y}^+(x,0)$;
2. there is no L^2_η function $\widetilde{\mathbf{y}}$ satisfying

$$\left(\frac{d}{dx} - A_1(x)\right)\widetilde{\mathbf{y}} = 2A_2(x)(\partial_\lambda\mathbf{y}^+)|_{\lambda=0}. \tag{4.128}$$

Recalling the reduction of the eigenvalue problem (4.15), it is straightforward to see that these statements imply that the algebraic multiplicity of the zero eigenvalue of $\mathcal{L}$ in L^2_η is two. The first statement has been already proved in Proposition 4.10.

To show the second statement, we suppose that there is a function $\widetilde{\mathbf{y}} \in L^2_\eta$ satisfying (4.128). Since $\partial_\lambda \mathbf{y}^+|_{\lambda=0} \in L^2_\eta$, we then have that $\partial_x \widetilde{\mathbf{y}} \in L^2_\eta$ from (4.128), and thus $e^{\eta x}\widetilde{\mathbf{y}}$ is bounded.

Let $\mathbf{y}^0 := \widetilde{\mathbf{y}} - \partial^2_\lambda \mathbf{y}^+|_{\lambda=0}$. Then, we see from (4.127) and (4.128) that $\mathbf{y}^0$ satisfies

$$\frac{d\mathbf{y}^0}{dx} = A_1(x)\mathbf{y}^0. \tag{4.129}$$

Since $e^{\eta x}\widetilde{\mathbf{y}}$ is bounded, and $e^{\eta x}\partial^2_\lambda \mathbf{y}^+$ decays to 0 as $x \to +\infty$, we have $\mathbf{y}^0 = O(e^{-\eta x})$ as $x \to +\infty$. Hence, from (4.51), $\mathbf{y}^0 = o(e^{(\mu_* - \theta)x})$ as $x \to +\infty$ for sufficiently small $\theta > 0$, and this implies that there is some constant $\alpha_0 \neq 0$ such that $\mathbf{y}^0 = \alpha_0 \mathbf{y}^+(x, 0)$ by Proposition 4.4.

Now we obtain that

$$\partial^2_\lambda \mathbf{y}^+|_{\lambda=0} = \widetilde{\mathbf{y}} - \alpha_0 \mathbf{y}^+(x, 0) = O(e^{-\eta x}) \quad \text{as } x \to -\infty.$$

This is a contradiction since $\operatorname{Re}\mu_1 < -\eta$ but the order of $\partial^2_\lambda \mathbf{y}^+$ is exactly $e^{\mu_1 x}$ as $x \to -\infty$. (See Proposition 5.46 and Remark 32.)

Remark 7. The similar argument shows that $\partial_c(n_c, u_c)^T$ is the generalized eigenvector of $\mathcal{L}$ in L^2_η space. Let $\mathbf{y}_2 := (\partial_c n_c, \partial_c u_c, \partial_c \phi_c, \partial_c \partial_x \phi)^T$ and $\mathbf{y}_1 := (\partial_x n_c, \partial_x u_c, \partial_x \phi_c, \partial^2_x \phi_c)^T$. From (4.17) and Lemma 4.16, we have that

$$\left(\frac{d}{dx} - A_1\right)\mathbf{y}_2 = -A_2\mathbf{y}_1 = -\alpha_1^{-1} A_2 \mathbf{y}^+|_{\lambda=0},$$

equivalently,

$$\left(\frac{d}{dx} - \eta - A_1\right)(e^{\eta x}\mathbf{y}_2) = -\alpha_1^{-1} A_2 (e^{\eta x}\mathbf{y}^+|_{\lambda=0}).$$

Since the operator $\frac{d}{dx} - \eta - A_1$ is Fredholm with index zero, by the result of [24], the ODE system $(\frac{d}{dx} - \eta - A_1)\mathbf{y} = 0$ has exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$ (but not on $\mathbb{R}$ since $D(0) = 0$, or equivalently, the kernel of $\frac{d}{dx} - \eta - A_1$ in L^2 is non-trivial). Since $e^{\eta x}\mathbf{y}^+(x, 0)$ is in L^2, we have that $e^{\eta x}\mathbf{y}_2 \in H^1$ by the generalized Young's inequality. Hence,

$$\mathbf{y}_2 = O(e^{-\eta x}) \quad \text{as } x \to +\infty.$$

Using (4.127), we have

$$(\tfrac{d}{dx} - A_1)(\mathbf{y}_2 + \alpha_1^{-1}\partial_\lambda \mathbf{y}^+|_{\lambda=0}) = 0.$$

We observe that $\mathbf{y}_2 = o(e^{(\mu_* - \theta)x})$ and $\partial_\lambda \mathbf{y}^+|_{\lambda=0} = o(e^{(\mu_* - \theta)x})$ as $x \to +\infty$ since $-\mu_* - \eta < 0$ and $\mu_1 - \mu_* < 0$. Therefore, we conclude that $\mathbf{y}_2 + \alpha_1^{-1}\partial_\lambda \mathbf{y}^+|_{\lambda=0}$ is a constant multiple of $\mathbf{y}^+(x, 0) = \alpha_1 \mathbf{y}_1$.

Uniform Resolvent Estimate We aim to obtain the uniform bound for the resolvent operator:

$$\sup_{\lambda \in \mathcal{D}_4, \operatorname{Re}\lambda > 0} \|(\lambda - \mathcal{L}^\eta)^{-1}\|_{(L^2)^2} \le M.$$

Then $(\lambda-\mathcal{L}^\eta)^{-1}$ is uniformly bounded on $\operatorname{Re}\lambda > 0$ in L^2 norm, outside any small neighbourhood of the origin since the resolvent is analytic on the resolvent set.

For each $\lambda \in \mathcal{D}_4$, the ODE system $(\frac{d}{dx} - \eta - A(x, \lambda, \varepsilon))\mathbf{y} = 0$ has an exponential dichotomy on $\mathbb{R}$. Indeed,

$$P(\lambda) := \frac{1}{D(\lambda)}\mathbf{y}^+(0,\lambda)\mathbf{z}^-(0,\lambda) \quad \text{and} \quad I - P(\lambda) = \left(\mathbf{y}^-(\mathbf{z}^+\mathbf{y}^-)^{-1}\mathbf{z}^+\right)(0,\lambda)$$

are the projection matrices onto the space of initial conditions at $x = 0$ of solutions to (4.32) satisfying

$$\mathbf{y}(x) = O(e^{\mu_1 x}) \quad \text{as } x \to +\infty \quad \text{and} \quad \mathbf{y}(x) = O(e^{(\mu_* - \theta)x}) \quad \text{as } x \to -\infty,$$

respectively for sufficiently small $\theta > 0$. Let $\Phi(x) = \Phi(x; \lambda, \varepsilon)$ be the fundamental solution of (4.32) satisfying $\Phi(0; \lambda, \varepsilon) = I$. Then $G^\eta(x, x') = G^\eta(x, x'; \lambda, \varepsilon)$ defined by

$$G^\eta(x, x') = \begin{cases} e^{\eta(x-x')}\Phi(x)P\Phi(x')^{-1} & x > x', \\ -e^{\eta(x-x')}\Phi(x)(I-P)\Phi(x')^{-1} & x' > x, \end{cases}$$

satisfies

$$(\partial_x - \eta)G^\eta = A(x, \lambda, \varepsilon)G^\eta \quad \text{for } x \neq x', \quad G^\eta(x'+0, x') - G^\eta(x'-0, x') = I,$$

and $G^\eta(x, x') \to 0$ exponentially fast as $|x| \to \infty$ since $\eta + \operatorname{Re}\mu_1 < 0 < \eta + \operatorname{Re}\mu_*$ on $\mathcal{D}_4$. In particular, the projection of an exponential dichotomy on $\mathbb{R}$ is unique. By applying the generalized Young's inequality, we see that for given $e^{\eta x}(f_1, f_2) \in (L^2)^2$ the L^2 solution $\mathbf{y}^\eta = e^{\eta x}(n, u, \phi, \psi)^T$ to the inhomogeneous ODE system

$$(\partial_x - \eta)\mathbf{y}^\eta = A(x, \lambda, \varepsilon)\mathbf{y}^\eta + \mathbf{f}^\eta, \quad \mathbf{f}^\eta := e^{\eta x}(f_1, f_2, 0, 0)^T,$$

is given by

$$\mathbf{y}^\eta = \int_{-\infty}^{\infty} G^\eta(x, x')\mathbf{f}^\eta(x')\,dx'.$$

We show that there is a constant $C > 0$, independent of $\lambda \in \mathcal{D}_4$ with $\operatorname{Re}\lambda > 0$, such that

$$\sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} |(G^\eta)_{jk}(x, x')|\,dx' + \sup_{x'\in\mathbb{R}} \int_{-\infty}^{\infty} |(G^\eta)_{jk}(x, x')|\,dx < C,$$

for $j, k = 1, 2$, then we would obtain the uniform bound

$$\|(n, u)\|_{L^2_\eta} = \|e^{\eta x}(n, u)\|_{L^2} \leq C\|e^{\eta x}(f_1, f_2)\|_{L^2} = C\|(f_1, f_2)\|_{L^2_\eta}.$$

Step 0: Diagonalization By letting $\widetilde{\mathbf{y}}_\eta := V_0^{-1}e^{\eta x}\mathbf{y}$, we obtain from (4.32) that

$$\begin{aligned} \partial_x\widetilde{\mathbf{y}}_\eta &= \eta W_0V_0\widetilde{\mathbf{y}}_\eta + W_0A^\infty V_0\widetilde{\mathbf{y}}_\eta + W_0(A - A^\infty)V_0\widetilde{\mathbf{y}}_\eta \\ &= [\operatorname{diag}(\eta + \mu_i) + W_0(A - A^\infty)V_0]\,\widetilde{\mathbf{y}}_\eta. \end{aligned} \tag{4.130}$$

The matrix $\widetilde{G^\eta}(x,x') = \widetilde{G^\eta}(x,x';\lambda,\varepsilon) := W_0 G^\eta(x,x')V_0$ satisfies

$$\begin{cases} \partial_x \widetilde{G^\eta} = [\mathrm{diag}(\mu_i+\eta) + W_0(A-A^\infty)V_0]\,\widetilde{G^\eta} & \text{for } x\neq x', \\ \widetilde{G^\eta}(x'+0,x') - \widetilde{G^\eta}(x'-0,x') = I. \end{cases}$$

One may check that by term by term computation using (4.102), (4.103), (4.104), (4.105),

$$|(V_0)_{jl}||(W_0)_{mk}| \le C \quad \text{for } j,k=1,2 \text{ and } l,m=1,2,3,4$$

uniformly in $\lambda \in \mathfrak{D}_4$. Hence, for $j,k=1,2$,

$$|(G^\eta)_{jk}| = |(V_0\widetilde{G^\eta}W_0)_{jk}| \le 64C \sum_{l,m=1,2,3,4} |(\widetilde{G^\eta})_{lm}|.$$

Here we note that

$$\begin{aligned} \widetilde{G^\eta}(x,x') &= W_0 G^\eta(x,x')V_0 \\ &= \begin{cases} \widetilde{\Phi}^\eta(x)P^\eta(\widetilde{\Phi}^\eta)^{-1}(x') & x>x', \\ -\widetilde{\Phi}^\eta(x)(I-P^\eta)(\widetilde{\Phi}^\eta)^{-1}(x') & x'>x, \end{cases} \end{aligned}$$

where $\widetilde{\Phi}^\eta(x) := W_0 e^{\eta x}\Phi(x)V_0$ is the fundamental solution of (4.130) with $\Phi^\eta(0) = I$ and $P^\eta := W_0PV_0 = (P^\eta)^2$ is a projection. In the next step, we show that there exists constants $\varepsilon_0, C_0, \alpha_0 > 0$, independent of λ, such that for all $(\varepsilon,\lambda) \in (0,\varepsilon_0] \times \mathcal{D}_4$, there holds that

$$|\widetilde{\Phi}^\eta(x)P^\eta(\widetilde{\Phi}^\eta)^{-1}(x')| \le C_0 e^{-\alpha_0(x-x')}, \quad x>x', \tag{4.131a}$$

$$|\widetilde{\Phi}^\eta(x)(I-P^\eta)(\widetilde{\Phi}^\eta)^{-1}(x')| \le C_0 e^{-\alpha_0(x'-x)}, \quad x'>x. \tag{4.131b}$$

Step 1: Roughness of Exponential Dichotomy of Simpler Equation We recall that from (4.111) and (4.115),

$$W_0(A-A^\infty)V_0 = \frac{\lambda}{2\sqrt{K}}S_1 + \widetilde{R}^{(1)} + W_0R^{(2)}V_0,$$

and from (4.114) and (4.117),

$$|\widetilde{R}^{(1)} + W_0R^{(2)}V_0| \le C\varepsilon e^{-C\varepsilon^{1/2}|x|}$$

uniformly in $\lambda \in \mathcal{D}_4$. We consider the simpler equation

$$\partial_x \widetilde{\mathbf{y}} = \mathrm{diag}\{\mu_i+\eta\}\widetilde{\mathbf{y}} + \frac{\lambda}{2\sqrt{K}}S_1\widetilde{\mathbf{y}}, \tag{4.132}$$

and show that (4.132) has an exponential dichotomy on $\mathbb{R}$ with uniform constants. Then, by the roughness of exponential dichotomies (Theorem 5.32), we conclude that the system (4.130) has an exponential dichotomy (4.131) on $\mathbb{R}$ with uniform constants.

Consider the fundamental solution $\widetilde{\Psi}(x)$ of (4.132) which satisfies

$$\partial_x\widetilde{\Psi}(x) = \mathrm{diag}\{\mu_i+\eta\}\widetilde{\Psi}(x) + \frac{\lambda}{2\sqrt{K}}S_1\widetilde{\Psi}(x), \quad \widetilde{\Psi}(0) = I.$$

From the form of S_1 (see (4.116)), we see that

$$\widetilde{\Psi}(x) = \operatorname{diag}(e^{(\operatorname{Re}\mu_1+\eta)x}, \widetilde{\Psi}_c(x), e^{(\operatorname{Re}\mu_4+\eta)x}),$$

where

$$\widetilde{\Psi}_c(x) := \begin{pmatrix} \widetilde{\Psi}_{22}(x) & \widetilde{\Psi}_{23}(x) \\ \widetilde{\Psi}_{32}(x) & \widetilde{\Psi}_{33}(x) \end{pmatrix}.$$

Let $\widetilde{P} := e_1 e_1^T$. Then, the Green function of (4.132) is

$$\widetilde{G}(x,x') = \begin{cases} \widetilde{\Psi}(x)P\widetilde{\Psi}^{-1}(x') & \text{for} \quad x > x', \\ -\widetilde{\Psi}(x)(I-P)\widetilde{\Psi}^{-1}(x') & \text{for} \quad x' > x, \end{cases}$$

provided that (4.132) has an exponential dichotomy. It is easy to see that

$$\widetilde{G}(x,x') = \begin{cases} \operatorname{diag}(e^{(\operatorname{Re}\mu_1+\eta)(x-x')}, \widetilde{G}_c(x,x'), 0) & \text{for } \ x > x', \\ \operatorname{diag}(0, \widetilde{G}_c(x,x'), e^{(\operatorname{Re}\mu_4+\eta)(x-x')}) & \text{for } \ x' > x, \end{cases}$$

where

$$\widetilde{G}_c(x,x') := \begin{cases} 0 & \text{for } \ x > x', \\ \widetilde{\Psi}_c(x)\widetilde{\Psi}_c^{-1}(x') & \text{for } \ x' > x. \end{cases}$$

Here $\widetilde{G}_c(x,x')$ satisfies for $x < x'$,

$$\partial_x \widetilde{G}_c(x;) = \operatorname{diag}(\mu_2+\eta, \mu_3+\eta)\widetilde{G}_c(x;) + \frac{\lambda}{2\sqrt{K}}\widetilde{S}_1\widetilde{G}_c(x;), \quad \widetilde{G}_c(x'-0,x') = I, \tag{4.133}$$

where $\widetilde{S}_1$, defined in (4.123), is the nonzero 2×2 submatrix of S_1. By taking the Frobenius inner product[24], we have that

$$\frac{1}{2}\partial_x\langle \widetilde{G}_c, \widetilde{G}_c\rangle_F \geq \min_{j=2,3}\{\operatorname{Re}\mu_j + \eta\}\langle \widetilde{G}_c, \widetilde{G}_c\rangle_F + \frac{\operatorname{Re}\lambda}{2\sqrt{K}}\langle \widetilde{S}_1\widetilde{G}_c, \widetilde{G}_c\rangle_F.$$

Since $\operatorname{Re}\mu_j \geq 0$ $(j = 2,3,4)$ when $\operatorname{Re}\lambda \geq 0$, and $\widetilde{S}_1$ is nonnegative, we obtain

$$\partial_x\langle \widetilde{G}_c, \widetilde{G}_c\rangle_F \geq 2\eta\langle \widetilde{G}_c, \widetilde{G}_c\rangle_F. \tag{4.134}$$

Multiplying (4.134) by $e^{-2\eta x}$ and then integrating the resultant in x argument over $[x,x']$, the jump condition in (4.133) yields that for $x' > x$,

$$\langle \widetilde{G}_c, \widetilde{G}_c\rangle_F(x,x') \leq e^{2\eta(x-x')}.$$

Since $\operatorname{Re}\mu_4 > 0$ on $\operatorname{Re}\lambda \geq 0$ and $\operatorname{Re}\mu_1 + \eta < -1/2$ for all sufficiently large $|\lambda|$ and small ε, the system (4.132) possesses an exponential dichotomy on $\mathbb{R}$ with constants uniform in λ.

[24] $\langle A,B\rangle_F := \operatorname{tr}(\overline{B^T}A) = \sum_{1\leq i,j\leq n}\overline{B_{ij}}A_{ij}$ for $A, B \in \mathbb{C}^{n\times n}$. It is easy to check that $2\operatorname{Re}\langle \partial_x A, A\rangle_F = \partial_x\langle A,A\rangle_F$ since $\langle A,B\rangle_F = \overline{\langle B,A\rangle_F}$.

4.7 Appendix

4.7.1 Specific form of $A_*(\xi,\Lambda,\varepsilon)$

Using the simplified form of $A_1(x,\varepsilon)$ (see (4.82)),

$$A\left(\frac{\xi}{\sqrt{\varepsilon}},\varepsilon^{3/2}\Lambda,\varepsilon\right)=\begin{pmatrix}\dfrac{\varepsilon^{1/2}n_*'}{1+n_*} & \dfrac{2\varepsilon^{1/2}u_*'(1+n_*)}{J} & 0 & \dfrac{1+n_*}{J}\\ 0 & \dfrac{\varepsilon^{1/2}(Kn_*'+cu_*')}{J(1+n_*)} & 0 & \dfrac{c-u_*}{J}\\ 0 & 0 & 0 & 1\\ -1 & 0 & e^{\phi_*} & 0\end{pmatrix}$$

$$+\frac{\varepsilon^{3/2}\Lambda}{J}\begin{pmatrix}c-u_* & 1+n_* & 0 & 0\\ \dfrac{K}{1+n_*} & c-u_* & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Let a_{ij} be the entry of $A\left(\frac{\xi}{\sqrt{\varepsilon}},\varepsilon^{3/2}\Lambda,\varepsilon\right)$. Then,

$$A_*(\xi,\Lambda,\varepsilon)=\begin{pmatrix}\dfrac{a_{22}-\mathsf{V}a_{12}}{\sqrt{\varepsilon}} & \dfrac{a_{21}+\mathsf{V}a_{22}-\mathsf{V}a_{11}-\mathsf{V}^2a_{12}}{\varepsilon^{3/2}} & \dfrac{a_{24}-\mathsf{V}a_{14}}{\varepsilon} & 0\\ \sqrt{\varepsilon}\,a_{12} & \dfrac{a_{11}+\mathsf{V}a_{12}}{\sqrt{\varepsilon}} & a_{14} & 0\\ 0 & \dfrac{e^{\phi_*}-1}{\varepsilon} & 0 & e^{\phi_*}\\ \dfrac{-a_{12}}{\sqrt{\varepsilon}} & -\dfrac{a_{11}+\mathsf{V}a_{12}}{\varepsilon^{3/2}} & \dfrac{1-a_{14}}{\varepsilon} & 0\end{pmatrix}. \tag{4.135}$$

4.7.2 Proof of Proposition 4.3

Proof of Proposition 4.3. It is enough to prove that for $k\in\mathbb{Z}$, $\lambda I-\mathcal{L}$ is Fredholm with index k if and only if $\mathcal{A}(\lambda)$ is Fredholm with index k. For a closed subspace $\mathcal{R}(\lambda-\mathcal{L})$ of a Hilbert space $\mathcal{H}$, we have

$$\mathcal{H}=\mathcal{R}(\lambda-\mathcal{L})\oplus\mathcal{R}(\lambda-\mathcal{L})^{\perp}=\mathcal{R}(\lambda-\mathcal{L})\oplus\mathcal{N}\left(\overline{\lambda}-\mathcal{L}^*\right)$$

and a similar decomposition holds for a closed subspace $\mathcal{R}(\mathcal{A}(\lambda))$. Hence we claim that

C1. $\mathcal{R}(\lambda-\mathcal{L})$ is closed if and only if $\mathcal{R}(\mathcal{A}(\lambda))$ is closed;

C2. $\mathcal{N}(\lambda-\mathcal{L})$ is isomorphic to $\mathcal{N}(\mathcal{A}(\lambda))$;

C3. $\mathcal{N}(\overline{\lambda}-\mathcal{L}^*)$ is isomorphic to $\mathcal{N}(\mathcal{A}^*(\lambda))$. ($^*:=$ Hermitian adjoint)

We see that C1-C3 imply not only (a), but (b) and (c). To check C1–C3, we recall from (4.27) and (4.28) that the first two rows of $\mathcal{A}(\lambda)=\partial_x-A(\lambda)$ is nothing but $(L_1)^{-1}(\lambda I-\mathcal{L})$.

We only prove the right direction of C1 since the converse is easier to check. We suppose that $\mathcal{R}(\lambda-\mathcal{L})$ is closed and consider a sequence $\mathbf{f}_i=(f_i^1,f_i^2,f_i^3,f_i^4)^T\in\mathcal{R}(\mathcal{A}(\lambda))$ such that

$\mathbf{f}_i \to \mathbf{f} = (f^1, f^2, f^3, f^4)^T \in (L^2)^4$ as $i \to \infty$. Let $\mathbf{y}_i$ be a solution of $\mathcal{A}(\lambda)\mathbf{y}_i = \mathbf{f}_i$ for each i. We may decompose the last two components of $\mathcal{A}(\lambda)\mathbf{y}_i = \mathbf{f}_i$ (corresponding to the Poisson equation (4.28)) into two parts:

$$\begin{cases} \partial_x \phi_i^f - \psi_i^f = f_i^3, \\ \partial_x \psi_i^f - e^{\phi_c}\phi_i^f = f_i^4, \end{cases} \qquad \begin{cases} \partial_x(\phi_i - \phi_i^f) - (\psi_i - \psi_i^f) = 0, \\ \partial_x(\psi_i - \psi_i^f) - e^{\phi_c}(\phi_i - \phi_i^f) + n_i = 0, \end{cases} \tag{4.136}$$

where $n_i \in H^1$ is the first component of $\mathbf{y}_i$. Indeed, since $\sup_{x\in\mathbb{R}} |\phi_c| \to 0$ as $\varepsilon \to 0$, by the roughness of exponential dichotomy of the linear ODE systems (Theorem 5.32) and Remark 18, the solution $(\phi_i^f, \psi_i^f) \in (H^1)^2$ to the LHS of (4.136) exists for all $(f_i^3, f_i^4) \in (L^2)^2$. For the same reason, the solution $(\phi^f, \psi^f) \in (H^1)^2$ to the LHS of (4.136) exists for $(f^3, f^4) \in (L^2)^2$. By using the generalized Young's inequality, we have that $(\phi_i^f, \psi_i^f) \to (\phi^f, \psi^f)$ in $(L^2)^2$. By letting

$$\widetilde{\mathbf{y}}_i := (n_i,\ u_i,\ \phi_i - \phi_i^f,\ \psi_i - \psi_i^f)^T,$$

we have $\mathcal{A}(\lambda)\widetilde{\mathbf{y}}_i = \widetilde{\mathbf{f}}_i := (f_i^1 + \frac{1+n_c}{J}\psi_i^f,\ f_i^2 + \frac{c-u_c}{J}\psi_i^f,\ 0,\ 0)^T$, equivalently,

$$(\lambda - \mathcal{L})(n_i, u_i)^T = L_1(f_i^1 + \frac{1+n_c}{J}\psi_i^f,\ f_i^2 + \frac{c-u_c}{J}\psi_i^f)^T.$$

Since $\mathcal{R}(\lambda - \mathcal{L})$ is closed, there is $(n, u) \in (H^1)^2$ such that

$$(\lambda - \mathcal{L})(n, u)^T = L_1(f^1 + \frac{1+n_c}{J}\psi^f,\ f^2 + \frac{c-u_c}{J}\psi^f)^T,$$

equivalently,

$$\mathcal{A}(\lambda)\widetilde{\mathbf{y}} = (f^1 + \frac{1+n_c}{J}\psi^f,\ f^2 + \frac{c-u_c}{J}\psi^f,\ 0,\ 0)^T,$$

where $\widetilde{\mathbf{y}} = (n, u, \widetilde{\phi}, \widetilde{\psi})^T$ and

$$\widetilde{\phi}_x = \widetilde{\psi}, \quad \widetilde{\psi}_x = e^{\phi_c}\widetilde{\phi} - n$$

By adding

$$\begin{cases} \partial_x \phi^f - \psi^f = f^3, \\ \partial_x \psi^f - e^{\phi_c}\phi^f = f^4, \end{cases}$$

we have

$$\begin{cases} \partial_x(\widetilde{\phi} + \phi^f) - (\widetilde{\psi} + \psi^f) = f^3, \\ \partial_x(\widetilde{\psi} + \psi^f) - e^{\phi_c}(\widetilde{\psi} + \phi^f) + n = f^4, \end{cases}$$

Hence, we have $\mathcal{A}(\lambda)\mathbf{y} = \mathbf{f}$, where $\mathbf{y} := (n,\ u,\ \widetilde{\phi} + \phi^f,\ \widetilde{\psi} + \psi^f)^T$, and conclude that $\mathcal{R}(\mathcal{A}(\lambda))$ is closed.

Since (ϕ, ψ) is determined by n through the linear Poisson equation, it is clear that the projection mapping

$$(n, u, \phi, \psi) \mapsto (n, u)$$

is an isomorphism between $\mathcal{N}(\partial_x - A)$ and $\mathcal{N}(\lambda - \mathcal{L})$, which proves C2.[25]

[25]We note that $(n, u, \phi, \psi)^T \in (H^1)^2 \times H^3 \times H^2$ as long as $(n, u, \phi, \psi)^T \in \mathcal{N}(\partial_x - A) \subset (H^1)^4$ due to the Poisson equation.

To prove C2, we observe that the adjoint operator of $(\lambda I - \mathcal{L})$ in the standard L^2 inner product is given by

$$(\lambda I - \mathcal{L})^*(\widetilde{n}, \widetilde{u})^T = (\overline{\lambda} - \partial_x L_1^T + L_2^T)(\widetilde{n}, \widetilde{u})^T + \Big((-\partial_x^2 + e^{\phi_c})^{-1}(-\partial_x \widetilde{u}), 0\Big)^T.$$

Thus,

$$\begin{aligned}(L_1^{-1}(\lambda I - \mathcal{L}))^*(\widetilde{n}, \widetilde{u})^T &= (\lambda I - \mathcal{L})^* \left[(L_1^{-1})^T(\widetilde{n}, \widetilde{u})^T\right] \\ &= (\overline{\lambda}(L_1^{-1})^T - \partial_x + L_2^T(L_1^{-1})^T)(\widetilde{n}, \widetilde{u})^T \\ &+ \begin{pmatrix} (-\partial_x^2 + e^{\phi_c})^{-1}(-\partial_x)\left(-\dfrac{1}{J}\left[(1+n_c)\widetilde{n} + (c-u_c)\widetilde{u}\right]\right) \\ 0 \end{pmatrix}.\end{aligned}$$

$$\begin{aligned}(\partial_x - A)^* &= -\partial_x - A^* \\ &= -\partial_x - \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$-\overline{\lambda} I_2 - L_2^T$}} & 0 & -1 \\ & & 0 & 0 \\ \hline 0 & 0 & 0 & e^{\phi_c} \\ 0 & -1 & 1 & 0 \end{array}\right] \begin{pmatrix} (L_1^{-1})^T & 0_2 \\ 0_2 & I_2 \end{pmatrix} \\ &= -\partial_x + \left[\begin{array}{c|cc} \multirow{2}{*}{$\overline{\lambda}(L_1^{-1})^T + L_2^T(L_1^{-1})^T$} & 0 & 1 \\ & 0 & 0 \\ \hline -\dfrac{1}{J}\begin{pmatrix} 0 & 0 \\ 1+n_c & c-u_c \end{pmatrix} & \begin{matrix} 0 \\ -1 \end{matrix} & \begin{matrix} -e^{\phi_c} \\ 0 \end{matrix} \end{array}\right].\end{aligned}$$

If $(\tilde{n}, \tilde{u}, \tilde{\phi}, \tilde{\psi})^T \in \mathrm{Ker}(\partial_x - A)^*$, then

$$-\partial_x \tilde{\phi} - e^{\phi_c}\tilde{\psi} = 0, \quad -\partial_x \tilde{\psi} - \frac{1}{J}((1+n_c)\tilde{n} + (c-u_c)\tilde{u}) - \tilde{\phi} = 0.$$

Taking ∂_x to the second equation and using the first equation, we have

$$(-\partial_x^2 + e^{\phi_c})\tilde{\psi} = \partial_x \left[\frac{1}{J}((1+n_c)\tilde{n} + (c-u_c)\tilde{u})\right].$$

Thus,

$$(\tilde{n}, \tilde{u}, \tilde{\phi}, \tilde{\psi})^T \to (L_1^{-1})^T(\tilde{n}, \tilde{u})^T, \quad (\tilde{n}, \tilde{u})^T \to (((L_1)^T(\tilde{n}, \tilde{u})^T)^T, \tilde{\phi}, \tilde{\psi})^T$$

gives an isomorphism between $\mathrm{Ker}(\partial_x - A)^*$ and $\mathrm{Ker}(\lambda I - \mathcal{L})^*$.

□

4.7.3 Proof of Lemma 4.30

Proof of Lemma 4.30. Since the matrix S_1 defined in (4.116) is symmetric, it is enough to show that the eigenvalues of S_1,

$$0, 0, 2\sqrt{K}R_{11}^{(1)} \pm \sqrt{2K^2(R_{12}^{(1)})^2 + 2(R_{21}^{(1)})^2},$$

are non-negative. Here $R_{11}^{(1)}$ is positive since $u_c(x)>0$ for all $x\in\mathbb{R}$ and

$$R_{11}^{(1)} = u_c\left(\frac{1}{c^2-K}u_c + O(|u_c|)\right)$$

for sufficiently small $\varepsilon>0$. We check that $2\sqrt{K}R_{11}^{(1)} - \sqrt{2K^2(R_{12}^{(1)})^2+2(R_{21}^{(1)})^2}$ is positive. Since $R_{11}^{(1)}>0$, it is enough to show that

$$\begin{aligned}
&4K(R_{11}^{(1)})^2 - 2K^2(R_{12}^{(1)})^2 - 2(R_{21}^{(1)})^2 \\
&= \frac{2K}{J_1}\Bigg[\underbrace{2\left((c-u_c)(c^2-K)-cJ\right)^2(1+n_c)^2}_{=:I_1}\underbrace{-K\left((c^2-K)(1+n_c)-J\right)^2(1+n_c)^2}_{=:I_2} \\
&\underbrace{-K\left(c^2-K-(1+n_c)J\right)^2}_{=:I_3}\Bigg],
\end{aligned}$$

is positive, where $J_1 := J^2(c^2-K)^2(1+n_c)^2>0$. Using the solitary wave identity (4.81a) and the definition of J (4.26), we have

$$\begin{aligned}
I_1 &= 2\left((c-u_c)(c^2-K) - c\left((c-u_c)^2-K\right)\right)^2(1+n_c)^2 \\
&= 2\left(c(c^2-K) - c\left(c(c-u_c)-K(1+n_c)\right)\right)^2 \\
&= 2c^2(cu_c+Kn_c)^2,
\end{aligned}$$

$$\begin{aligned}
I_2 &= -K\Big((c^2-K)(1+n_c) - \left((c-u_c)^2-K\right)\Big)^2(1+n_c)^2 \\
&= -K\Big((c^2-K)(1+n_c)^2 - \left(c(c-u_c)-K(1+n_c)\right)\Big)^2 \\
&= -K\left(n_c(c^2-K)(2+n_c) + (cu_c+Kn_c)\right)^2 \\
&= -K\left(cu_c+Kn_c\right)^2 - Kn_c^2(c^2-K)^2(2+n_c)^2 - 2Kn_c(c^2-K)(2+n_c)\left(cu_c+Kn_c\right) \\
&= -K\left(cu_c+Kn_c\right)^2 - 4Kn_c^2(c^2-K)^2 - 4Kn_c(c^2-K)\left(cu_c+Kn_c\right) \\
&\quad + O\big(|n_c|^3+|n_c|^2|u_c|\big),
\end{aligned}$$

$$\begin{aligned}
I_3 &= -K\left(c^2-K-(1+n_c)\left((c-u_c)^2-K\right)\right)^2 \\
&= -K\left(cu_c+Kn_c\right)^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
I_1+I_2+I_3 &= 2(c^2-K)(cu_c+Kn_c)^2 - 4Kn_c^2(c^2-K)^2 - 4Kn_c(c^2-K)\left(cu_c+Kn_c\right) \\
&\quad + O\big(|n_c|^3+|n_c|^2|u_c|\big) \\
&= 2(c^2-K)(cu_c+Kn_c)\left(cu_c-Kn_c\right) - 4Kn_c^2(c^2-K)^2 \\
&\quad + O\big(|n_c|^3+|n_c|^2|u_c|\big) \\
&= 2(c^2-K)(c^2u_c^2+K^2n_c^2-2Kc^2n_c^2) + O\big(|n_c|^3+|n_c|^2|u_c|\big).
\end{aligned}$$

Since $\dfrac{cn_c}{1+n_c} = u_c$ from (4.81a), we obtain that

$$c^2u_c^2 + K^2n_c^2 - 2Kc^2n_c^2 = (c^2-K)^2n_c^2\big(1+O(|n_c|)\big).$$

Therefore, we conclude that $I_1 + I_2 + I_3 > 0$ for all sufficiently small $\varepsilon > 0$. □

4.7.4 Proof of Proposition 4.29

Proof of Proposition 4.29. We set $\widetilde{\mathcal{P}}(\mu;\lambda) = (\mu^2-1)((\lambda-c\mu)^2 - K\mu^2+1)$ and $\widetilde{\mathcal{L}}(\mu) = 1$. Then, $(c^2-K)d(\mu) = \widetilde{\mathcal{P}}(\mu) + \widetilde{\mathcal{L}}(\mu)$. For all λ with sufficiently large $|\lambda|$, $\widetilde{\mathcal{P}}(\mu)$ has four simple zeros

$$\widetilde{\mu}_1 = -1, \quad \widetilde{\mu}_4 = 1, \quad \widetilde{\mu}_2 = \frac{c\lambda - \sqrt{-c^2+K+K\lambda^2}}{c^2-K}, \quad \widetilde{\mu}_3 = \frac{c\lambda + \sqrt{-c^2+K+K\lambda^2}}{c^2-K}.$$

In particular, it is clear that for each pair of $\widetilde{\mu}_j$, there is a positive lower bound, uniform in large $|\lambda|$, for the distance between them. Since the derivative of $\widetilde{\mathcal{P}}(\mu)$ in μ is

$$\widetilde{\mathcal{P}}'(\mu) = 2\mu((\lambda-c\mu)^2 - K\mu^2+1) + 2(\mu^2-1)(-c\lambda + (c^2-K)\mu),$$

we obtain

$$\begin{aligned}\widetilde{\mathcal{P}}'(\widetilde{\mu}_1) &= -2((\lambda+c)^2 - K+1), \quad \widetilde{\mathcal{P}}'(\widetilde{\mu}_4) = 2((\lambda-c)^2-K+1),\\ \widetilde{\mathcal{P}}'(\widetilde{\mu}_2) &= -2(\widetilde{\mu}_2^2-1)\sqrt{-c^2+K+K\lambda^2}, \quad \widetilde{\mathcal{P}}'(\widetilde{\mu}_3) = 2(\widetilde{\mu}_3^2-1)\sqrt{-c^2+K+K\lambda^2}.\end{aligned}$$

Thus, we may take some constant $\rho_0 > 1$, independent of ε and λ, and positive functions $\rho_j(\lambda)$ such that as $|\lambda| \to \infty$,

$$\rho_j(\lambda) = O(|\lambda|^{-2}) \ \text{ for } j=1,4, \quad \rho_j(\lambda) = O(|\lambda|^{-3}) \ \text{ for } j=2,3, \tag{4.137}$$

and for all λ with sufficiently large $|\lambda|$,

$$\rho_j(\lambda) > \rho_0 \frac{1}{|\widetilde{\mathcal{P}}'(\widetilde{\mu}_j)|} \quad \text{for } j=1,2,3,4. \tag{4.138}$$

Moreover, ρ_j can be taken so that (4.137) and (4.138) hold uniformly in $\varepsilon \in [0,\varepsilon_0]$ for some sufficiently small ε_0 since $c = \sqrt{1+K}+\varepsilon$. From (4.137)–(4.138) and the Taylor theorem, we have that on the circle $|\mu - \widetilde{\mu}_j| = \rho_j$,

$$\begin{aligned}|\widetilde{P}(\mu)| &= |\widetilde{P}'(\widetilde{\mu}_j)||\mu-\widetilde{\mu}_j||1+O(|\mu-\widetilde{\mu}_j|)|\\ &= \rho_j|\widetilde{P}'(\widetilde{\mu}_j)||1+O(|\mu-\widetilde{\mu}_j|)|\\ &> \rho_0|1+O(|\mu-\widetilde{\mu}|)|\\ &> 1 = |\widetilde{\mathcal{L}}(\mu)|\end{aligned}$$

for all λ with sufficiently large $|\lambda|$. Now Rouché's theorem implies that there is exactly one simple root μ_j of $d(\mu)$ such that $|\mu_j - \widetilde{\mu}_j| < \rho_j$. The proof is finished from (4.137). □

4.7.5 Proof of Proposition 4.27

Proof of Proposition 4.27. By expanding $d_-(\mu)$ near $\mu = 0$, (4.43b) is equivalent to

$$\frac{\mu^3}{2\sqrt{1+K}} + \lambda = (c - \sqrt{1+K})\mu + \mu^5\mathcal{R}(\mu) = \varepsilon\mu + \mu^5\mathcal{R}(\mu), \tag{4.139}$$

where $\mathcal{R}(\mu)$ is analytic near $\mu = 0$ and $\mathcal{R}(\mu) = O(1)$ as $|\mu|$ tends to 0. The RHS of (4.139) is *presumably* negligible for small ε and μ. We let $\widetilde{\mu}_j = (-2\sqrt{1+K}\lambda)^{1/3}e^{2\pi i j/3}$ for $j = 1, 2, 3$, and then plug the Ansatz $\mu_j = \widetilde{\mu}_j(1-\beta_j)^{1/3}$ into (4.139). Then we obtain

$$\beta_j = \frac{\varepsilon\widetilde{\mu}_j}{\lambda}(1-\beta_j)^{1/3} + \frac{\widetilde{\mu}_j^5(1-\beta_j)^{5/3}}{\lambda}\mathcal{R}\left(\widetilde{\mu}_j(1-\beta_j)^{1/3}\right)$$

Since $\dfrac{\varepsilon\widetilde{\mu}_j}{\lambda} = O(\varepsilon|\lambda|^{-2/3})$ and $\dfrac{\widetilde{\mu}_j^5(1-\beta)^{5/3}}{\lambda} = O(|\lambda|^{2/3})$, employing the fixed point argument, β_j exists and $\beta_j = O(\varepsilon|\lambda|^{-2/3} + |\lambda|^{2/3}) = o(1)$ as $\delta \to 0$.

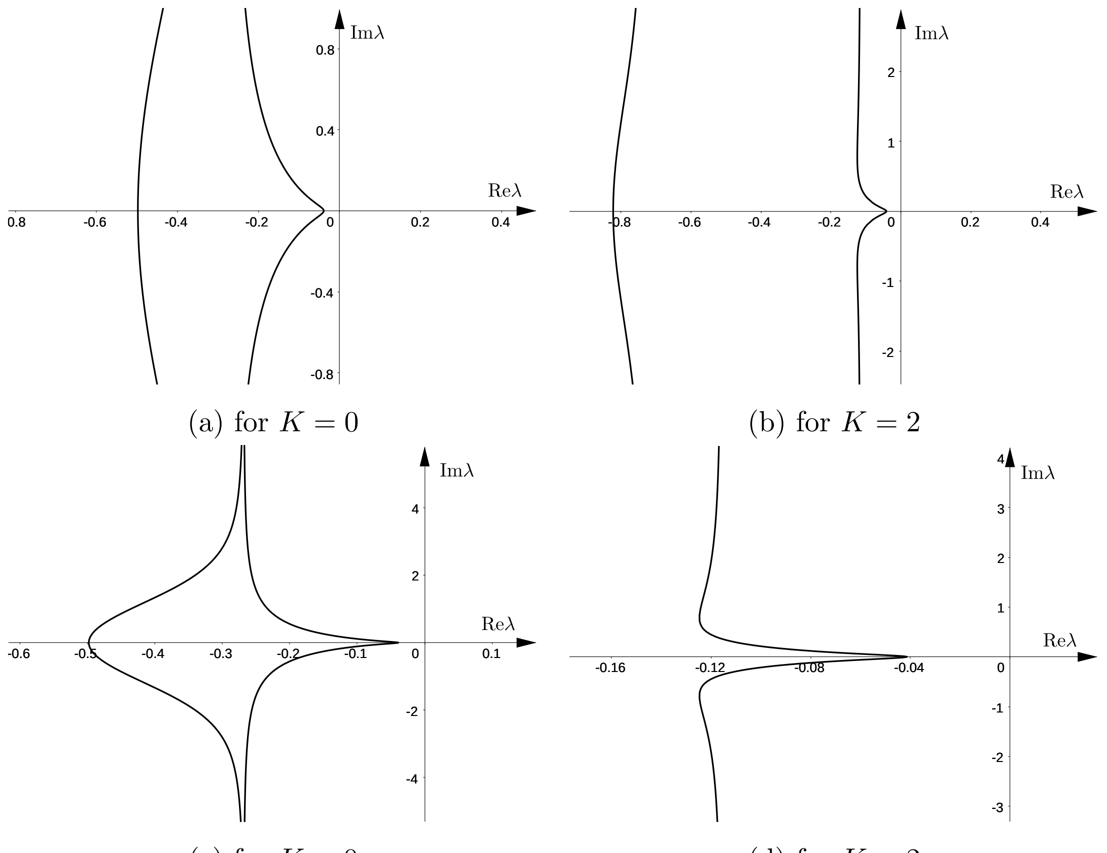
We prove the second assertion. we note that at $\lambda = 0$, $d_+(\mu) = \lambda$ has a (unique) solution $\mu_4 = 0$. Hence, for small $|\lambda|$, $|\mu_4|$ is small, thus, from the form of $d_+(\mu)$, we have $\mu_4 = O(\lambda)$. By expanding d_+,

$$\lambda = (c + \sqrt{1+K})\mu + \frac{\mu^3}{2\sqrt{1+K}} + \mu^5\mathcal{R}(\mu)$$

if and only if

$$\mu = \frac{\lambda}{c+\sqrt{1+K}} - \frac{\mu^3}{c+\sqrt{1+K}}\left(\frac{1}{2\sqrt{1+K}} + \mu^2\mathcal{R}(\mu).\right)$$

□



(a) for $K = 0$ (b) for $K = 2$

(c) for $K = 0$ (d) for $K = 2$

Figure 6: The images of curves $d_{\pm}(ik - \mu)$ for $c_0 = 0.5$, $\varepsilon = 0.2$, $\mu = c_0\varepsilon^{1/2}$ in different scales.

5 Linear Stability and Instability of Nonlinear Waves

This section concerns linear (in)stability of nonlinear waves.

Notation: We denote the kernel of a linear (bounded or unbounded) operator $\mathcal{L}$ by $\mathcal{N}(\mathcal{L})$, and the range of $\mathcal{L}$ by $\mathcal{R}(\mathcal{L})$.

5.1 Linear Systems of ODEs with Constant Matrices

We consider the linear systems of ordinary differential equations

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}, \tag{5.1}$$

where $A \in \mathbb{C}^{n\times n}$ is a constant matrix. We review some basic results of linear algebra on finite-dimensional vector spaces and asymptotic behaviors of solutions of the system (5.1).

For a constant matrix $A \in \mathbb{C}^{n\times n}$, we define

$$e^{At} := \sum_{n=0}^{\infty} \frac{(At)^n}{n!}, \quad x \in \mathbb{R} \tag{5.2}$$

The summation (5.2) is absolutly convergent on $\mathbb{R}$, and it has the following properties.

Lemma 5.1. *For all constant matrices $A, B \in \mathbb{C}^{n\times n}$, and $t, s \in \mathbb{R}$, the following statements hold.*

1. $\frac{d}{dt}e^{At} = Ae^{At}$.
2. $(e^{At})^{-1} = e^{-At}$.
3. $e^{A(t+s)} = e^{At}e^{As} = e^{As}e^{At}$.
4. $e^{(A+B)t} = e^{At}e^{Bt} = e^{Bt}e^{At}$ *if* $AB = BA$.
5. $e^{BAB^{-1}t} = Be^{At}B^{-1}$ *for an invertible matrix* B.
6. e^{At} *is a matrix-valued solution of* (5.1) *satisfying* $e^{At}|_{t=0} = I_n$.

The spectral information of A gives the asymptotic behaviors of e^{At}.

Definition 5.1. Let $A \in \mathbb{C}^{n\times n}$ be a constant matrix. μ is called an *eigenvalue* of A if $\det(A - \mu I_n) = 0$. The set of all eigenvalues of A is called the *spectrum of* A, denoted by $\sigma(A)$. Any non-zero vector of $\mathcal{N}(A-\mu I_n)$ is called an *eigenvector of* A corresponding to μ. The order of μ as a zero of the characteristic polynomial is called the *algebraic multiplicity of* μ, denoted by $m_a(\mu)$. The dimension of $\mathcal{N}(A-\mu I_n)$ is called the *geometric multiplicity of* μ, denoted by $m_g(\mu)$. We say that $\mu \in \sigma(A)$ is *simple* if $m_a(\mu) = m_g(\mu) = 1$, and *semi-simple* if $m_a(\mu) = m_g(\mu)$.

In general, we have $m_a(\mu) \geq m_g(\mu)$. When $m_a(\mu) > m_g(\mu) = 1$, for an eigenvector $\mathbf{v}_1$ of A, there is $\mathbf{v}_2 \in \mathcal{N}(A-\mu I_n)^2 \setminus \mathcal{N}(A-\mu I_n)$ such that $(A-\mu I_n)\mathbf{v}_2 = \mathbf{v}_1$.

Definition 5.2. For $\mu \in \sigma(A)$ and $\mathbf{v}_1 \in \mathcal{N}(A-\mu I_n)$, a set of vectors $\{\mathbf{v}_j\}_{j=1}^{k}$ is called the *Jordan chain* (of full length k) generated by $\mathbf{v}_k$, if we can choose $\mathbf{v}_j$ satisfying $(A-\mu I_n)\mathbf{v}_j = \mathbf{v}_{j-1}$ for $j = 2, \cdots, k$, but there is no vector $\mathbf{v}$ satisfying $(A-\mu I_n)\mathbf{v} = \mathbf{v}_k$. For $\{\mathbf{v}_1^l\}_{l=1}^r$, a basis of $\mathcal{N}(A-\mu I_n)$, let $\{\mathbf{v}_j^l\}_{1\le j\le k_l, 1\le l\le r}$ be the set of all elements of the Jordan chains (of full length k_l) generated by $\mathbf{v}_{k_l}^l$. The subspace

$$\mathbb{E}_\mu := \text{the linear span of } \{\mathbf{v}_j^l\}_{1\le j\le k_l, 1\le l\le r}$$

is called the *generalized eigenspace* of A corresponding to μ. The vectors $\mathbf{v} \in \mathbb{E}_\mu \setminus \mathcal{N}(A-\mu I_n)$ are called the *generalized eigenvectors* of A corresponding to μ.

The set $\{\mathbf{v}_j^l\}_{1\le j\le k_l, 1\le l\le r}$ is a linearly independent set of vectors. Moreover, the dimension of $\mathbb{E}_\mu$, $\sum_{l=1}^r k_l$, coincides with $m_a(\mu)$.

Definition 5.3. For a matrix $A \in \mathbb{C}^{n\times n}$, we decompose the spectrum $\sigma(A)$ as follows:

$$\begin{aligned}\sigma^s(A) &:= \{\mu \in \sigma(A) : \operatorname{Re}\mu < 0\},\\ \sigma^c(A) &:= \{\mu \in \sigma(A) : \operatorname{Re}\mu = 0\},\\ \sigma^u(A) &:= \{\mu \in \sigma(A) : \operatorname{Re}\mu > 0\}.\end{aligned}$$

We call $\sigma^s(A)$ the *stable spectrum*, $\sigma^c(A)$ the *center spectrum*, and $\sigma^u(A)$ the *unstable spectrum*. Let

$$\begin{aligned}\mathbb{E}^s &:= \oplus\{\mathbb{E}_{\mu_j} : \mu_j \in \sigma^s(A)\},\\ \mathbb{E}^c &:= \oplus\{\mathbb{E}_{\mu_j} : \mu_j \in \sigma^c(A)\},\\ \mathbb{E}^u &:= \oplus\{\mathbb{E}_{\mu_j} : \mu_j \in \sigma^u(A)\}.\end{aligned}$$

We call $\mathbb{E}^s$ the *stable eigenspace*, $\mathbb{E}^c$ the *center eigenspace*, and $\mathbb{E}^u$ the *unstable eigenspace*.

We define the operator

$$P^s := \frac{1}{2\pi i}\int_C (\lambda I - A)^{-1} d\lambda, \tag{5.3}$$

where C is a simple closed curve lies in the open left half-plane and contains the stable spectrum $\sigma^s(A)$ in its interior. P^s satisfies $P^s = P^sP^s$ and the range of P^s is $\mathbb{E}^s$. The operator P^s is called the spectral projection onto the stable eigenspace $\mathbb{E}^s$. The spectral projections P^c and P^u onto the center eigenspace $\mathbb{E}^c$ and the unstable eigenspace $\mathbb{E}^u$ are defined in a similar way.

Theorem 5.2. *For a matrix $A \in \mathbb{C}^{n\times n}$, the following properties hold.*

1. $\mathbb{C}^n = \mathbb{E}^s \oplus \mathbb{E}^c \oplus \mathbb{E}^u$.
2. $\dim \mathbb{E}^s = \sum_{\mu\in\sigma^s(A)} m_a(\mu)$, $\dim \mathbb{E}^c = \sum_{\mu\in\sigma^c(A)} m_a(\mu)$, $\dim \mathbb{E}^u = \sum_{\mu\in\sigma^u(A)} m_a(\mu)$.
3. $AP^{s,c,u} = P^{s,c,u}A$, *respectively. In other words, A commutes with the spectral projections.*
4. $e^{At}\mathbb{E}^{s,c,u} \subset \mathbb{E}^{s,c,u}$, *respectively. In other words, $\mathbb{E}^s$, $\mathbb{E}^c$, and $\mathbb{E}^u$ are invariant subspaces under the multiplication by e^{At}.*

Since any vector $\mathbf{y}_0 \in \mathbb{C}^n$ can be decomposed as

$$\mathbf{y}_0 = P^s\mathbf{y}_0 + P^c\mathbf{y}_0 + P^u\mathbf{y}_0,$$

the solution of (5.1) with the initial value $\mathbf{y}_0$ has the form

$$e^{At}\mathbf{y}_0 = e^{At}P^s\mathbf{y}_0 + e^{At}P^c\mathbf{y}_0 + e^{At}P^u\mathbf{y}_0.$$

The asymptotic behavior of the solution is obtained from the following proposition. We let

$$\sigma_M^s := \max\{\operatorname{Re}\mu : \mu \in \sigma^s(A)\}, \qquad \sigma_M^u := \max\{\operatorname{Re}\mu : \mu \in \sigma^u(A)\},$$
$$\sigma_m^s := \min\{\operatorname{Re}\mu : \mu \in \sigma^s(A)\}, \qquad \sigma_m^u := \min\{\operatorname{Re}\mu : \mu \in \sigma^u(A)\}.$$

Proposition 5.3. *For a matrix $A \in \mathbb{C}^{n\times n}$, the following hold.*

1. *For all sufficiently small $\varepsilon > 0$, there exist constants $M(\varepsilon), m(\varepsilon) > 0$ such that*

$$\begin{aligned}
me^{\sigma_m^s t}|P^s\mathbf{y}_0| &\le |e^{At}P^s\mathbf{y}_0| \le Me^{(\sigma_M^s t+\varepsilon|t|)}|P^s\mathbf{y}_0|, \quad t>0,\\
me^{\sigma_M^s t}|P^s\mathbf{y}_0| &\le |e^{At}P^s\mathbf{y}_0| \le Me^{(\sigma_m^s t+\varepsilon|t|)}|P^s\mathbf{y}_0|, \quad t<0,\\
me^{\sigma_m^u t}|P^u\mathbf{y}_0| &\le |e^{At}P^u\mathbf{y}_0| \le Me^{(\sigma_M^u t+\varepsilon|t|)}|P^u\mathbf{y}_0|, \quad t>0,\\
me^{\sigma_M^u t}|P^u\mathbf{y}_0| &\le |e^{At}P^u\mathbf{y}_0| \le Me^{(\sigma_m^u t+\varepsilon|t|)}|P^u\mathbf{y}_0|, \quad t<0,
\end{aligned}$$

   for all $\mathbf{y}_0 \in \mathbb{C}^n$.

2. *There exists some integer k with $0 \le k \le n-1$ such that for all $\mathbf{y}_0 \in \mathbb{C}^n$*

$$m|P^c\mathbf{y}_0| \le |e^{At}P^c\mathbf{y}_0| \le M(1+|t|^k)|P^c\mathbf{y}_0|, \quad t \in \mathbb{R}.$$

Remark 8. If the eigenvalue μ with $\operatorname{Re}\mu = \sigma_M^s$ (or σ_m^s, σ_M^u, σ_m^u) is semi-simple, then $\varepsilon = 0$ can be chosen for the upper bound estimate. If all $\mu \in \sigma^c(A)$ are semi-simple, then $k = 0$ can be chosen.

We say that a matrix A is *hyperbolic* if $\sigma^c(A) = \emptyset$. From the above discussion, every solution of the system (5.1) with a hyperbolic coefficient matrix is decomposed into two linearly independent solutions: one exponentially decays as $t \to +\infty$, and the other exponentially grows as $t \to +\infty$ (decays as $t \to -\infty$). In such a case, we say that the system possesses an *exponential dichotomy*. This concept will be discussed further in a following section.

If $\sigma^c(A) \neq \emptyset$ and $\sigma^u(A) = \emptyset$, then the asymptotic behavior of the solution $e^{At}\mathbf{y}_0$ to the system (5.1) for a large time $t > 0$ is described by the dynamics of $e^{At}P^c\mathbf{y}_0$ in the sense that

$$e^{At}\mathbf{y}_0 - e^{At}P^c\mathbf{y}_0 \to 0 \quad \text{as } t \to +\infty.$$

In the next two sections, we study the asymptotic behavior of the solution to the infinite-dimensional version of (5.1).

5.2 Spectral Theory of Linear Operators

Notation: Throughout this section, $\mathcal{X}$ (or $\mathcal{Y}$) denotes a Banach space.

We consider linear operators $\mathcal{L} : D(\mathcal{L}) \subset \mathcal{X} \to \mathcal{Y}$ defined on a subspace $D(\mathcal{L})$, the *domain* of $\mathcal{L}$. If $D(\mathcal{L})$ is dense in $\mathcal{X}$, we say that $\mathcal{L}$ is *densely defined.* We say that $\mathcal{L}$ is *closed* if for any sequence $\{x_j\} \subset D(\mathcal{L})$ such that

$$x_j \to x \quad \text{in } \mathcal{X} \quad \text{and} \quad \mathcal{L}x_j \to y \quad \text{in } \mathcal{Y}, \tag{5.4}$$

then we have $x \in D(\mathcal{L})$ and $\mathcal{L}x = y = \lim_{j\to\infty} \mathcal{L}x_j$. $\mathcal{L}$ is closed if and only if the graph of $\mathcal{L}$, $\{(x, \mathcal{L}x) \in \mathcal{X} \times \mathcal{Y} : x \in D(\mathcal{L})\}$, is closed in $\mathcal{X} \times \mathcal{Y}$.

Let $\mathcal{L} : D(\mathcal{L}) \subset \mathcal{X} \to \mathcal{Y}$ be a closed linear operator. The kernel of $\mathcal{L}$ is a closed subspace of $\mathcal{X}$. For a bounded linear operator $\mathcal{B} : \mathcal{X} \to \mathcal{Y}$, the operator $\mathcal{B} + \mathcal{L}$ is closed with the domain $D(\mathcal{B} + \mathcal{L}) = D(\mathcal{L})$. If $\mathcal{B} : \mathcal{Z} \to \mathcal{X}$ is bounded with $\mathcal{R}(\mathcal{B}) \subset D(\mathcal{L})$, then $\mathcal{L}\mathcal{B}$ is also closed. If $\mathcal{B} : \mathcal{Y} \to \mathcal{Z}$ has the inverse, then $\mathcal{B}\mathcal{L}$ is a closed operator. If $\mathcal{L}$ is invertible, then $\mathcal{L}^{-1}$ is also a closed operator.

Theorem 5.4 (Closed graph theorem[26]). *Let $\mathcal{L} : \mathcal{D}(\mathcal{L}) = \mathcal{X} \to \mathcal{Y}$ be a linear operator. Then $\mathcal{L}$ is continuous if and only if $\mathcal{L}$ is closed.*

We require densely defined operators to define the adjoint operator.[27] In terms of semigroup theory, we will see that generators of strongly continuous semigroups are necessarily closed and densely defined operators.

5.2.1 Projection Operators

A bounded linear operator $P : \mathcal{X} \to \mathcal{X}$ with $P^2 = P$ is called a *projection.* For a projection P, $I - P$ is also a projection. We introduce elementary properties of projection operators.

Lemma 5.5. *For a projection operator P on $\mathcal{X}$, the following hold.*

(a) $Px = x$ for all $x \in \mathcal{R}(P)$.

(b) $\mathcal{X} = \mathcal{N}(P) \oplus \mathcal{R}(P)$.

(c) $\mathcal{R}(I - P) = \mathcal{N}(P)$ and $\mathcal{N}(I - P) = \mathcal{R}(P)$.

(d) $\mathcal{R}(P)$ and $\mathcal{R}(I - P)$ are closed subspaces of $\mathcal{X}$.

(e) If $P, \widetilde{P} : \mathcal{X} \to \mathcal{X}$ are projections such that $\mathcal{N}(P) = \mathcal{N}(\widetilde{P})$ and $\mathcal{R}(P) = \mathcal{R}(\widetilde{P})$, then $P = \widetilde{P}$.

[26] The proof invokes the axiom of choice.

[27] See [20], Chapter 3.

Proof. (a) is trivial. To prove (b), we first check that $\mathcal{N}(P)\cap\mathcal{R}(P)=\{0\}$. If $x\in\mathcal{N}(P)\cap\mathcal{R}(P)$, then $Px=0$ and $x=Py$ for some $y\in\mathcal{X}$. Hence we have

$$0=Px=P^2y=Py=x.$$

Now (b) follows from that for any $x\in\mathcal{X}$,

$$Px\in\mathcal{R}(P),\quad x-Px\in\mathcal{N}(P),\quad\text{and}\quad x=Px+(x-Px)\,.$$

To prove (c), it suffices to show the first equality since $I-P$ is also a projection. For $x\in\mathcal{N}(P)$, (b) implies that $x=x_1+x_2$ for some $x_1\in\mathcal{N}(I-P)$ and $x_2\in\mathcal{R}(I-P)$. On the other hand,

$$x=(I-P)x=(I-P)x_1+(I-P)x_2=x_2\in\mathcal{R}(I-P).$$

Conversely, for $x\in\mathcal{R}(I-P)$, we have $x=(I-P)x=x-Px$. Thus, $Px=0$. Now (d) follows from (c) since the kernel of a bounded operator is closed. To prove (e), for all $x\in\mathcal{X}$ we let $x=x_1+x_2$, where $x_1\in\mathcal{N}(P)=\mathcal{N}(\widetilde{P})$ and $x_2\in\mathcal{R}(P)=\mathcal{R}(\widetilde{P})$. Then we have

$$Px-\widetilde{P}x=Px_2-\widetilde{P}x_2=x_2-x_2=0.$$

□

Remark 9. For a projection P on a Hilbert space $\mathcal{H}$, let $P^*:\mathcal{H}\to\mathcal{H}$ be the adjoint operator of P. Then, P^* is also a projection since $P^*=(PP)^*=P^*P^*$. We remark that the direct sum in Lemma 5.5 is orthogonal if and only if the projection P is self-adjoint. Also, we have

$$\mathcal{N}(P^*)=\mathcal{R}(P)^\perp,\quad\mathcal{N}(P)=\mathcal{R}(P^*)^\perp.$$

5.2.2 Spectrum of Linear Operators

For $\mathcal{L}:D(\mathcal{L})\subset\mathcal{X}\to\mathcal{Y}$ a densely defined closed linear operator, $\lambda I-\mathcal{L}:D(\mathcal{L})\subset\mathcal{X}\to\mathcal{Y}$ is also a densely defined closed linear operator for any $\lambda\in\mathbb{C}$. We are interested in the invertibility of $\lambda I-\mathcal{L}$.

We say that $\lambda\in\mathbb{C}$ is in the *resolvent set*, $\rho(\mathcal{L})$, if $\lambda I-\mathcal{L}$ is has the bounded inverse operator $(\lambda I-\mathcal{L})^{-1}:\mathcal{Y}\to D(\mathcal{L})\subset\mathcal{X}$.[28,29] The inverse operator $(\lambda I-\mathcal{L})^{-1}$ is called the *resolvent*. We say that $\lambda\in\mathbb{C}\setminus\rho(\mathcal{L})$ is an *eigenvalue* of $\mathcal{L}$ if $\mathcal{N}(\lambda I-\mathcal{L})$ is a non-trivial subspace of the domain $D(\mathcal{L})$. Unlike the finite dimensional case, however, $\mathbb{C}\setminus\rho(\mathcal{L})$ is not the set of all eigenvalues of $\mathcal{L}$ in general. We call $\sigma(\mathcal{L}):=\mathbb{C}\setminus\rho(\mathcal{L})$ the *spectrum* of $\mathcal{L}$, and decompose it in terms of the Fredholm properties of the operator $\lambda I-\mathcal{L}$.

[28] $(\lambda I-\mathcal{L})^{-1}:\mathcal{Y}\to D(\mathcal{L})\subset\mathcal{X}$ exists and there is a constant $C>0$ such that $\|(\lambda I-\mathcal{L})^{-1}f\|_{\mathcal{X}}\le C\|f\|_{\mathcal{Y}}$ for all $f\in\mathcal{Y}$.

[29] Indeed, invertibility implies bounded invertibility. This follows from the closed graph theorem and that the inverse of a closed operator is also closed.

Definition 5.4 (Fredholm operator)**.** Let $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ be a closed and densely defined operator. We say that $\mathcal{L}$ is *Fredholm* if

1. the range of $\mathcal{L}$, $\mathcal{R}(\mathcal{L})$, is closed[30],
2. the dimension of $\mathcal{N}(\mathcal{L})$ is finite,
3. the codimension of $\mathcal{R}(\mathcal{L})$[31] is finite.

For a Fredholm operator $\mathcal{L}$, we define the *Fredholm index*

$$\operatorname{ind}(\mathcal{L}) = \dim \mathcal{N}(\mathcal{L}) - \operatorname{codim} \mathcal{R}(\mathcal{L}).$$

The Fredholm index of an operator tells us how far an operator is from being invertible. If $\operatorname{ind}(\mathcal{L}) = 0$, the operator $\mathcal{L}$ is injective if and only if it is surjective. If $\operatorname{ind}(\mathcal{L}) \neq 0$, the operator $\mathcal{L}$ cannot be bijective.

We decompose $\sigma(\mathcal{L})$ as follows:

1. we say that $\lambda \in \sigma(\mathcal{L})$ is in the *point spectrum*, $\sigma_{\text{pt}}(\mathcal{L})$, if $\lambda I - \mathcal{L}$ is Fredholm with $\operatorname{ind}(\lambda I - \mathcal{L}) = 0$, but it is not invertible;
2. we call $\sigma_{\text{ess}}(\mathcal{L}) := \sigma(\mathcal{L}) \setminus \sigma_{\text{pt}}(\mathcal{L})$ the *essential spectrum* of $\mathcal{L}$.

By the above classification, the essential spectrum of $\mathcal{L}$ is the set of all $\lambda \in \mathbb{C}$ such that

1. $\lambda I - \mathcal{L}$ is not Fredholm, or
2. $\lambda I - \mathcal{L}$ is Fredholm with $\operatorname{ind}(\lambda I - \mathcal{L}) \neq 0$.

We note that $\lambda I - \mathcal{L}$ is Fredholm with $\operatorname{ind}(\lambda I - \mathcal{L}) = 0$ for all $\lambda \in \rho(\mathcal{L})$ since $\mathcal{R}(\lambda I - \mathcal{L}) = \mathcal{Y}$ and $\mathcal{N}(\lambda I - \mathcal{L}) = 0$; $\mathcal{R}(\lambda I - \mathcal{L})$ is closed in $\mathcal{Y}$, $\dim \mathcal{N}(\lambda I - \mathcal{L}) = 0$, and $\operatorname{codim} \mathcal{R}(\lambda I - \mathcal{L}) = 0$.

Remark 10. For each $k \in \mathbb{Z}$, the set of $\lambda \in \mathbb{C}$ for which $(\lambda I - \mathcal{L})$ is Fredholm with index k is open since small bounded perturbations of a Fredholm operator do not change the Fredholm index.[32] The set of $\lambda \in \mathbb{C}$ for which $\lambda I - \mathcal{L}$ is Fredholm is called the *Fredholm domain* for $\mathcal{L}$. The complement set of the Fredholm domain for $\mathcal{L}$ is called the *Fredholm border* for $\mathcal{L}$. The Fredholm domain is open and the Fredholm border is closed. In general the Fredholm domain is the union of countable open connected components. In many applications, however, the Freholm domain is the union of a finite number of open connected components, and the Fredholm border is the union of some parametrized curves.

[30] Invoking the axiom of choice, this condition is implied by the other two conditions.
[31] The dimension of $\mathcal{Y}/\mathcal{R}(\mathcal{L})$.
[32] See [20], Chapter 4, Section 5.3

Remark 11. In some literatures[33], the essential spectrum of an operator is defined by the set of $\lambda \in \mathbb{C}$ for which $\lambda I - \mathcal{L}$ is not Fredholm. Our definition of the essential spectrum is larger, and it takes an advantage in that the point spectrum consists of isolated eigenvalues of $\mathcal{L}$ with finite algebraic multiplicities. This can be explained from properties of the Evans function. The set of λ for which $\lambda I - \mathcal{L}$ has the Fredholm index zero is open. On an open connected subset Ω of this set, either the zeros of the Evans function, which is analytic, must be a discrete set, or the Evans function is identically zero. However, the Evans function typically tends to a non-zero value as $\operatorname{Re}\lambda$ tends to $+\infty$. The relative complement set of the zero set of the Evans function with respect to Ω lies in the resolvent set.

5.2.3 Resolvent Operators

Notation: We denote the resolvent operator $(\lambda - \mathcal{L})^{-1}$ by $R(\lambda, \mathcal{L})$ or $R(\lambda)$.[34],

Proposition 5.6 (Resolvent identities)**.** *For $\lambda, \mu \in \rho(\mathcal{L})$ with $\lambda \neq \mu$, the following hold:*

(a) $R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu)$.

(b) $R(\lambda)R(\mu) = R(\mu)R(\lambda)$.

Proof. (a) is obtained by subtracting two equations

$$\begin{aligned} R(\lambda) &= R(\lambda)(\mu - \mathcal{L})R(\mu), \\ R(\mu) &= R(\lambda)(\lambda - \mathcal{L})R(\mu). \end{aligned}$$

Since $\mathcal{L}R(\lambda) = \lambda R(\lambda) - I$ is a bounded operator, (a) holds for all $\mathcal{X}$. (b) follows from (a). □

Proposition 5.7. *For a closed operator $\mathcal{L}$ on $\mathcal{X}$, the following hold.*

(a) The resolvent set $\rho(\mathcal{L})$ is open in $\mathbb{C}$, and the resolvent $R(\lambda)$ is (piecewise[35]) analytic in $\lambda \in \rho(\mathcal{L})$. In particular, for fixed $\lambda_0 \in \rho(\mathcal{L})$, we have

$$R(\lambda) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0)^{n+1} \tag{5.5}$$

for all $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_0| < 1/\|R(\lambda_0)\|$.

(b) $\|R(\lambda)\| \geq \dfrac{1}{\operatorname{dist}(\lambda, \sigma(\mathcal{L}))}$.

(c) For a sequence $\lambda_n \in \rho(\mathcal{L})$ such that $\lim_{n\to\infty} \lambda_n = \lambda_0$, we have that $\lambda_0 \in \sigma(\mathcal{L})$ if and only if $\lim_{n\to\infty} \|R(\lambda_n)\| = \infty$.

[33] See [20], Chapter 4, Section 5.6, for instance

[34] In some literatures such as [20], the notation $R(\lambda) = (\mathcal{L} - \lambda)^{-1}$ is used.

[35] The resolvent set is not connected in general.

Proof. For fixed $\lambda_0 \in \rho(\mathcal{L})$, we have that

$$\lambda - \mathcal{L} = [I - (\lambda_0 - \lambda)R(\lambda_0)]\,(\lambda_0 - \mathcal{L}) = (\lambda_0 - \mathcal{L})\,[I - (\lambda_0 - \lambda)R(\lambda_0)]$$

for all $\lambda \in \mathbb{C}$ and $x \in D(\mathcal{L})$. Since $\lambda_0 - \mathcal{L}$ is invertible, $\lambda - \mathcal{L}$ is boundedly invertible if $I - (\lambda_0 - \lambda)R(\lambda_0)$ has the bounded inverse, which is true for all λ with $|\lambda - \lambda_0| \leq 1/\|R(\lambda_0)\|$. Hence, $\rho(\mathcal{L})$ is open and we have

$$[I - (\lambda_0 - \lambda)R(\lambda_0)]^{-1} = \sum_{n=0}^{\infty}(\lambda_0 - \lambda)^n R(\lambda_0)^n.$$

In particular, $[I - (\lambda_0 - \lambda)R(\lambda_0)]^{-1} : \mathcal{X} \to D(\mathcal{L})$ since $\mathcal{L}$ is closed. Indeed, for $x \in \mathcal{X}$, $R(\lambda_0)^n x \in D(\mathcal{L})$ for all $n \in \mathbb{N}$, and thus $\sum_{n=0}^{\infty}(\lambda_0 - \lambda)^n R(\lambda_0)^n x \in D(\mathcal{L})$. Therefore, we obtain

$$\begin{aligned} R(\lambda) &= R(\lambda_0)\,[I - (\lambda_0 - \lambda)R(\lambda_0)]^{-1} = [I - (\lambda_0 - \lambda)R(\lambda_0)]^{-1}\,R(\lambda_0) \\ &= \sum_{n=0}^{\infty}(\lambda_0 - \lambda)^n R(\lambda_0)^{n+1}. \end{aligned}$$

To prove (b), we fix $\lambda_0 \in \rho(\mathcal{L})$. We showed in the proof of (a) that if $|\lambda - \lambda_0| \leq 1/\|R(\lambda_0)\|$, then $\lambda \in \rho(\mathcal{L})$. Hence, for all $\lambda \in \sigma(\mathcal{L}) = \mathbb{C} \setminus \rho(\mathcal{L})$, we have $|\lambda - \lambda_0| > 1/\|R(\lambda_0)\|$. Taking the infimum in $\lambda \in \sigma(\mathcal{L})$, we obtain (b). To prove (c), we assume that $\lambda \in \rho(\mathcal{L})$. Then, since the set $\{\lambda_n : n \geq 0\}$ is compact and $R(\lambda)$ is continuous, $R(\lambda_n)$ must be uniformly bounded for all $n \geq 0$. The converse follows from (b). □

Remark 12. As a direct consequence of Proposition 5.7, $\sigma(\mathcal{L})$ is closed. In general, the spectrum of an unbounded operator is not bounded. However, for a bounded operator $\mathcal{L}$, $\sigma(\mathcal{L})$ is bounded (hence compact) since

$$R(\lambda) = \frac{1}{\lambda}\left(1 - \frac{\mathcal{L}}{\lambda}\right)^{-1} = \sum_{n=0}^{\infty}\frac{\mathcal{L}^n}{\lambda^{n+1}} < \infty$$

for all $|\lambda| > \|\mathcal{L}\|$. Also, by the Liouville's theorem, we see that $\sigma(\mathcal{L}) \neq \emptyset$ since

$$\|R(\lambda)\| \leq \frac{1}{|\lambda|}\left(1 - \frac{\|\mathcal{L}\|}{|\lambda|}\right)^{-1} = (|\lambda| - \|\mathcal{L}\|)^{-1} \to 0 \quad \text{as } |\lambda| \to \infty.$$

Moreover, $R(\lambda)$ is analytic at the infinity.

This remark introduces the following definition for bounded operators.

Definition 5.5. For a bounded operator $\mathcal{L}$ on $\mathcal{X}$, we define the *spectral radius* of $\mathcal{L}$ by

$$r(\mathcal{L}) := \sup\{|\lambda| : \lambda \in \sigma(\mathcal{L})\}.$$

Corollary 5.8. *For a bounded operator $\mathcal{L}$ on $\mathcal{X}$, we have $r(\mathcal{L}) \leq \|\mathcal{L}\|$.*

5.2.4 Commutativity and Spectral Decomposition

Definition 5.6. Let $\mathcal{L}$ be an operator on $\mathcal{X}$. We say that $\mathcal{L}$ commutes with A (or A commutes with $\mathcal{L}$) if for all $x \in D(\mathcal{L})$, we have that $Ax \in D(\mathcal{L})$ and

$$\mathcal{L}Ax = A\mathcal{L}x.$$

From Lemma 5.5, we know that a projection P decomposes a Banach space $\mathcal{X}$ in such a way that

$$\mathcal{X} = M_1 \oplus M_2, \tag{5.6}$$

where $M_1 := \mathcal{R}(P)$ and $M_2 := \mathcal{R}(I - P)$.

Definition 5.7. Let $\mathcal{L}$ be an operator on $\mathcal{X}$ and P be a projection such that the decomposition (5.6) holds. We say that $\mathcal{L}$ is decomposed according to $\mathcal{X} = M_1 \oplus M_2$ if

$$PD(\mathcal{L}) \subset D(\mathcal{L}), \quad \mathcal{L}M_1 \subset M_1, \quad \mathcal{L}M_2 \subset M_2. \tag{5.7}$$

We observe that (5.7) is equivalent to that $\mathcal{L}$ commutes with P. Indeed, from (5.7), we have $Px \in D(\mathcal{L})$, $\mathcal{L}Px \in M_1$ and $\mathcal{L}(I - P)x \in M_2$ for all $x \in D(\mathcal{L})$. Thus $(I - P)\mathcal{L}Px = 0$ and $P\mathcal{L}(I - P)x = 0$ from Lemma 5.5, and this implies $\mathcal{L}Px = P\mathcal{L}x$. The converse is easy.

Definition 5.8. Let $\mathcal{L}$ be an operator on $\mathcal{X}$ and M be a subspace of $\mathcal{X}$. The *part* $\mathcal{L}|_M$ of $\mathcal{L}$ in M is defined by $\mathcal{L}|_M x := \mathcal{L}x$ with the domain $D(\mathcal{L}|_M) := \{x \in D(\mathcal{L}) \cap M : \mathcal{L}x \in M\}$.

If $\mathcal{L}$ is a closed operator and M is a closed subspace, then $\mathcal{L}|_M$ is also closed since the graph of $\mathcal{L}|_M$ is the intersection of the graph of $\mathcal{L}$ and the closed set $M \times M$.

$\mathcal{L}R(\lambda) = \lambda R(\lambda) - I$ is a bounded operator. Since $\lambda R(\lambda) - I = R(\lambda)\mathcal{L}$, $\mathcal{L}$ commutes with the resolvent $R(\lambda)$.

Proposition 5.9. *Assume that $\rho(\mathcal{L}) \neq \phi$. If $\mathcal{L}$ commutes with a bounded operator A, then we have*

$$R(\lambda, \mathcal{L})A = AR(\lambda, \mathcal{L})$$

for all $\lambda \in \rho(\mathcal{L})$. Conversely, if $R(\lambda_0)$ commutes with A for some $\lambda_0 \in \rho(\mathcal{L})$, then $\mathcal{L}$ commutes with A.

Proof. We first claim that if an invertible operator $T : D(T) \subset \mathcal{X} \to \mathcal{X}$ commutes with a bounded operator A, then T^{-1} also commutes with A. Since A is bounded, $Ax \in D(T^{-1}) = \mathcal{X}$ for all $x \in \mathcal{X}$. Since

$$TAT^{-1}x = ATT^{-1}x = Ax$$

for all $x \in \mathcal{X}$, we prove the claim by taking T^{-1}. If $\mathcal{L}$ commutes with A, then $(\lambda - \mathcal{L})$, which is invertible for all $\lambda \in \rho(\mathcal{L})$, commutes with A for all $\lambda \in \rho(\mathcal{L})$. The result follows from the claim. The converse also follows from the claim. □

Theorem 5.10 (Spectral Decomposition)**.** *Let $\mathcal{L}$ be a closed operator on $\mathcal{X}$. Suppose that the spectrum of $\mathcal{L}$ is decomposed as $\sigma(\mathcal{L}) = \sigma_c(\mathcal{L}) \cup \sigma_u(\mathcal{L})$ in such a way that $\sigma_c(\mathcal{L})$ is enclosed by a simple closed curve C and $\sigma_u(\mathcal{L})$ lies in the exterior of C. Then we have a decomposition of $\mathcal{L}$ according to a decomposition $\mathcal{X} = M_c \oplus M_u$ such that the following hold:*

(a) $P\mathcal{L} = \mathcal{L}P$ and $(I-P)\mathcal{L} = \mathcal{L}(I-P)$.

(b) M_c and M_u are closed.

(c) The parts $\mathcal{L}|_{M_c}$ and $\mathcal{L}|_{M_u}$ are closed, and $\mathcal{L}|_{M_c}$ is bounded.

(d) $\sigma(\mathcal{L}|_{M_c}) = \sigma_c(\mathcal{L})$ and $\sigma(\mathcal{L}|_{M_u}) = \sigma_u(\mathcal{L})$.

Proof. Let

$$P = \frac{1}{2\pi i}\int_C R(\lambda)\, d\lambda.$$

One may check that P is a bounded operator with $P^2 = P$. Let $M_c := P\mathcal{X}$ and $M_u := (I-P)\mathcal{X}$. Since the resolvents commutes, $PR(\lambda) = R(\lambda)P$ for all $\lambda \in \rho(\mathcal{L})$. Thus, from Proposition 5.9, P commutes with $\mathcal{L}$. This implies that (see below (5.7)) the parts $\mathcal{L}|_{M_c}$ and $\mathcal{L}|_{M_u}$ are defined.

It is easy to see that $R(\lambda)|_{M_c}$ and $R(\lambda)|_{M_u}$ are the inverses of $\lambda - \mathcal{L}|_{M_c}$ and $\lambda - \mathcal{L}|_{M_c}$, respectively, for all $\lambda \in \rho(\mathcal{L})$. Hence, $\rho(\mathcal{L}|_{M_c})$ and $\rho(\mathcal{L}|_{M_u})$ contain $\rho(\mathcal{L})$. We show that $\rho(\mathcal{L}|_{M_c}) \supset \sigma_u(\mathcal{L})$. To see this, we note that $R(\lambda)|_{M_c}x = R(\lambda)x = R(\lambda)Px$ for all $x \in M_c$ and $\lambda \in \rho(\mathcal{L})$. For any $\lambda \in \rho(\mathcal{L}) \setminus C$, we have

$$R(\lambda)P = \frac{1}{2\pi i}\int_C R(\lambda)R(\lambda')\, d\lambda' = \frac{1}{2\pi i}\int_C \big(R(\lambda) - R(\lambda')\big)\frac{d\lambda'}{\lambda' - \lambda}$$

by the resolvent identity. If λ lies outside of C, we have

$$R(\lambda)P = -\frac{1}{2\pi i}\int_C R(\lambda')\frac{d\lambda'}{\lambda' - \lambda}.$$

Since the RHS is analytic in λ outside of C,[36] $R(\lambda)P$, and hence $R(\lambda)|_{M_c}$ also, has an analytic extension outside C. This analytic extension is the resolvent of $\mathcal{L}|_{M_c}$. Similarly, one may check that $\rho(\mathcal{L}|_{M_u}) \supset \sigma_c(\mathcal{L})$. Therefore, we conclude that $\sigma(\mathcal{L}|_{M_c}) \subset \sigma_c(\mathcal{L})$ and $\sigma(\mathcal{L}|_{M_u}) \subset \sigma_u(\mathcal{L})$.

If $\lambda \in \sigma_c(\mathcal{L}) \setminus \sigma(\mathcal{L}|_{M_c})$, then $\lambda \in \rho(\mathcal{L}|_{M_c})$ and $\lambda \in \rho(\mathcal{L}|_{M_u})$. But, this implies that

$$R(\lambda)|_{M_c}P + R(\lambda)|_{M_u}(I-P)$$

is the inverse of $\lambda - \mathcal{L}$. This shows that $\sigma(\mathcal{L}|_{M_c}) = \sigma_c(\mathcal{L})$ and in a similar fashion, $\sigma(\mathcal{L}|_{M_u}) = \sigma_u(\mathcal{L})$.

Lastly, by using that $\mathcal{L}R(\lambda) = \lambda R(\lambda) - I$ is bounded, $\mathcal{L}$ is closed, and the summation representaion of $R(\lambda)$, we obtain that

$$\mathcal{L}P = \frac{1}{2\pi i}\int_C \mathcal{L}R(\lambda)\, d\lambda = \frac{1}{2\pi i}\int_C \lambda R(\lambda)\, d\lambda,$$

where the last one is a bounded on $\mathcal{X}$. Thus, $\mathcal{L}|_{M_c}$ is a bounded on M_c. $\square$

[36] Since $R(\lambda)$ is continuous on C, it is also analytic inside of C.

5.3 Semigroups and Application to Linear Stability of Nonlinear Waves

5.3.1 Strongly Continuous Semigroups

Definition 5.9. A family of bounded linear operators $(T(t))_{t\geq 0} : \mathcal{X} \to \mathcal{X}$ is called a *(one-parameter) semigroup* if it satisfies the functional equation

$$T(t+s) = T(t)T(s) \quad \text{for all } t, s \geq 0, \quad T(0) = I. \tag{5.8}$$

A semigroup $T(t)$ is called a *strongly continuous (one-paramter) semigroup* (or C_0-*semigroup*), if the *orbit* map

$$\xi_x : t \in [0, \infty) \mapsto \xi_x(t) := T(t)x \in \mathcal{X} \tag{5.9}$$

is continuous for every $x \in \mathcal{X}$.

In fact, combined with (5.8), (5.9) is equivalent to a weaker condition. To see this, we first observe that a C_0-semigroup $T(t)$ is uniformly bounded on any compact set $[0, t_0]$, that is,

$$\sup_{t\in[0,t_0]} \|T(t)\| < \infty. \tag{5.10}$$

Since the orbit map $T(t)x$ is continuous for each $x \in \mathcal{X}$, the image of a compact set $[0, t_0]$ under $T(t)x$ is also compact, and hence $\sup_{t\in[0,t_0]} \|T(t)x\| < \infty$ for each $x \in \mathcal{X}$. Now (5.10) follows from the uniform boundedness principle.

Proposition 5.11. *For a semigroup $T(t)$ on $\mathcal{X}$, the following are equivalent.*

(a) $t \mapsto T(t)x$ is continuous on $[0, \infty)$ for all $x \in \mathcal{X}$.

(b) $\lim_{h\downarrow 0} T(h)x = x$ for all $x \in \mathcal{X}$.

Proof. It is enough to show that (b) implies (a). For $t \geq 0$ and $h > 0$,

$$\|T(t+h)x - T(t)x\| \leq \|T(t)\|\|T(h)x - x\| \to 0 \quad \text{as } h \downarrow 0$$

for all $x \in \mathcal{X}$. On the other hands, for $t > 0$ and $h < 0$,

$$\begin{aligned}\|T(t+h)x - T(t)x\| &= \|T(t+h)\left(I - T(-t-h)T(t)\right)x\| \\ &\leq \|T(t+h)\|\|x - T(-h)x\| \\ &\to 0 \quad \text{as } h \uparrow 0\end{aligned}$$

using (5.10). □

The fact that a C_0-semigroup $T(t)$ is uniformly bounded on any compact set $[0, t_0]$ implies that the uniform norm of $T(t)$ is controlled by some exponential function on $[0, \infty)$.

Proposition 5.12. *For all C_0-semigroup $T(t)$, there exist some constants $w \in \mathbb{R}$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{wt} \tag{5.11}$$

for all $t \geq 0$.

Proof. We can choose $M \geq 1$ such that $\|T(s)\| \leq M$ for all $s \in [0,1]$. We write $t \geq 0$ as $t = s+k$ for some $k \in \mathbb{N}$ and $s \in [0,1]$. Let $w := \log M \geq 0$. Then, we have

$$\|T(t)\| \leq \|T(s)\|\|T(1)\|^k \leq M^{k+1} = Me^{k \log M} = Me^{w(t-s)} \leq Me^{wt}.$$

□

Definition 5.10. For a C_0-semigroup $T(t)$, its *growth bound* w_0 is defined by

$$\begin{aligned} w_0 &:= \inf\{w \in \mathbb{R} : \text{there exists } M_w \geq 1 \text{ such that } \|T(t)\| \leq M_w e^{wt} \text{ for all } t \geq 0\} \\ &= \inf\{w \in \mathbb{R} : \lim_{t\to\infty} e^{wt}\|T(t)\| = 0\}. \end{aligned}$$

$T(t)$ is called *bounded* if $w = 0$, and *contractive* if $w = 0$ and $M = 1$ can be chosen in (5.11).

Proposition 5.13. [37] *Let $T(t)$ be a C_0-semigroup with the growth bound w_0. Then the spectral radius of $T(t)$ is given by*

$$r\left(T(t)\right) = e^{w_0 t} \quad \text{for all } t \geq 0.$$

5.3.2 Generators of Semigroups

The differentiability of a C_0-semigroup is equivalent to a weaker condition.

Lemma 5.14. *Let $T(t)$ be a C_0-semigroup on $\mathcal{X}$ and x be an element of $\mathcal{X}$. For the orbit map $\xi_x : t \mapsto T(t)x$, the following are equivalent.*

(a) $\xi_x(t)$ is differentiable on $[0,\infty)$.

(b) $\xi_x(t)$ is right differentiable at $t = 0$.

Proof. It is enought to show that (b) implies (a). For $t \geq 0$ and $h > 0$, we have

$$\frac{\xi_x(t+h) - \xi_x(h)}{h} = T(t)\frac{T(h)x - x}{h}$$

and the RHS converges to $T(t)\xi_x'(0)$ as $h \downarrow 0$. For $-t < h < 0$, we have

$$\frac{\xi_x(t+h) - \xi_x(t)}{h} - T(t)\xi_x'(0) = T(t+h)\left(\frac{T(-h)x - x}{-h} - \xi_x'(0)\right) + \left(T(t+h) - T(t)\right)\xi_x'(0).$$

Since $\|T(t+h)\|$ is uniformly bounded in h for all small $h < 0$,

$$\left\|T(t+h)\left(\frac{T(-h)x - x}{-h} - \xi_x'(0)\right)\right\| \to 0 \quad \text{as } h \uparrow 0.$$

[37] For the proof, we refer to [10], Proposition 2.2, Chapter 4

Since $T(t)$ is strongly continuous,

$$\left\|(T(t+h)-T(t))\,\xi_x'(0)\right\| \to 0 \quad \text{as } h \uparrow 0.$$

Therefore, we conclude that $\xi_x(t)$ is differentiable for all $t \geq 0$ and we have

$$\xi_x'(t) = T(t)\xi_x'(0). \tag{5.12}$$

□

Definition 5.11. For a C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathcal{X}$, let

$$D(\mathcal{L}) := \{x \in \mathcal{X} : \lim_{h\to+0} \tfrac{1}{h}\left(T(h)x - x\right) \text{ exists}\}.$$

The *generator* $\mathcal{L} : D(\mathcal{L}) \subset \mathcal{X} \to \mathcal{X}$ of $(T(t))_{t\geq 0}$ is the linear operator

$$\mathcal{L}x := \xi_x'(0) = \lim_{h\to+0} \tfrac{1}{h}\left(T(h)x - x\right)$$

and $D(\mathcal{L})$ is called its *domain.*

Proposition 5.15. *For the generator $\mathcal{L}$ of a C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathcal{X}$, the following properties hold.*

(a) For $x \in D(\mathcal{L})$, we have $T(t)x \in D(\mathcal{L})$ and

$$\frac{d}{dt}T(t)x = T(t)\mathcal{L}x = \mathcal{L}T(t)x \quad \textit{for all } t \geq 0.$$

(b) For $t \geq 0$ and $x \in \mathcal{X}$, we have

$$\int_0^t T(s)x\,ds \in D(\mathcal{L}).$$

(c) For $t \geq 0$, we have

$$\begin{aligned} T(t)x - x &= \mathcal{L}\int_0^t T(s)x\,ds \quad \textit{for } x \in \mathcal{X} \\ &= \int_0^t T(s)\mathcal{L}x\,ds \quad \textit{for } x \in D(\mathcal{L}). \end{aligned}$$

Proof. By the definition of $D(A)$, $x \in D(A)$ means $\xi_x'(0)$ exists. From Lemma 5.14, $\xi_x'(t)$ exists for all $t \in [0,\infty)$ and in particular, from (5.12), we have

$$T(t)Ax = \xi_x'(t) = \lim_{h\downarrow 0} \frac{T(h)T(t)x - T(t)x}{h} = AT(t)x,$$

which proves (a). To prove (c), we see that for $x \in \mathcal{X}$, $t \geq 0$ and $h > 0$,

$$\begin{aligned}\frac{1}{h}(T(h)-I)\int_0^t T(s)x\,ds &= \frac{1}{h}\left(\int_0^t T(h+s)x\,ds - \int_0^t T(s)x\,ds\right)\\ &= \frac{1}{h}\left(\int_h^{h+t} T(s')x\,ds' - \int_0^t T(s')x\,ds'\right)\\ &= \frac{1}{h}\left(\int_t^{h+t} T(s')x\,ds' - \int_0^h T(s')x\,ds'\right)\\ &\to T(t)x + x \quad \text{as } h \downarrow 0.\end{aligned}$$

Thus, the LHS converges as $h \downarrow 0$, which implies the first equality of (c) as well as (b). For $x \in D(\mathcal{L})$,

$$\begin{aligned}\lim_{h\downarrow 0}\frac{1}{h}(T(h)-I)\int_0^t T(s)x\,ds &= \lim_{h\downarrow 0}\int_0^t T(s)\frac{1}{h}(T(h)-I)\,x\,ds\\ &= \int_0^t T(s)\lim_{h\downarrow 0}\frac{1}{h}(T(h)-I)\,x\,ds\\ &= \int_0^t T(s)\mathcal{L}x\,ds,\end{aligned}$$

where the second line is from that $s \mapsto T(s)\frac{1}{h}(T(h)-I)\,x$ uniformly converses to $T(s)\mathcal{L}x$ on $[0,t]$. (Recall (5.10).) □

Remark 13 (Rescaled Semigroups). For a C_0-semigroup $T(t)$ with the generator $\mathcal{L}$, $S(s) := e^{-\lambda s}T(s)$ is also a C_0-semigroup with the generator $\mathcal{L}-\lambda$ and domain $D(\mathcal{L}-\lambda) := D(\mathcal{L})$. This scaling will be used frequently.

Theorem 5.16. *The generator of a C_0-semigroup is a closed and densely defined linear operator that determines the semigroup uniquely.*

Theorem 5.17. *Let $\mathcal{L}$ be the generator of a C_0- semigroup $(T(t))_{t\geq 0}$ on $\mathcal{X}$ such that $\|T(t)\| \leq Me^{wt}$ for all $t \geq 0$ (see (5.11)). Then the following statements hold.*

(a) If there is $\lambda \in \mathbb{C}$ such that $\widetilde{R}(\lambda)x := \lim_{t\to\infty}\int_0^t e^{-\lambda s}T(s)x\,ds$ exists for all $x \in \mathcal{X}$, then $\lambda \in \rho(\mathcal{L})$ and $R(\lambda,\mathcal{L}) = \widetilde{R}(\lambda)$.

(b) If $\operatorname{Re}\lambda > w$, then $\lambda \in \rho(\mathcal{L})$ and the resolvent $R(\lambda,\mathcal{L})$ is given by $\widetilde{R}(\lambda)$.

(c) $\|R(\lambda,\mathcal{L})\| \leq \frac{M}{\operatorname{Re}\lambda - w}$ for all $\operatorname{Re}\lambda > w$.

Proof. We first prove (b) and (c) using (a). If $\operatorname{Re}\lambda > w$, then

$$\|\int_0^t e^{-\lambda s}T(s)x\,ds\| \leq M\int_0^t e^{-\operatorname{Re}\lambda s + ws}\,ds = M\left[\frac{1}{w-\operatorname{Re}\lambda}e^{(-\operatorname{Re}\lambda + w)s}\right]_0^t$$

Since $-\operatorname{Re}\lambda + w < 0$, the RHS converges to $\frac{M}{\operatorname{Re}\lambda - w}$ as $t \to \infty$. Thus, (b) and (c) follows from (a). To prove (a), we assume that $\lambda = 0$ without loss of generality (see Remark 13). For $x \in \mathcal{X}$

and $h > 0$, we have

$$\begin{aligned}\frac{T(h)-I}{h}\widetilde{R}(0)x &= \frac{T(h)-I}{h}\int_0^\infty T(s)x\,ds \\ &= \frac{1}{h}\int_0^\infty T(h+s)x\,ds - \frac{1}{h}\int_0^\infty T(s)x\,ds \\ &= \frac{1}{h}\int_h^\infty T(s)x\,ds - \frac{1}{h}\int_0^\infty T(s)x\,ds \\ &= -\frac{1}{h}\int_0^h T(s)x\,ds \\ &\to -x \quad \text{as } h \downarrow 0.\end{aligned}$$

Hence, $\widetilde{R}(0)x \in D(\mathcal{L})$ and $\mathcal{L}\widetilde{R}(0) = -I$ for all $x \in \mathcal{X}$. For $x \in D(\mathcal{L})$,

$$\lim_{t\to\infty}\mathcal{L}\int_0^t T(s)x\,ds = \lim_{t\to\infty}\int_0^t T(s)\mathcal{L}x\,ds = \widetilde{R}(0)\mathcal{L}x,$$

where we have used Proposition 5.15 (c). On the other hand, we have

$$\mathcal{L}\widetilde{R}(0) = \mathcal{L}\lim_{t\to\infty}\int_0^t T(s)x\,ds = \lim_{t\to\infty}\mathcal{L}\int_0^t T(s)x\,ds,$$

where one can justify the interchange of limit from that $\mathcal{L}$ is closed and Proposition 5.15 (b). Since $-x = \mathcal{L}\widetilde{R}(0)x = \widetilde{R}(0)\mathcal{L}x$, we conclude that $\widetilde{R}(0) = R(0,\mathcal{L})$. □

Definition 5.12. For a linear operator $\mathcal{L}$, its *spectral bound* is defined by

$$s(\mathcal{L}) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(\mathcal{L})\}.$$

As a direct consequence of Theorem 5.17, we have the following.

Corollary 5.18. *For a C_0-semigroup with the generator $\mathcal{L}$, we have*

$$-\infty \le s(\mathcal{L}) \le w_0 < +\infty.$$

Remark 14. Theorem 5.16 and Theorem 5.17 imply some necessary properties for the generator of a C_0-semigroup : it is closed and densely defined, and its spectrum lies in some left half-plane. Moreover, the resolvent of the generator can be represented by the Laplace transform of $T(t)$.

5.3.3 Inversion Formulas

Theorem 5.19. *Let $(T(t))_{t\ge 0}$ be a bounded C_0-semigroup on $\mathcal{X}$ generated by $\mathcal{L}$. Then for all $\delta > 0$ and $x \in \mathcal{X}$,*

$$\int_0^t T(s)x\,dx = \lim_{n\to+\infty}\frac{1}{2\pi i}\int_{\delta-in}^{\delta+in}\frac{e^{\lambda t}}{\lambda}R(\lambda,\mathcal{L})x\,d\lambda.$$

Here the convergence is uniform in t on compact intervals.

Corollary 5.20. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on $\mathcal{X}$ generated by $\mathcal{L}$. Then for all $x \in D(\mathcal{L})$ and $w > w_0$,*

$$T(t)x = \lim_{n\to+\infty} \frac{1}{2\pi i}\int_{w-in}^{w+in} e^{\lambda t} R(\lambda,\mathcal{L})x\, d\lambda.$$

Here the convergence is uniform in t on compact intervals of $(0,\infty)$.

If we require more regularity on $x \in \mathcal{X}$, then the inversion formula absolutely converges.

Corollary 5.21. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on $\mathcal{X}$ generated by $\mathcal{L}$. Then for all $x \in D(\mathcal{L}^2)$, $w > w_0$, $k \in \mathbb{N}$, $t > 0$,*

$$T(t)x = \frac{(k-1)!}{t^{k-1}} \frac{1}{2\pi i} \lim_{n\to+\infty} \int_{w-in}^{w+in} e^{\lambda t} R(\lambda,\mathcal{L})^k x\, d\lambda.$$

For $k \geq 2$, the integral converges absolutely and uniformly for $t > 0$.

5.3.4 Generation Theorems

The following theorem is due to Hille and Yosida. We recall that the generator of a C_0-semigroup is necessarily closed and densely defined.

Theorem 5.22. *For a densely defined closed linear operator $\mathcal{L}$ on $\mathcal{X}$, the following are equivalent:*

(a) $\mathcal{L}$ generates a C_0-contraction semigroup.

(b) for every $\lambda > 0$ we have $\lambda \in \rho(\mathcal{L})$ and

$$\|\lambda(\lambda - \mathcal{L})^{-1}\| \leq 1. \tag{5.13}$$

(c) for all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > 0$ we have

$$\|(\lambda - \mathcal{L})^{-1}\| \leq \frac{1}{\operatorname{Re}\lambda}. \tag{5.14}$$

Remark 13 on the rescaled semigroup leads the following corollary.

Corollary 5.23. *For $w \in \mathbb{R}$ and a a densely defined closed linear operator $\mathcal{L}$ on $\mathcal{X}$, the following are equivalent.*

1. *$\mathcal{L}$ generates a C_0-semigroup $T(t)$ satisfying*

$$\|T(t)\| \leq e^{wt} \quad \text{for all } t \geq 0.$$

2. *for every $\lambda > w$, we have $\lambda \in \rho(\mathcal{L})$ and*

$$\|(\lambda - w)(\lambda - \mathcal{L})^{-1}\| \leq 1.$$

3. for every $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > w$, we have $\lambda \in \rho(\mathcal{L})$ and

$$\|(\lambda - \mathcal{L})^{-1}\| \leq \frac{1}{\operatorname{Re}\lambda - w}.$$

Definition 5.13. A linear operator $\mathcal{L}$ on $\mathcal{X}$ is called *dissipative* if

$$\lambda\|x\| \leq \|(\lambda - \mathcal{L})x\| \tag{5.15}$$

for all $\lambda > 0$ and $x \in D(\mathcal{L})$.

Proposition 5.24. *For a dissipative operator $\mathcal{L}$, we have the following:*

(a) $\lambda - \mathcal{L}$ is injective for all $\lambda > 0$ and

$$\|\lambda(\lambda - \mathcal{L})^{-1}z\| \leq \|z\|$$

for all $z \in \mathcal{R}(\lambda - \mathcal{L}) := (\lambda - \mathcal{L})D(\mathcal{L})$.

(b) $\lambda - \mathcal{L}$ is surjective for some $\lambda > 0$ if and only if it is surjective for all $\lambda > 0$. In this case, we have $(0, \infty) \subset \rho(\mathcal{L})$.

Proof. From (5.15), we see that the kernel of $\lambda - \mathcal{L}$ is trivial for all $\lambda > 0$. We substitute $x = (\lambda - \mathcal{L})^{-1}z \in D(\mathcal{L})$ ito (5.15). Suppose that $\lambda_0 - \mathcal{L}$ is surjective for some $\lambda_0 > 0$. From (a), $\lambda_0 \in \rho(\mathcal{L})$ and $\|(\lambda_0 - \mathcal{L})^{-1}\| \leq 1/\lambda_0$. From (5.5), we have $(0, 2\lambda_0) \subset \rho(\mathcal{L})$ since all λ satisfying $|\lambda - \lambda_0| < \lambda_0 \leq \|(\lambda_0 - \mathcal{L})^{-1}\|$ are included in $\rho(\mathcal{L})$. In this way, one can check that $(0, \infty) \subset \rho(\mathcal{L})$. □

The following theorem is practically useful since it does not require the resolvent bounds.

Theorem 5.25 (Lumer-Phillips)**.** *For a densely defined closed linear operator $\mathcal{L}$ on $\mathcal{X}$, the following are equivalent:*

(a) $\mathcal{L}$ generates a C_0-contraction semigroup.

(b) $\mathcal{L}$ is dissipative and $\mathcal{R}(\lambda_0 - \mathcal{L}) = X$ for some $\lambda_0 > 0$.

Proof. (a) implies (b) by Theorem 5.22. Suppose that (b) holds. Then, from Proposition 5.24, $(0, \infty) \subset \rho(\mathcal{L})$. From (5.15), we have $\|\lambda(\lambda - \mathcal{L})^{-1}\| \leq 1$ for all $\lambda > 0$. Hence, (a) follows from Theorem 5.22. □

We introduce a practically useful characterization of dissipative operators.

Proposition 5.26. *Let $\mathcal{H}$ be a Hilbert space. An operator $\mathcal{L}$ on $\mathcal{H}$ is dissipative if and only if*

$$\operatorname{Re}\langle \mathcal{L}x, x\rangle \leq 0 \quad \textit{for all } x \in D(\mathcal{L}). \tag{5.16}$$

Proof. We only prove the 'if' statement.[38] Suppose that (5.16) holds. Then, for all $\lambda > 0$ and $x \in D(\mathcal{L})$ with $x \neq 0$,

$$\begin{aligned} \|\lambda x - \mathcal{L}x\|\|x\| &\geq |\langle \lambda x - \mathcal{L}x, x\rangle| \\ &\geq \operatorname{Re}\langle \lambda x - \mathcal{L}x, x\rangle \\ &= \lambda\|x\|^2 - \operatorname{Re}\langle Ax, x\rangle \\ &\geq \lambda\|x\|^2. \end{aligned}$$

□

Theorem 5.27 (Bounded Perturbation of Generators)**.** *Let $\mathcal{L}$ be the generator of a C_0- semigroup $T(t)$ on $\mathcal{X}$ satisfying*

$$\|T(t)\| \leq Me^{wt} \text{ for all } t \geq 0$$

and some $w \in \mathbb{R}$, $M \geq 1$. If B is a linear bounded operator on $\mathcal{X}$, then $\widetilde{\mathcal{L}} := \mathcal{L} + B$ with $D(\widetilde{\mathcal{L}}) := D(\mathcal{L})$ generates a C_0-semigroup $S(t)$ satisfying

$$\|S(t)\| \leq Me^{(w+M\|B\|)t} \quad \text{for all } t \geq 0.$$

Proof. We only prove the contraction case ($w = 0$ and $M = 1$).[39] In this case, we have $(0, \infty) \subset \rho(\mathcal{L})$ from Theorem 5.22, and $\lambda - \widetilde{\mathcal{L}}$ can be written as

$$\lambda - \widetilde{\mathcal{L}} = (I - BR(\lambda, \mathcal{L}))(\lambda - \mathcal{L})$$

for all $\lambda > 0$. From Theorem 5.22, we see that $\|BR(\lambda, \mathcal{L})\| \leq \|B\|/\lambda$ for all $\lambda > 0$, and hence $I - BR(\lambda, \mathcal{L})$ is boundedly invertible for all $\lambda > \|B\|$. This implies that $\lambda - \widetilde{\mathcal{L}}$ is invertible, and we have

$$R(\lambda, \widetilde{\mathcal{L}}) = R(\lambda, \mathcal{L})(I - BR(\lambda, \mathcal{L}))^{-1}$$

for all $\lambda > \|B\|$. Moreover, for all $\lambda \geq \|B\|$,

$$\|R(\lambda, \widetilde{\mathcal{L}})\| \leq \frac{1}{\lambda}\frac{1}{1 - \|B\|/\lambda} = \frac{1}{\lambda - \|B\|}.$$

From Corollary 5.23, we finish the proof for the case that $w = 0$ and $M = 1$. □

5.3.5 Asymptotic Behavior of Semigroups

We introduce some notions of stability for C_0-semigroups.

Definition 5.14. We say that a C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathcal{X}$ is

1. *uniformly exponentially stable* if for some $\varepsilon > 0$

$$\lim_{t\to+\infty} e^{\varepsilon t}\|T(t)\| = 0, \tag{5.17}$$

[38] For the converse direction, we refer to [10], Chapter 2.

[39] For the general case, we refer to [10], Chapter 3.

2. *uniformly stable* if

$$\lim_{t\to+\infty} \|T(t)\| = 0. \tag{5.18}$$

The uniform exponential stability is equivalent to weaker conditions.

Proposition 5.28. *For a C_0-semigroup $T(t)$, the following are equivalent:*

(a) $T(t)$ is uniformly exponentially stable,

(b) $T(t)$ is uniformly stable,

(c) there exists $\varepsilon > 0$ such that $\lim\limits_{t\to+\infty} e^{\varepsilon t}\|T(t)x\| = 0$ for all $x \in \mathcal{X}$.

Proof. It is obvious that (a) implies (b) and (c). From Proposition 5.13 and Corollary 5.8. we have $e^{w_0 t} = r\left(T(t)\right) \leq \|T(t)\|$. Since (b) implies that $e^{w_0 t}$ decreases as $t \to 0$, we must have $w_0 < 0$, which means (a). Suppose that (c) holds. Then, $\sup\limits_{t\in[0,\infty)} \|e^{\varepsilon t}T(t)x\| < \infty$ for each x. By the uniform boundedness principle, we obtain $\sup\limits_{t\in[0,\infty)} \|e^{\varepsilon t}T(t)\| < \infty$, which implies that $\lim\limits_{t\to\infty} e^{\varepsilon/2t}\|T(t)\| = 0$. □

The following theorem is very useful in that it only requires the uniform boundedness of the resolvent.

Theorem 5.29 (Gearhart-Prüss-Greiner)**.** *Let $\mathcal{L}$ be a generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. Then, $T(t)$ is uniformly exponentially stable if and only if*

$$M := \sup_{\operatorname{Re}\lambda>0} \|(\lambda - \mathcal{L})^{-1}\| < \infty. \tag{5.19}$$

Remark 15. Theorem 5.29 does not hold without the uniform bound (5.19). Also the theorem is not true for arbitrary Banach spaces. See ([10], Chapter 5) for some related examples.

Proof. If $T(t)$ is uniformly exponentially stable, then we have $w_0 < 0$, and the uniform bound (5.19) follows from Theorem 5.17. We note that the uniform bound (5.19) and Proposition 5.7 imply that the imaginary axis is also included in $\rho(\mathcal{L})$. Thus, (5.19) also holds for $\operatorname{Re}\lambda \geq 0$ from continuity of $R(\lambda, \mathcal{L})$. We consider the rescaled semigroup $T_{-w}(t) := e^{-wt}T(t)$ for some $w > |w_0| + 1$. By Theorem 5.17, we have

$$R(w + is, \mathcal{L})x = \int_0^\infty e^{-(w+is)t}T(t)x\,dt = \int_0^\infty e^{-ist}T_{-w}(t)x\,dt = R(is, \mathcal{L} - w)x$$

for all $x \in \mathcal{H}$ and $s \in \mathbb{R}$. We extend $T_{-w}(t)$ to $\mathbb{R}$ by letting $T_{-w}(t) := 0$ for $t < 0$. Since $T_{-w}(t)$ is exponentially stable, we have $T_{-w}(\cdot)x \in L^2(\mathbb{R}, \mathcal{H})$. Hence we can represent the above integral in terms of the Fourier transform $\mathcal{F} : L^2(\mathbb{R}, \mathcal{H}) \to L^2(\mathbb{R}, \mathcal{H})$, and we obtain

$$R(w + is, \mathcal{L})x = \mathcal{F}\left(T_{-w}(\cdot)x\right)(s).$$

By using the Plancherel theorem[40] and the fact that $T_{-w}(t)$ is exponentially stable, we get

$$\begin{aligned}\int_{-\infty}^{+\infty} \|R(w+is,\mathcal{L})x\|^2\,ds &= \int_{-\infty}^{+\infty} \|\mathcal{F}\left(T_{-w}(\cdot)x\right)\|^2\,ds \\ &= \int_0^{\infty} \|T_{-w}(t)x\|^2\,dt \\ &\le C\|x\|^2\end{aligned}$$

for all $x \in \mathcal{H}$ and some constant $C > 0$. From the resolvent identity (Proposition 5.6 (a)), we have

$$R(is,\mathcal{L}) = R(w+is,\mathcal{L}) + wR(is,\mathcal{L})R(w+is,\mathcal{L}),$$

and thus the estimate

$$\|R(is,\mathcal{L})x\| \le (1+wM)\|R(w+is,\mathcal{L})x\|$$

holds for all $x \in \mathcal{H}$ and $s \in \mathbb{R}$. From these estimates, we have for all $x \in \mathcal{H}$,

$$\int_{-\infty}^{+\infty} \|R(is,\mathcal{L})x\|^2\,ds \le (1+wM)^2 \int_{-\infty}^{+\infty} \|R(w+is,\mathcal{L})x\|^2\,ds \le (1+wM)^2\|x\|^2.$$

Let $T^*(t)$ be the adjoint C_0-semigroup of $T(t)$ with generator $\mathcal{L}^*$.[41] Since $T(t)$ is bounded on a Hilbert space, we have $\|T(t)\| = \|T^*(t)\|$. Thus, by symmetry, we have for all $y \in \mathcal{H}$,

$$\int_{-\infty}^{+\infty} \|R(is,\mathcal{L}^*)x\|^2\,ds \le (1+wM)^2\|x\|^2.$$

By the inversion formula of $T(t)$ for $k = 2$, we have

$$\begin{aligned}\langle tT(t)x, y\rangle &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(w+is)t}\left\langle R(w+is,\mathcal{L})^2x, y\right\rangle\,ds \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ist}\left\langle R(is,\mathcal{L})x, R(-is,\mathcal{L}^*)y\right\rangle\,ds\end{aligned}$$

for all $x \in D(\mathcal{L}^2)$ and $y \in \mathcal{H}$, where we used the Cauchy integral theorem in the second line. This shifting of contour is possible since, from the definition of $R(\lambda,\mathcal{L})$ and that $R(\lambda,\mathcal{L})$ is uniformly bounded by M on $\operatorname{Re}\lambda \ge 0$, we have for all $x \in D(\mathcal{L})$ and $\lambda \ne 0$ with $\operatorname{Re}\lambda \ge 0$,

$$\|R(\lambda,\mathcal{L})x\| \le \frac{1}{|\lambda|}\|R(\lambda,\mathcal{L})\mathcal{L}x + x\| \le \frac{1}{|\lambda|}\left(M\|\mathcal{L}x\| + \|x\|\right),$$

which implies that $\|R(w+is,\mathcal{L})x\| \to 0$ as $|s| \to \infty$. By the Cauchy-Schwarz inequity, we obtain for all $x, y \in D(A^2)$,

$$\begin{aligned}|\langle tT(t)x, y\rangle| &\le \frac{1}{2\pi}\left(\int_{-\infty}^{\infty}\|R(is,\mathcal{L})x\|^2\,ds\right)^{1/2}\left(\int_{-\infty}^{\infty}\|R(is,\mathcal{L}^*)y\|^2\,ds\right)^{1/2}. \\ &\le \frac{(1+Mw)^2L^2}{2\pi}\|x\|\|y\|.\end{aligned}$$

By the density of $D(\mathcal{L}^2)$ in $\mathcal{H}$,[42] we obtain

$$\|tT(t)\| = \sup\{|\langle tT(t)x, y\rangle| : x, y \in D(\mathcal{L}^2), \|x\| = \|y\| = 1\} \le \frac{(1+Mw)^2L^2}{2\pi}.$$

Since $\|T(t)\| \to 0$ as $t \to \infty$, $T(t)$ is uniformly exponentially stable by Proposition 5.28. □

[40] This theorem holds only for Hilbert space valued functions.

[41] See [25] Chapter 1, Section 10.

[42] See [10], Chapter 2, Section 1

5.3.6 Abstract Cauchy Problem

We consider the *abstract*[43] *Cauchy problem*

$$\begin{cases} \partial_t u(t) = \mathcal{L}u(t) \quad (t \geq 0), \\ u(0) = u_0 \end{cases} \tag{5.20}$$

where $\mathcal{L}$ is a linear operator on $\mathcal{X}$.

(a) A function $u : [0, \infty) \to \mathcal{X}$ is called a *(classical) solution* of (5.20) if $u \in C^1([0, \infty); \mathcal{X})$, $u(t) \in D(\mathcal{L})$ for all $t \geq 0$, and (5.20) holds.

(b) A function $u : [0, \infty) \to \mathcal{X}$ is called a *mild solution* of (5.20) if $u \in C([0, \infty); \mathcal{X})$, $\int_0^t u(s)\, ds \in D(\mathcal{L})$ for all $t \geq 0$, and

$$u(t) - u_0 = \mathcal{L} \int_0^t u(s)\, ds.$$

From Proposition 5.15, we see that the C_0-semigroup with generator $\mathcal{L}$ yields solutions of (5.20).

Proposition 5.30. *Let $\mathcal{L}$ be the generator of a C_0-semigroup $T(t)$. Then, the following statements hold.*

(a) For every $u_0 \in D(\mathcal{L})$, $u(t) := T(t)u_0$ is the unique (classical) solution of (5.20).

(b) For every $u_0 \in \mathcal{X}$, $u(t) := T(t)u_0$ is the unique mild solution of (5.20).

5.3.7 Application: Linear Stability of Nonlinear Waves

Typically, a family of nonlinear traveling waves are realized as a stationary point of the nonlinear PDEs in the moving frame with the speed of the family of traveling waves. Let $\mathcal{L}$ be the linearized operator around a fixed point of a nonlinear operator $\mathcal{F}$. We consider the initial value problem (5.20) with $\mathcal{L}$ on a Hilbert space $\mathcal{H}$. For simplicity, we assume that $\sigma_{\text{pt}}(\mathcal{L}) = \{0\}$. Such a case is typical when the PDE under consideration has the translation invariance. We further assume that

(i) $\mathcal{L} : D(\mathcal{L}) \subset \mathcal{H} \to \mathcal{H}$ generates a C_0-semigroup.

(ii) $\lambda = 0$ is an isolated eigenvalue of $\mathcal{L}$ with algebraic multiplicity $k \geq 1$.

(iii) $(\lambda - \mathcal{L})^{-1}$ is uniformly bounded on $\operatorname{Re} \lambda > 0$, outside any small neighbourhood of the origin.

[43] $u(t)$ is considered as a Banach space valued function.

From the assumption (ii), we can choose a sufficiently small disk U_0 containing $\lambda = 0$ so that $U_0 \setminus \{0\} \subset \rho(\mathcal{L})$. We define the spectral projection

$$P := \frac{1}{2\pi i} \oint_{C_0} (\lambda I - \mathcal{L})^{-1} \, d\lambda, \tag{5.21}$$

where C_0 is a positively oriented simple closed curve enclosing the interior of U_0. The range of P is the k-dimensional generalized eigenspace of $\mathcal{L}$ associated with $\lambda = 0$.

From the spectral decomposition theorem (Theorem 5.10), P and its complementary projection $Q := I - P$ satisfy the following properties:

(a) $P : \mathcal{H} \to \mathcal{R}(P), \quad PP = P$ on $\mathcal{H}, \quad P\mathcal{L} = \mathcal{L}P$ on $D(\mathcal{L})$;

(b) $Q : \mathcal{H} \to \mathcal{R}(Q), \quad QQ = Q$ on $\mathcal{H}, \quad Q\mathcal{L} = \mathcal{L}Q$ on $D(\mathcal{L})$;

(c) $\mathcal{L}|_{\mathcal{R}(P)} = \mathcal{L} = \mathcal{L}P$ on $\mathcal{R}(P) \cap D(\mathcal{L}), \quad \mathcal{L}|_{\mathcal{R}(Q)} = \mathcal{L} = \mathcal{L}Q$ on $\mathcal{R}(Q) \cap D(\mathcal{L})$;

(d) $\sigma\left(\mathcal{L}|_{\mathcal{R}(P)}\right) = \{0\} = \sigma_{\mathrm{pt}}(\mathcal{L}), \quad \sigma\big(\mathcal{L}|_{\mathcal{R}(Q)}\big) = \sigma(\mathcal{L}) \setminus \{0\} = \sigma_{\mathrm{ess}}(\mathcal{L})$;

Moreover, the operator P satisfying these properties is unique since $\mathcal{N}(P)$ and $\mathcal{R}(P)$ are uniquely determined.

From the assumption (i), $u(t) := e^{\mathcal{L}t} u_0$ is the solution to (5.20) in a suitable sense depending the regularity of the initial data u_0. Since $P + Q = I$, we have

$$\partial_t (Pu + Qu) = \mathcal{L}(Pu + Qu), \quad u(0) = Pu_0 + Qu_0.$$

Applying P and Q, and then using the commutativity of $\mathcal{L}$ with P and Q, we get

$$\begin{cases} \partial_t Pu = P\mathcal{L}(Pu + Qu) = \mathcal{L}Pu, & Pu(0) = Pu_0, \\ \partial_t Qu = Q\mathcal{L}(Pu + Qu) = \mathcal{L}Qu, & Qu(0) = Qu_0, \end{cases}$$

since $PQ = QP = 0$, $P^2 = P$, and $Q^2 = Q$. Since $\mathcal{R}(P) =$ and $\mathcal{R}(Q)$ are invariant subspaces under $\mathcal{L}$, we obtain two decoupled system for $u^+ := Pu$ and $u^- := Qu$,

$$\partial_t u^+ = \mathcal{L}|_{\mathcal{R}(P)} u^+, \quad u^+(0) = u_0^+ := Pu_0 \in \mathcal{R}(P), \tag{5.22a}$$

$$\partial_t u^- = \mathcal{L}|_{\mathcal{R}(Q)} u^-, \quad u^-(0) = u_0^- := Qu_0 \in \mathcal{R}(Q). \tag{5.22b}$$

The equation (5.22a) describes the dynamics on the k-dimensional invariant subspace $\mathcal{R}(P)$, and the equation (5.22b) represents the dynamics on the complementary infinite-dimensional invariant subspace $\mathcal{R}(Q)$.

The C_0-semigroup $e^{\mathcal{L}t}$ is also decomposed as follows:

$$e^{\mathcal{L}t} = e^{\mathcal{L}|_{\mathcal{R}(P)} t} + e^{\mathcal{L}|_{\mathcal{R}(Q)} t},$$

where $e^{\mathcal{L}|_{\mathcal{R}(P)} t}$ and $e^{\mathcal{L}|_{\mathcal{R}(Q)} t}$ are the C_0-semigroups generated by the parts $\mathcal{L}|_{\mathcal{R}(P)}$ and $\mathcal{L}|_{\mathcal{R}(Q)}$, respectively, and thus, the solution operators of (5.22a) and (5.22b), respectively, in a suitable sense.

The resolvent of the part $\mathcal{L}|_{\mathcal{R}(Q)}$ is given as follows: on $\mathcal{R}(Q)$,

$$(\lambda - \mathcal{L}|_{\mathcal{R}(Q)})^{-1} = \begin{cases} (\lambda - \mathcal{L})^{-1}(I-P) = (\lambda - \mathcal{L})^{-1} & \text{for } \lambda \in \rho(\mathcal{L}), \\ \\ \text{analytic extension of } (\lambda - \mathcal{L})^{-1}(I-P) \text{ inside the curve } C_0 \\ \text{for } \lambda \in \{\text{the inside region of } C_0\}. \end{cases}$$

Since the resolvent is analytic, the assumption (iii) implies that $(\lambda - \mathcal{L}|_{\mathcal{R}(Q)})^{-1} : \mathcal{R}(Q) \to \mathcal{R}(Q)$ satisfies the uniform boundedness condition (5.19), and hence we conclude that there exists $C > 0$ such that for $u_0^- \in \mathcal{R}(Q)$,

$$\|e^{\mathcal{L}|_{\mathcal{R}(Q)}t} u_0^-\|_{\mathcal{H}} \le e^{-Ct}\|u_0^-\|_{\mathcal{H}} \quad \text{for all } t \ge 0. \tag{5.23}$$

To illustrate the finite-dimensional dynamics, we suppose for instance that $\lambda = 0$ is an isolated eigenvalue with algebraic multiplicity two, and that

$$\mathcal{L}u_1 = 0, \quad \mathcal{L}u_2 = u_1.$$

Then, the behavior of two-dimensional dynamics $u^+(t)$ is characterized by

$$u^+(t) = c_1 u_1 + c_2(u_2 + tu_1).$$

where c_1 and c_2 is determined by solving $c_1u_1 + c_2u_2 = u^+(0) = Pu_0$.

Definition 5.15. Let $\mathcal{L}$ be the linearized operator around a fixed point of a nonlinear operator $\mathcal{F}$. We say that the fixed point of $\mathcal{F}$ is *spectrally stable* if $\sigma(\mathcal{L}) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \le 0\}$. We say that the fixed point of $\mathcal{F}$ is *linearly asymptotically stable modulo* $\mathcal{R}(P)$ if the C_0-semigroup generated by $\mathcal{L}|_{\mathcal{R}(\mathcal{Q})}$ satisfies (5.23).

Remark 16. If $\sigma_{\text{pt}}(\mathcal{L}) = \{\lambda_0\}$, then the point spectrum of the adjoint operator $\mathcal{L}^*$ is $\sigma_{\text{pt}}(\mathcal{L}^*) = \{\overline{\lambda}_0\}$. The generalized eigenspace $\mathbb{E}$ of $\mathcal{L}$ corresponding to λ_0 is the range of the spectral projection operator P. The adjoint operator of P, denoted by P^*, is the spectral projection onto the generalized eigenspace $\mathbb{E}^*$ of $\mathcal{L}^*$, and we have $\dim\mathbb{E} = \dim\mathbb{E}^*$ (see [20]). Moreover, since

$$\mathcal{R}(Q) = \mathcal{R}(I-P) = \mathcal{N}(P) = \mathcal{R}(P^*)^{\perp} = \mathbb{E}^{*\perp},$$

we have a decomposition

$$\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{R}(Q) = \mathbb{E} \oplus \mathbb{E}^{*\perp}.$$

5.4 Reformulation of Eigenvalue Problem

It is often useful to study the eigenvalue problem of an operator $\mathcal{L}$ by reformulating it to the associated system of linear first-order ODEs (5.24). We consider the operator

$$\mathcal{A}(\lambda) := \tfrac{d}{dx} - A(x,\lambda) : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X},$$

where $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$, and $A \in \mathbb{C}^{n\times n}$ is a matrix. Here either $\mathcal{X} = L^2(\mathbb{R},\mathbb{C}^n)$ with $D(\mathcal{A}) = H^1(\mathbb{R},\mathbb{C}^n)$ or $\mathcal{X} = C_b(\mathbb{R},\mathbb{C}^n)$ with $D(\mathcal{A}) = C_b^1(\mathbb{R},\mathbb{C}^n)$ can be taken. In many applications, the similar statements as Proposition 4.3, which relates the Fredholm properties of $\lambda - \mathcal{L}$ and those of $\mathcal{A}(\lambda)$, hold true. Typically, $\mathcal{A}(\lambda)$ has the form of

$$\mathcal{A}(\lambda) = \tfrac{d}{dx} - A_1(x) - \lambda A_2(x).$$

For simplicity, we suppose that $\mathcal{A}(\lambda)$ is Fredholm with index zero and $\dim \mathcal{N}(\mathcal{A}(\lambda)) = 1$. We further assume that there is a set of functions $\{\mathbf{y}_j\}_{j=1}^k \subset D(\mathcal{A})$ such that

$$\left(\tfrac{d}{dx} - A_1 - \lambda A_2\right)\mathbf{y}_j = A_2\mathbf{y}_{j-1} \quad \text{for } j = 2,\cdots,k,$$

but there is no function $\mathbf{y} \in D(\mathcal{A})$ satisfying

$$\left(\tfrac{d}{dx} - A_1 - \lambda A_2\right)\mathbf{y} = A_2\mathbf{y}_k.$$

Then the algebraic multiplicity of an eigenvalue of $\mathcal{L}$ typically coincides with the length of the longest possible chain $\{\mathbf{y}_1,\cdots,\mathbf{y}_k\}$.

Now our interest is to study the invertibility of the operator $\mathcal{A}(\lambda)$. The Fredholm properties of $\mathcal{A}(\lambda)$ will be characterized in terms of exponential dichotomy.

5.4.1 Exponential Dichotomies

We consider the linear systems of ODEs

$$\frac{d\mathbf{y}}{dx} = A(x,\lambda)\mathbf{y}, \tag{5.24}$$

where $x \in I = (a,b) \subseteq \mathbb{R}$, $\lambda \in \mathbb{C}$, $A \in \mathbb{C}^{n\times n}$.

Theorem 5.31 (Initial value problem)**.** *Suppose that for fixed λ, $A(x,\lambda)$ is continuous in $x \in I$. Then for each $x_0 \in I$, and $\mathbf{y}_0 \in \mathbb{C}^n$, there exists a unique solution to the initial value problem of (5.24) with the initial value $\mathbf{y}(x_0,\lambda) = \mathbf{y}_0$. If $A(x,\lambda)$ and $\mathbf{y}_0(\lambda)$ are analytic in λ for each x, then the solution $\mathbf{y}(x,\lambda)$ is also analytic in λ for each x.*

We refer to ([18], Chapter 5) for more details on the initial value problem of the system of ODEs such as dependence on parameters and initial conditions. Throughout Section 5.4.1, we will suppress the λ dependence whenever it is not important.

Let $\{\mathbf{v}_1,\cdots,\mathbf{v}_n\}$ be a basis of $\mathbb{C}^n$ and $\{\mathbf{y}_1(x),\cdots,\mathbf{y}_n(x)\}$ be a set of solutions of (5.24) with $\mathbf{y}_i(x_0) = \mathbf{v}_i$. By the uniqueness, this set of solutions is a basis of $\mathbb{C}^n$ for each x. The matrix defined by

$$\Phi(x) := (\mathbf{y}_1(x),\cdots,\mathbf{y}_n(x))$$

is a matrix-valued solution to (5.24), and it is called a *fundamental matrix*. We note that

$$\Phi(x;x_0) := \Phi(x)\Phi^{-1}(x_0)$$

is also a matrix-valued solution to (5.24) satisfying $\Phi(x_0; x_0) = I$. If $\Phi(x)$ is a fundamental matrix of (5.24) satisfying $\Phi(x_0) = I$, then for any vector $\mathbf{y}_0$,

$$\mathbf{y}(x) := \Phi(x; x_0)\mathbf{y}_0$$

is the unique solution to the initial value problem (5.24) with $\mathbf{y}(x_0) = \mathbf{y}_0$.

Definition 5.16. Let $\Phi(x)$ be a fundamental matrix of (5.24). We say that the system (5.24) has an *exponential dichotomy on* I, if there is a projection $P : \mathbb{C}^n \to \mathbb{C}^n$ and some constants $K, \alpha > 0$ such that

$$|\Phi(x)P\Phi(y)^{-1}| \le Ke^{-\alpha(x-y)}, \quad x > y, \tag{5.25a}$$

$$|\Phi(x)(I-P)\Phi(y)^{-1}| \le Ke^{-\alpha(y-x)}, \quad y > x, \tag{5.25b}$$

for all $x, y \in I$.

Roughly speaking, the existence of an exponential dichotomy implies that the solution space of the system is decomposed into two subspaces: a subspace of solutions exponentially decays to zero as $x \to +\infty$ and a subspace of solutions exponentially grows as $x \to +\infty$. In Section 5.1, we have observed that when the coefficient matrix A is independent of x, the system (5.24) has an exponential dichotomy on I if and only if A is hyperbolic. In this case, $\Phi(x) = e^{Ax}$ and the projection P can be taken as the spectral projection P^s onto the stable eigenspace $\mathbb{E}^s$ defined in (5.3).

To see the meaning of an exponential dichotomy more precisely, we observe that (5.25) implies that (apply $\Phi(y)P\mathbf{y}$ and $\Phi(y)(I-P)\mathbf{y}$)

$$|\Phi(x)P\mathbf{y}| \le Ke^{-\alpha(x-y)}|\Phi(y)P\mathbf{y}| \quad \text{for} \quad x > y, \tag{5.26a}$$

$$|\Phi(x)(I-P)\mathbf{y}| \le Ke^{-\alpha(y-x)}|\Phi(y)(I-P)\mathbf{y}| \quad \text{for} \quad y > x, \tag{5.26b}$$

where $\mathbf{y} \in \mathbb{C}^n$ is an arbitrary constant vector.[44]

Suppose that $I = \mathbb{R}_+$ and P has rank k. Then, the first condition (5.26a) says that there is a k-dimensional subspace of solutions exponentially decays to zero as $x \to +\infty$. The second condition (5.26b) says that there is a complementary $(n-k)$-dimensional subspace of solutions exponentially grows to infinity as $x \to +\infty$.[45]

We suppose that the system (5.24) has exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$ with projections, denoted by P_+ and P_-, respectively, and the fundamental matrix $\Phi(x)$ satisfying $\Phi(0) = I$. On $\mathbb{R}_+$, the set of initial conditions $\mathbf{y}_0$ such that $\mathbf{y}(x) = \Phi(x)\mathbf{y}_0$ decays exponentially as $x \to +\infty$ is given by $\mathcal{R}(P_+)$. The set of initial conditions $\mathbf{y}_0$ such that $\mathbf{y}(x) = \Phi(x)\mathbf{y}_0$ exponentially grows as $x \to +\infty$ is given by $\mathcal{N}(P_+)$. On $\mathbb{R}_-$, the set of initial conditions $\mathbf{y}_0$ such that $\mathbf{y}(x) = \Phi(x)\mathbf{y}_0$ decays exponentially as $x \to -\infty$ is given by $\mathcal{R}(I - P_-) = \mathcal{N}(P_-)$.

[44] Indeed, (5.26) is a part of the conditions equivalent to (5.25). See Coppel [7], Chapter 2.

[45] Fix x and let $y \to +\infty$.

The set of initial conditions $\mathbf{y}_0$ such that $\mathbf{y}(x) = \Phi(x)\mathbf{y}_0$ exponentially grows as $x \to -\infty$ is given by $\mathcal{N}(I - P_-) = \mathcal{R}(P_-)$.

The dimension of the kernel of the projection is reffered to as the *Morse index* of the exponential dichotomy. If the system (5.24) has exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$, we denote the Morse indices by $m_+(\lambda) := \dim \mathcal{N}(P_+)$ and $m_-(\lambda) := \dim \mathcal{N}(P_-)$, respectively.

Remark 17. If the ODE system has an exponential dichotomy on $I = [x_0, \infty)$, where $x_0 > 0$, with a projection P, then it has an exponential dichotomy on $I = \mathbb{R}_+$ with the same projection P. (See [7], Chapter 2, p.13.) If $\widetilde{P}$ is another projection with $\mathcal{R}(P) = \mathcal{R}(\widetilde{P})$, then the ODE system also has an exponential dichotomy with the projection $\widetilde{P}$. (See [7], Chapter 2, p.15.)

The projection P of an exponential dichotomy is not unique in general. While $\mathcal{R}(P)$ is uniquely determined as the subspace of initial conditions of bounded solutions, there is no unique choice for $\mathcal{N}(P)$ since the direct sum decomposition of a vector space is not unique. However, if the ODE system has an exponential dichotomy on $I = \mathbb{R}$, then the projection P is uniquely determined. This is because $\mathcal{R}(P)$ must be the subspace of initial conditions of bounded solutions on $\mathbb{R}_+$ and $\mathcal{N}(P)$ must be the subspace of initial conditions of bounded solutions on $\mathbb{R}_-$. In other word, every projection satisfying (5.25) has the same kernel and the same range. Lemma 5.5 completes the claim.

Remark 18. We suppose that the system (5.24) has exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$. It is clear that $\mathcal{N}(P_-) \cap \mathcal{R}(P_+) \neq \{\mathbf{0}\}$ is equivalent to that the system (5.24) has a non-trivial solution decaying exponentially as $|x| \to +\infty$. On the other hand, if $\mathcal{N}(P_-) \cap \mathcal{R}(P_+) = \{\mathbf{0}\}$ and $\dim \mathcal{N}(P_-) + \dim \mathcal{R}(P_+) = n$, then we have $P = P_- = P_+$ due to Lemma 5.5. Hence, (5.24) possesses an exponential dichotomy on $\mathbb{R}$ with the (unique) projection P. The converse is also true. In this case,

$$G(x, y) := \begin{cases} \Phi(x) P \Phi^{-1}(y), & x > y, \\ -\Phi(x)(I - P)\Phi^{-1}(y), & y > x \end{cases} \tag{5.27}$$

is the Green's function of the system (5.24) on $I = \mathbb{R}$ with the boundary condition $G(x, \cdot) \to 0$ as $|x| \to +\infty$. Indeed, $G(x, y)$ satisfies

$$\begin{cases} \dfrac{d}{dx} G(x, y) = A(x) G(x, y), & x \neq y, \\ G(y^+, y) - G(y^-, y) = I. \end{cases} \tag{5.28}$$

Moreover, we have

$$\sup_{y \in \mathbb{R}} \int |G(x, y)|\, dx \le C, \quad \sup_{x \in \mathbb{R}} \int |G(x, y)|\, dy \le C$$

for some constant $C > 0$. By the generalized Young's inequality for integral operator, for any given $\mathbf{f} \in L^2(\mathbb{R})$ (or $C_b(\mathbb{R})$),

$$\mathbf{y}(x) := \int_{-\infty}^{\infty} G(x, \cdot)\mathbf{f}(\cdot)\, dy = \int_{-\infty}^{x} \Phi(x) P \Phi(\cdot)^{-1}\mathbf{f}(\cdot)\, dy - \int_{x}^{\infty} \Phi(x)(I - P)\Phi(\cdot)^{-1}\mathbf{f}(\cdot)\, dy$$

satisfies

$$\|\mathbf{y}\|_{L^2} \leq C\|\mathbf{f}\|_{L^2} \quad (\text{or } \|\mathbf{y}\|_{L^\infty} \leq C\|\mathbf{f}\|_{L^\infty}),$$

and

$$\frac{d\mathbf{y}}{dx} - A(x)\mathbf{y} = \mathbf{f}. \tag{5.29}$$

Hence, $\mathbf{y}$ is a unique solution to the problem (5.29), and $\mathbf{y} \in H^1(\mathbb{R})$ (or $C_b^1(\mathbb{R})$).

The following theorem says that the exponential dichotomy is stable under small perturbations of the system (5.24). In particular, the Morse indicies $i_\pm(\lambda)$ are preserved under small perturbations.

Theorem 5.32 (Roughness of exponential dichotomies, [7], Chapter 4)**.** *Suppose that the system* (5.24) *has an exponential dichotomy* (5.25) *on* $\mathbb{R}_+$ *with the fundamental matrix* $\Phi(x)$ *such that* $\Phi(0) = I$. *If*

$$\delta = \sup_{x\in\mathbb{R}_+} |B(x)| < \alpha/4K^2,$$

then the perturbed system

$$\frac{d\mathbf{y}}{dx} = (A(x) + B(x))\mathbf{y}$$

also has an exponential dichotomy on $\mathbb{R}_+$*:*

$$|\tilde{\Phi}(x)\tilde{P}\tilde{\Phi}(y)^{-1}| \leq (5/2)K^2 e^{-(\alpha-2K\delta)(x-y)}, \quad x > y \geq 0,$$
$$|\tilde{\Phi}(x)(I-\tilde{P})\tilde{\Phi}(y)^{-1}| \leq (5/2)K^2 e^{-(\alpha-2K\delta)(y-x)}, \quad y > x \geq 0,$$

where $\tilde{\Phi}$ *is the fundamental matrix for the perturbed system with* $\tilde{\Phi}(0) = I$*, and* $\tilde{P}$ *is a projection such that* $\mathcal{N}(\tilde{P}) = \mathcal{N}(P)$*. Moreover, for all* $x \geq 0$*,*

$$|\tilde{\Phi}(x)\tilde{P}\tilde{\Phi}(x)^{-1} - \Phi(x)P\Phi(x)^{-1}| \leq 4\alpha^{-1}K^3\delta.$$

Remark 19. Similar statements to Theorem 5.32 also holds on the intervals $I = \mathbb{R}_-$, $[x_0, +\infty)$, $(-\infty, x_0]$, $\mathbb{R}$. We refer to [23] for the roughness of the exponential dichotomy (Theorem 5.32) under the perturbation $B(x)$ with $\lim_{x\to+\infty} B(x) \to 0$.

5.4.2 Exponential Dichotomies and the Fredholm Properties

Theorem 5.33 (Palmer,[23],[24])**.** *Consider the operator*

$$\mathcal{A}(\lambda) = \tfrac{d}{dx} - A(x,\lambda) : H^1(\mathbb{R},\mathbb{C}^n) \subset L^2(\mathbb{R},\mathbb{C}^n) \to L^2(\mathbb{R},\mathbb{C}^n),$$

where $\lambda \in \mathbb{C}$*. Then* $\mathcal{A}(\lambda)$ *is Fredholm if and only if the system* (5.24) *possesses exponential dichotomies on* $\mathbb{R}_\pm$*. Moreover, the Fredholm index of* $\mathcal{A}(\lambda)$ *is given by*

$$\operatorname{ind}(\mathcal{A}(\lambda)) = \dim\mathcal{R}(P_+(\lambda)) + \dim\mathcal{N}(P_-(\lambda)) - n.$$

Remark 20. The following proof is a modification of the Palmer's original result, where the space $\mathcal{X} = C_b(\mathbb{R}, \mathbb{R}^n)$ with $D(\mathcal{A}) = C_b^1(\mathbb{R}, \mathbb{R}^n)$ is considered. We apply the generalized Young's inequality with different exponents and consider the complex adjoint system. The similar statement for the case $\mathcal{X} = C_b(\mathbb{R}, \mathbb{C}^n)$ with $D(\mathcal{A}) = C_b^1(\mathbb{R}, \mathbb{C}^n)$ also holds.

Proof. We only prove the '*if*' statement. We refer to [24] for the other direction. We omit λ dependence for simplicity.

Suppose that the system (5.24) possesses exponential dichotomies (5.25) on $\mathbb{R}_\pm$ with the fundamental matrix $\Phi(x)$ satisfying $\Phi(0) = I$ and the associated projection operators $P_\pm$, respectively. Then, we have

$$\dim \mathcal{N}(\mathcal{A}) = \dim \left(\mathcal{N}(P_-) \cap \mathcal{R}(P_+)\right) < \infty \tag{5.31}$$

since all bounded solutions of the linear ODE system (5.24) (or equivalently, all elements of the kernel of $\mathcal{A}$) with linearly independent initial values must be linearly independent by the uniqueness. To prove that $\mathcal{R}(\mathcal{A})$ is closed and that $\text{codim}\mathcal{R}(\mathcal{A})$ is finite, we first claim that $f \in \mathcal{R}(\mathcal{A})$ if and only if

$$\int_{-\infty}^{\infty} \psi^* f \, dx = 0 \tag{5.32}$$

for all bounded (hence exponentially decaying) solution $\psi(x)$ of the (complex) adjoint system

$$\frac{d\mathbf{z}}{dx} = -A^*(x)\mathbf{z}, \tag{5.33}$$

where $A^* = \overline{A^T}$.

By taking the derivative of $\Phi(x)\Phi^{-1}(x) = I$, one can check that $(\Phi^{-1})^*(x)$ is the fundamental matrix of (5.33). Moreover, by taking the adjoint of (5.25), we see that (5.33) possesses exponential dichotomies (5.25) on $\mathbb{R}_\pm$ with projections $I - (P_\pm)^*$.[46] Thus, the subspace of initial values of bounded solutions to the adjoint system (5.33) is

$$\mathcal{N}(I - P_-^*) \cap \mathcal{R}(I - P_+^*) = \mathcal{R}(P_+)^\perp \cap \mathcal{N}(P_-)^\perp, \tag{5.34}$$

where the equality comes from

$$\mathcal{N}(I - P_-^*) = \mathcal{R}(P_-^*) = \mathcal{N}(P_-)^\perp \quad \text{and} \quad \mathcal{R}(I - P_+^*) = \mathcal{N}(P_+^*) = \mathcal{R}(P_+)^\perp.$$

If $f \in \mathcal{R}(\mathcal{A}) \in L^2$, then there is $\mathbf{y} \in H^1$ (hence bounded on $\mathbb{R}$) such that

$$\frac{d\mathbf{y}}{dx} - A(x)\mathbf{y} = f(x).$$

For all bounded (hence exponentially decaying) solutions $\psi(x)$ of (5.33) we have

$$\begin{aligned}\int_{-\infty}^{\infty} \psi^* f \, dx &= \int_{-\infty}^{\infty} \psi^* \partial_x \mathbf{y} - \psi^* A\mathbf{y} \, dx \\ &= \int_{-\infty}^{\infty} \psi^* \partial_x \mathbf{y} + \partial_x \psi^* \mathbf{y} \, dx = 0.\end{aligned}$$

[46] $\mathcal{R}P_+$(stable), $\mathcal{R}(I - P_+)$(unstable) $\leftrightarrow$ $\mathcal{R}P_+^*$(unstable), $\mathcal{R}(I - P_+^*)$(stable)

Conversely, we suppose that $f \in L^2$. Since (5.24) has exponential dichotomies, for any given initial value $\mathbf{y}_0$,

$$\mathbf{y}^+(x) := \Phi(x)P_+\mathbf{y}_0 + \int_0^x \Phi(x)P_+\Phi^{-1}(\cdot)f(\cdot)\,dy - \int_x^\infty \Phi(x)(I-P_+)\Phi^{-1}(\cdot)f(\cdot)\,dy,$$
$$\mathbf{y}^-(x) := \Phi(x)(I-P_-)\mathbf{y}_0 + \int_{-\infty}^x \Phi(x)P_-\Phi^{-1}(\cdot)f(\cdot)\,dy - \int_x^0 \Phi(x)(I-P_-)\Phi^{-1}(\cdot)f(\cdot)\,dy,$$

are the solutions of (5.24) in $C^1(\mathbb{R}_+) \cap H^1(\mathbb{R}_+)$ and $C^1(\mathbb{R}_-) \cap H^1(\mathbb{R}_-)$ respectively. (See Remark 18.) Our goal is now to find $\mathbf{y}_0$ such that $\mathbf{y}^-(0) = \mathbf{y}^+(0)$ holds. Equivalently, we find $\mathbf{y}_0$ satisfying

$$\left(P_+ - (I-P_-)\right)\mathbf{y}_0 = \int_0^\infty (I-P_+)\Phi^{-1}f\,dy + \int_{-\infty}^0 P_-\Phi^{-1}f\,dy \in \mathbb{C}^n. \tag{5.36}$$

Such $\mathbf{y}_0$ exists when the RHS of (5.36) is in $\left[\mathcal{N}\left(P_+^* - (I-P_-^*)\right)\right]^\perp$.

For all $\mathbf{q}$ with $\left(P_+ - (I-P_-)\right)\mathbf{q} = 0$, we define

$$\psi(x) := \begin{cases} (\Phi^{-1})^*(x)(I-P_+^*)\mathbf{q}, & x \geq 0, \\ (\Phi^{-1})^*(x)P_-^*\mathbf{q}, & x \leq 0. \end{cases} \tag{5.37}$$

Then, $\psi(x)$ is a exponentially decaying solution of (5.33) (see (5.34)). From the assumption (5.32), it follows that

$$0 = \int_{-\infty}^\infty \psi^* f\,dx = \mathbf{q}^*\left[\int_0^\infty (I-P_+)\Phi^{-1}f\,dx + \int_{-\infty}^0 P_-\Phi^{-1}f\,dx\right]$$

for all $\mathbf{q} \in \mathcal{N}\left(P_+ - (I-P_-)\right)$. This proves the claim.

To finish the proof, we take a basis $\{\psi_i\}$ of the finite dimensional subspace $\mathcal{N}(d/dx + A^*(x))$, which is isomorphic to (5.34), and define a linear bounded mapping

$$\mathcal{T} : f \in L^2 \mapsto \left(\int_{-\infty}^\infty \psi_1 f\,dx, \cdots, \int_{-\infty}^\infty \psi_j f\,dx\right) \in \mathbb{C}^n.$$

From what we just have proved, the kernel of $\mathcal{T}$ is exactly $R(\mathcal{A})$. Since $\mathcal{T}$ is continuous, the kernel of $\mathcal{T}$ is closed, and hence $R(\mathcal{A})$ is closed. Since $\mathcal{R}(\mathcal{A})^\perp = \mathcal{N}(\mathcal{A}^*)$, we have[47]

$$\text{codim}\mathcal{R}(\mathcal{A}) = \dim\mathcal{R}(\mathcal{A})^\perp = \dim\mathcal{N}(\mathcal{A}^*) = \dim\left[\mathcal{N}(P_-)^\perp \cap \mathcal{R}(P_+)^\perp\right] < \infty,$$

where the last equality is from (5.34). Since

$$\begin{aligned} \text{codim}\mathcal{R}(\mathcal{A}) &= \dim\left[\mathcal{N}(P_-)^\perp \cap \mathcal{R}(P_+)^\perp\right] \\ &= \dim\left[\mathcal{N}(P_-) + \mathcal{R}(P_+)\right]^\perp \\ &= n - \dim\left[\mathcal{N}(P_-) + \mathcal{R}(P_+)\right] \\ &= n - \left(\dim\mathcal{N}(P_-) + \dim\mathcal{R}(P_+) - \dim\left[\mathcal{N}(P_-) \cap \mathcal{R}(P_+)\right]\right), \end{aligned} \tag{5.38}$$

[47] See [20], Chapter 3, Lemma 1.40, for the first equality.

we finally obtain using (5.31) that

$$\dim \mathcal{N}(\mathcal{A}) - \operatorname{codim}\mathcal{R}(\mathcal{A}) = \dim \mathcal{N}(P_-) + \dim \mathcal{R}(P_+) - n.$$

□

We note that the Fredholm index and the Morse indicies are related as follows:

$$\operatorname{ind}(\mathcal{A}(\lambda)) = \dim \mathcal{N}(P_-) - \dim \mathcal{N}(P_+) = m_-(\lambda) - m_+(\lambda).$$

From Theorem 5.33, we have the following statements for the system (5.24):

1. $\mathcal{A}(\lambda)$ is not Fredholm $\iff$ (5.24) does not have an exponential dichotomy on $\mathbb{R}_+$ or $\mathbb{R}_-$;
2. $\operatorname{ind}(\mathcal{A}(\lambda)) = k \neq 0$ $\iff$ (5.24) has exponential dichotomies on $\mathbb{R}_\pm$ and $m_-(\lambda) - m_+(\lambda) = k \neq 0$;
3. $\operatorname{ind}(\mathcal{A}(\lambda)) = 0$ and $\mathcal{N}(\mathcal{A}(\lambda)) = \{\mathbf{0}\}$ $\iff$ (5.24) has exponential dichotomies on $\mathbb{R}_\pm$ with $m_+(\lambda) = m_-(\lambda)$ and $\mathcal{R}(P_+(\lambda)) \cap \mathcal{N}(P_-(\lambda)) = \{\mathbf{0}\}$;
4. $\operatorname{ind}(\mathcal{A}(\lambda)) = 0$ and $\mathcal{N}(\mathcal{A}(\lambda)) \neq \{\mathbf{0}\}$ $\iff$ (5.24) has exponential dichotomies on $\mathbb{R}_\pm$ with $m_+(\lambda) = m_-(\lambda)$ and $\mathcal{R}(P_+(\lambda)) \cap \mathcal{N}(P_-(\lambda)) \neq \{\mathbf{0}\}$.

Different asymptotic matrices: front type waves Suppose that $A(x, \lambda)$ has different asymptotic matrices as $x \to \pm\infty$. This is typical when one consider the stability of front-type traveling waves such as viscous shock waves. Let $A_\infty^\pm(\lambda) := \lim_{x\to\pm\infty} A(x, \lambda)$. By the roughness of exponential dichotomies (Theorem 5.32), we have the following statements:

1. $\mathcal{A}(\lambda)$ is not Fredholm $\iff$ one of $A_\infty^\pm(\lambda)$ is not hyperbolic;
2. $\operatorname{ind}(\mathcal{A}(\lambda)) = k \neq 0$ $\iff$ $A_\infty^\pm(\lambda)$ are both hyperbolic with $m_-(\lambda) - m_+(\lambda) = k \neq 0$;
3. $\operatorname{ind}(\mathcal{A}(\lambda)) = 0$ and $\mathcal{N}(\mathcal{A}(\lambda)) = \{\mathbf{0}\}$ $\iff$ $A_\infty^\pm(\lambda)$ are both hyperbolic with $m_+(\lambda) = m_-(\lambda)$ and $\mathcal{R}(P_+(\lambda)) \cap \mathcal{N}(P_-(\lambda)) = \{\mathbf{0}\}$;
4. $\operatorname{ind}(\mathcal{A}(\lambda)) = 0$ and $\mathcal{N}(\mathcal{A}(\lambda)) \neq \{\mathbf{0}\}$ $\iff$ $A_\infty^\pm(\lambda)$ are both hyperbolic with $m_+(\lambda) = m_-(\lambda)$ and $\mathcal{R}(P_+(\lambda)) \cap \mathcal{N}(P_-(\lambda)) \neq \{\mathbf{0}\}$.

Same asymptotic matrix: pulse type waves When one consider the stability of pulse-type traveling waves such as solitary waves, $A(x, \lambda)$ has the same asymptotic matrix as $x \to \pm\infty$. Let $A_\infty(\lambda) := \lim_{|x|\to+\infty} A(x, \lambda)$. In this case, we always have $m_+(\lambda) = m_-(\lambda)$, and hence, we have the following:

1. $\mathcal{A}(\lambda)$ is not Fredholm $\iff$ $A_\infty(\lambda)$ is not hyperbolic;

2. $\operatorname{ind}(\mathcal{A}(\lambda)) = 0$ and $\mathcal{N}(\mathcal{A}(\lambda)) = \{\mathbf{0}\} \iff A_\infty(\lambda)$ is hyperbolic and $\mathcal{R}(P_+(\lambda)) \cap \mathcal{N}(P_-(\lambda)) = \{\mathbf{0}\}$;

3. $\operatorname{ind}(\mathcal{A}(\lambda)) = 0$ and $\mathcal{N}(\mathcal{A}(\lambda)) \neq \{\mathbf{0}\} \iff A_\infty(\lambda)$ is hyperbolic and $\mathcal{R}(P_+(\lambda)) \cap \mathcal{N}(P_-(\lambda)) \neq \{\mathbf{0}\}$.

Remark 21. If one of $A^{\pm}_\infty(\lambda)$ is not hyperbolic, one may show that $\mathcal{R}(\partial_x - A^{\pm}_\infty(\lambda))$ is not closed using an oscillating solution. (See [35], Chapter 3)

5.5 The Evans Function

The *Evans function* was first introduced by Evans ([11]–[14]) for the study of stability of some class of traveling waves. The Evans function is an analytic function in the parameter λ, which is particularly useful for detecting eigenvalues and their algebraic multiplicities. Its zeros are related to the values of λ such that the subspace of solutions exponentially decays to zero as $x \to +\infty$ and the subspace of solutions exponentially decays as $x \to -\infty$ intersect non-trivially, that is, $\mathcal{R}(P_+(\lambda)) \cap \mathcal{N}(P_-(\lambda)) \neq \{\mathbf{0}\}$. We introduce the Evans function and its properties summarizing the formulation given in [26]. As an application, we discuss the instability of solitary waves for the generalized KdV equation in the last section.

We consider a linear ODE system

$$\frac{d\mathbf{y}}{dx} = A(x,\lambda)\mathbf{y} \tag{5.39}$$

and its associated transposed system

$$\frac{d\mathbf{z}}{dx} = -\mathbf{z}A(x,\lambda), \tag{5.40}$$

Here, $\lambda \in \mathbb{C}$ is a parameter, $A(x,\lambda)$ is a $n \times n$ matrix, $\mathbf{y}(x,\lambda)$ is a column vector, and $\mathbf{z}(x,\lambda)$ is a row vector.

We assume that on a simply connected domain $\Omega \subset \mathbb{C}$, **H1**–**H4** hold.

H1. $A(x,\lambda)$ is jointly continuous in $(x,\lambda) \in \mathbb{R} \times \Omega$ and is analytic in λ for each fixed x.

H2. $\lim_{x\to\pm\infty} A(x,\lambda) = A_\infty^\pm(\lambda)$ exist, and the limit is uniform in λ on any compact subset of Ω.

H3. $\int_{-\infty}^{\infty} |R(x,\lambda)|\,dx$ converges for all λ, uniformly on any compact subsets of Ω, where

$$R(x,\lambda) := \begin{cases} A(x,\lambda) - A_\infty^-(\lambda) & \text{for} \quad x < 0, \\ A(x,\lambda) - A_\infty^+(\lambda) & \text{for} \quad x \geq 0. \end{cases}$$

H4. For every $\lambda \in \Omega$, $A_\infty^\pm(\lambda)$ has a unique eigenvalue of smallest real part, which is simple. Equivalently, the eigenvalues $\mu_j^\pm(\lambda)$ of $A_\infty^\pm(\lambda)$ can be labelled so that

$$\operatorname{Re}\mu_1^\pm < \mu_*^\pm := \min\{\operatorname{Re}\mu_j^\pm : i = 2, \cdots, n\}. \tag{5.41}$$

Proposition 5.34. *For any solution* $\mathbf{y}$ *of* (5.39) *and* $\mathbf{z}$ *of* (5.40), $\mathbf{zy}$ *is independent of* x .

Indeed, we have

$$\frac{d(\mathbf{zy})}{dx} = -\mathbf{z}A\mathbf{y} + \mathbf{z}A\mathbf{y} = 0.$$

This is one of the advantages of considering the transposed system.

Notation: $A^+_\infty(\lambda)$ and $A^-_\infty(\lambda)$ are different in general. For notational simplicity, we suppress $\pm$ signs as long as there is no confusion. We use the notation $f(x) \sim g(x)$ as $x \to +\infty$ for $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = 1$.

5.5.1 Analytic Eigenvectors

By Morera's theorem, **H2** implies that $A_\infty(\lambda)$ is analytic on Ω. It is easy to see that $\mu_1(\lambda)$ is analytic on Ω since μ_1 simple from **H4**.[48]

We show that we can make an *analytic* choice of the normalized left and right eigenvectors of $A^\pm_\infty(\lambda)$ corresponding to the simple eigenvalue $\mu^\pm_1(\lambda)$ so that for all $\lambda \in \Omega$,

$$\left[A^\pm_\infty - \mu^\pm_1 I\right] \mathbf{v}^\pm(\lambda) = 0, \quad \mathbf{w}^\pm(\lambda)\left[A^\pm_\infty - \mu^\pm_1 I\right] = 0, \quad \mathbf{w}^\pm(\lambda)\mathbf{v}^\pm(\lambda) = 1. \tag{5.42}$$

We need the following lemma[49]:

Lemma 5.35 ([20])**.** *Let $P(\lambda)$ be a projection on a finite-dimensional vector space $\mathcal{X}$, analytic on a simply connected domain $\Omega \subset \mathbb{C}$. Then, for $\lambda_0 \in \Omega$, there exists an operator-valued function $U(\lambda)$ such that*

1. *$U(\lambda)$ is analytic in $\lambda \in \Omega$, and its inverse $U^{-1}(\lambda)$ exists for all $\lambda \in \Omega$ and is also analytic in $\lambda \in \Omega$,*
2. *$U(\lambda)P(\lambda_0) = P(\lambda)U(\lambda)$ for all $\lambda \in \Omega$.*

From **H4**, for any fixed $\lambda_0 \in \Omega$, we may choose the left and right eigenvectors, $\mathbf{w}_0 = \mathbf{w}(\lambda_0)$ and $\mathbf{v}_0 = \mathbf{v}(\lambda_0)$, of $A_\infty(\lambda_0)$ corresponding to the simple eigenvalue $\mu_1(\lambda_0)$ so that $\mathbf{w}_0\mathbf{v}_0 = 1$.[50] We define the operator

$$P(\lambda) := \int_\Gamma (A_\infty(\lambda) - \nu I)^{-1}\, d\nu,$$

where Γ is a positively oriented circle around $\mu_1(\lambda)$ excluding the other eigenvalues inside the circle. $P(\lambda)$ is analytic[51] in λ and is a projection onto the eigenspace of the simple eigenvalue $\mu_1(\lambda)$ with $\dim \mathrm{Im} P(\lambda) = 1$. From Lemma 5.35, we see that

$$\mathbf{v}(\lambda) := U(\lambda)\mathbf{v}_0, \quad \mathbf{w}(\lambda) := \mathbf{w}_0 U^{-1}(\lambda)$$

are the desired eigenvectors satisfying (5.42).

---

[48] For any fixed $\lambda_0 \in \Omega$ and $\mu_1 = \mu_1(\lambda_0)$, we have $\det(A_\infty(\lambda) - \mu I) = (\mu - \mu_1)\bar{d}(\mu, \lambda)$, where $\bar{d}(\mu_1, \lambda_0) \neq 0$. Hence, $\mu_1(\lambda)$ is analytic from the implicit function theorem since $\lim_{\mu\to\mu_1} \frac{\det(A_\infty(\lambda) - \mu I)}{\mu - \mu_1} = \bar{d}(\mu_1, \lambda_0) \neq 0$.

[49] The proof of Lemma 5.35 does not require the assumptions **H1**–**H4**.

[50] This is possible whenever the eigenvalue is semi-simple. Consider the Jordan normal form.

[51] Analyticity of $P(\lambda)$ has nothing to do with the multiplicity of the eigenvalue $\mu_1(\lambda)$. See [20], p.68.

Proof of Lemma 5.35. Let us denote $\frac{d}{d\lambda}$ by $'$. Since $P^2 = P$, we have

$$P'P + PP' = P'. \tag{5.43}$$

Multiplying (5.43) by P , we obtain $PP' = PP'P + PP'$. Hence, we get $PP'P = 0$. Let $Q(\lambda) := P'P - PP'$. We note that $PQ = -PP'$ and $QP = P'P$. From (5.43), we obtain

$$P' = QP - PQ. \tag{5.44}$$

Now we consider the operator-valued ODE

$$X' = Q(\lambda)X, \quad X(\lambda_0) = X_0. \tag{5.45}$$

Since Q is analytic on a simply connected domain D and the problem is linear, from the usual iteration argument and the analytic continuation, one may show that this problem has a unique analytic solution $X(\lambda)$ on D.

Let $U(\lambda)$ be the solution of the operator-valued ODE $U' = Q(\lambda)U$ with $U(\lambda_0) = I$, and let $V(\lambda)$ be the solution of $V' = -VQ(\lambda)$ with $V(\lambda_0) = I$. We note that $(VU)' = -VQU + VQU = 0$. Hence, $V(\lambda)U(\lambda) = V(\lambda_0)U(\lambda_0) = I$.

To prove the second statement, using (5.44), we see that

$$(PU)' = P'U + PQU = (P' + PQ)U = QPU.$$

Thus, $P(\lambda)U(\lambda)$ is a solution to the ODE (5.45) with $X_0 = P(\lambda_0)U(\lambda_0) = P(\lambda_0)I = P(\lambda_0)$. On the other hand, $U(\lambda)P(\lambda_0)$ is also a solution to (5.45) with $X_0 = U(\lambda_0)P(\lambda_0) = P(\lambda_0)$. The proof is done by the uniqueness of the ODE. □

Remark 22. Lemma 5.35 imples that if $\{\mathbf{v}_j\}$ is a basis of $P(\lambda_0)\mathcal{X}$, then $\{U(\lambda)\mathbf{v}_j\}$ forms an analytic basis of $P(\lambda)\mathcal{X}$.[52] Suppose that $\mu_1(\lambda)$ is not simple. In this case, for an eigenvector $\mathbf{v}_0$ of $A_\infty(\lambda)$ and a projection $P(\lambda)$ onto the eigenspace corresponding to $\mu_1(\lambda)$, we cannot guarantee that $U(\lambda)\mathbf{v}_0$ is an eigenvector of $A_\infty(\lambda)$.

Remark 23. $P(\lambda)\mathbf{v}_0$ is analytic, and it lies in the space spanned by the eigenvector of $\mu_1(\lambda)$. But, it may vanish at some $\lambda \in \Omega$. Thus, even if μ_1 is simple, $P(\lambda)\mathbf{v}_0$ may not be the eigenvector.

Remark 24. Consider the matrix

$$A(\lambda) = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix},$$

where $\lambda \in \mathbb{C}$. The eigenvalue of A is 0 with multiplicity two. When $\lambda = 0$, there are two linearly independent eigenvectors $(1, 0)^T$ and $(0, 1)^T$. When $\lambda \neq 0$, the only eigenvector is $(0, 1)^T$. Even if $A(\lambda)$ and its eigenvalues are all analytic, the eigenvector may not be analytic in general. As this example shows, eigenvectors may degenerate at some point.

[52] If $P(\lambda)$ is continuous on a connected domain Ω, then $P(\lambda)\mathcal{X}$ is isomorphic to $P(\lambda_0)\mathcal{X}$. In particular, $\dim P(\lambda)\mathcal{X} = \dim P(\lambda_0)\mathcal{X}$ for all $\lambda \in \Omega$. See [20] for more details.

5.5.2 Asymptotic Behavior of Solutions

Proposition 5.36. *There exist unique solutions* $\mathbf{y}^+$ *of* (5.39) *and* $\mathbf{z}^-$ *of* (5.40) *satisfying the following properties:*

(a) $\mathbf{y}^+(x,\lambda) \sim e^{\mu_1^+(\lambda)x}\mathbf{v}^+(\lambda)$ *as* $x \to +\infty$ *and* $\mathbf{z}^-(x,\lambda) \sim e^{-\mu_1^-(\lambda)x}\mathbf{w}^-(\lambda)$ *as* $x \to -\infty$*, and the limits are uniform in* λ *on any compact sets of* Ω*;*

(b) $\mathbf{y}^+(x,\lambda)$ *and* $\mathbf{z}^-(x,\lambda)$ *are analytic in* λ *for each* x*.*

Moreover, any solution of (5.39) *((5.40)) satisfying* $\mathbf{y} = O(e^{\mu_1^+ x})$ *as* $x \to +\infty$ *(*$\mathbf{z} = O(e^{-\mu_1^- x})$ *as* $x \to -\infty$*) is a constant multiple of* $\mathbf{y}^+$ *(*$\mathbf{z}^-$ *respectively.).*

Remark 25. One may expect that $\partial_\lambda \mathbf{y}^+ \sim \partial_\lambda \mu_1^+ x e^{\mu_1^+ x}\mathbf{v}^+$ as $x \to +\infty$. This is true and is justified as follows. Let $\mu = \mu_1^+(\lambda)$ for simplicity. Since the limit $e^{-\mu x}\mathbf{y}^+(x,\lambda) \to \mathbf{v}^+(\lambda)$ is uniform in λ (on any compact sets of Ω from Proposition 5.36), we have that $(e^{-\mu x}\mathbf{y}^+)_\lambda \to \mathbf{v}^+_\lambda$ as $x \to +\infty$ uniformly in λ. On the other hand, we have

$$0 = \lim_{x\to+\infty} \frac{(e^{-\mu x}\mathbf{y}^+)_\lambda - \mathbf{v}^+_\lambda}{x} = \lim_{x\to+\infty}\left(\frac{\mathbf{y}^+_\lambda e^{-\mu x}}{x} - \mathbf{y}^+\mu_\lambda e^{-\mu x} - \frac{\mathbf{v}^+_\lambda}{x}\right).$$

Since

$$\lim_{x\to+\infty}\left(\mathbf{y}^+\mu_\lambda e^{-\mu x} + \frac{\mathbf{v}^+_\lambda}{x}\right) = \mu_\lambda \mathbf{v}^+,$$

we get

$$\partial_\lambda \mathbf{y}^+(x,\lambda) \sim x e^{\mu_1^+ x}\partial_\lambda \mu_1^+ \mathbf{v}^+ \quad \text{as } x \to +\infty.$$

A similar procedure yields that for each $k \in \mathbb{N}$,

$$\partial_\lambda^j \mathbf{y}^+(x,\lambda) = O(e^{(\mu_1^+ + \delta)x}) \quad \text{as } x \to +\infty \tag{5.46}$$

for all sufficiently small $\delta > 0$.

Proof of Proposition 5.36. Let $\widetilde{\mathbf{y}}(x) := e^{-\mu_1^+ x}\mathbf{y}(x)$. Then, (5.39) becomes

$$\frac{d\widetilde{\mathbf{y}}}{dx} = [B(\lambda) + R(x,\lambda)]\,\widetilde{\mathbf{y}}, \tag{5.47}$$

wehre $B(\lambda) := A^+_\infty(\lambda) - \mu_1^+ I$ and $R(x,\lambda) := A(x,\lambda) - A^+_\infty(\lambda)$. From **H4**, one of the eigenvalues of $B(\lambda)$ is 0 and the real part of the other eigenvalues are strictlfy positive. Hence, for all $x \leq 0$, we have $\|e^{B(\lambda)x}\| \leq C(\lambda)$, where $C(\lambda)$ is bounded on any compact subsets of Ω.

We let

$$(\mathcal{F}\widetilde{\mathbf{y}})(x) := -\int_x^\infty e^{B(\lambda)(x-s)}R(s,\lambda)\widetilde{\mathbf{y}}(s)\,ds.$$

For any fixed $x_0 \in \mathbb{R}$ and $\lambda \in \Omega$, $\mathcal{F}$ is a bounded linear operator on $C_b([x_0,\infty))$. Indeed,

$$\begin{aligned}|(\mathcal{F}\widetilde{\mathbf{y}})(x)| &\leq \int_x^\infty C(\lambda)|R(s,\lambda)||\widetilde{\mathbf{y}}(s)|\,ds \\ &\leq \sup_{x\geq x_0}|\widetilde{\mathbf{y}}(x)|\,C(\lambda)\int_x^\infty |R(s,\lambda)|\,ds.\end{aligned} \tag{5.48}$$

We fix a compact domain $\Omega_1 \subset \Omega$. From **H3**, we can choose sufficiently large $x_0 > 0$ so that

$$\theta := \sup_{\lambda \in \Omega_1} C(\lambda) \int_{x_0}^{\infty} |R(s,\lambda)|\, ds < 1.$$

Hence, $\mathcal{F}$ is a contraction on $C_b([x_0,\infty))$. For any given bounded function $\widetilde{\mathbf{y}}_0$, we define a sequence

$$\widetilde{\mathbf{y}}_{i+1} := \widetilde{\mathbf{y}}_0 + \mathcal{F}\widetilde{\mathbf{y}}_i, \quad (i \geq 1) \tag{5.49}$$

Using that $\mathcal{F}$ is a contraction, one may show that the sequence (5.49) is Cauchy in $C_b([x_0,\infty))$, and thus there exists a unique[53] $\widetilde{\mathbf{y}}(x,\lambda) \in C_b([x_0,\infty))$ satisfying

$$\widetilde{\mathbf{y}} = \widetilde{\mathbf{y}}_0 + \mathcal{F}\widetilde{\mathbf{y}} \tag{5.50}$$

Here, we note that if $\widetilde{\mathbf{y}}_0$ is differentiable in x, then $\widetilde{\mathbf{y}}$ is also differentiable in x since $\mathcal{F}\widetilde{\mathbf{y}}$ is differentiable in x.

Taking the derivative (formally) of (5.50), we see that

$$\begin{aligned}\frac{d(\widetilde{\mathbf{y}} - \widetilde{\mathbf{y}}_0)}{x} &= R(x,\lambda)\widetilde{\mathbf{y}}(x) + B(\lambda)(\mathcal{F}\widetilde{\mathbf{y}})(x) \\ &= R(x,\lambda)\widetilde{\mathbf{y}}(x) + B(\lambda)(\widetilde{\mathbf{y}} - \widetilde{\mathbf{y}}_0).\end{aligned}$$

Thus, if $\widetilde{\mathbf{y}}_0$ is a bounded solution of $\dfrac{d\widetilde{\mathbf{y}}_0}{dx} = B(\lambda)\widetilde{\mathbf{y}}_0$, then $\widetilde{\mathbf{y}}$, uniquely determined by (5.50), is a bounded solution of (5.47). Conversely, if $\widetilde{\mathbf{y}}$ is a bounded solution of (5.47), then $\widetilde{\mathbf{y}}_0 := \widetilde{\mathbf{y}} - \mathcal{F}\widetilde{\mathbf{y}}$ is a bounded solution of $\dfrac{d\widetilde{\mathbf{y}}_0}{dx} = B(\lambda)\widetilde{\mathbf{y}}_0$. Any bounded solution of $\dfrac{d\widetilde{\mathbf{y}}_0}{dx} = B(\lambda)\widetilde{\mathbf{y}}_0$ on $C_b([x_0,\infty))$ is a constant multiple of $\mathbf{v}^+(\lambda)$. We choose $\widetilde{\mathbf{y}}_0(x,\lambda) := \mathbf{v}^+(\lambda)$.

To finish the proof, we note that since the sequence (5.49) converges uniformly in λ on Ω_1, $\widetilde{\mathbf{y}}(x,\lambda)$ is also analytic in λ on Ω_1. From (5.48) and **H3**,

$$\widetilde{\mathbf{y}} - \mathbf{v}^+ = \widetilde{\mathbf{y}} - \widetilde{\mathbf{y}}_0 = (\mathcal{F}\widetilde{\mathbf{y}})(x) \to 0 \quad \text{as } x \to +\infty$$

uniformly in $\lambda \in \Omega_1$. $\widetilde{\mathbf{y}}(x,\lambda)$ is uniquely and analytically (in λ) extended on the half line $(-\infty, x_0]$ by solving the initial value problem. From the implicit function theorem, $\mu_1^+(\lambda)$ is analytic since μ_1^+ is simple. Hence, $\mathbf{y}^+ := e^{\mu_1^+ x}\widetilde{\mathbf{y}}$ is the desired solution.

We extend this local (in λ) result to Ω. We note that this extension is possible since Ω is simply connected. For any two region $\Omega_1, \Omega_2 \subset \Omega$ with $\Omega_1 \cap \Omega_2 \neq \emptyset$, there exist unique $\widetilde{\mathbf{y}}_1 = \widetilde{\mathbf{y}}_0 + \mathcal{F}\widetilde{\mathbf{y}}_1$ on $[x_1,\infty) \times \Omega_1$ and $\widetilde{\mathbf{y}}_2 = \widetilde{\mathbf{y}}_0 + \mathcal{F}\widetilde{\mathbf{y}}_2$ on $[x_2,\infty) \times \Omega_2$. By the uniqueness, $\widetilde{\mathbf{y}}_1 = \widetilde{\mathbf{y}}_2$ on $[x_3,\infty) \times (\Omega_1 \cap \Omega_2)$, where $x_3 = \max\{x_1, x_2\}$. By solving the initial value problem, $\widetilde{\mathbf{y}}_1 = \widetilde{\mathbf{y}}_2$ on $(-\infty,\infty) \times (\Omega_1 \cap \Omega_2)$.

□

In the following proposition, we characterize the asymptotic behavior of any solution of (5.39) in terms of the solution $\mathbf{z}^-$ of the transposed equation (5.40). We note that any solution $\mathbf{y}$ of (5.39) satisfies $\mathbf{y} = O(e^{\mu_1^- x})$ as $x \to -\infty$ since μ_1^- is the smallest simple eigenvalue.

[53] Uniqueness follows from that $\mathcal{F}$ is a constraction.

Proposition 5.37. *Suppose that* $\mathbf{y}$ *is a solution* (5.39) *and* $\mathbf{z}$ *is a solution of* (5.40). *Then, there holds that*

$$\mathbf{y}(x,\lambda) \sim (\mathbf{z}^- \, \mathbf{y})(\lambda) e^{\mu_1^-(\lambda)x} \mathbf{v}^-(\lambda) \quad \text{as} \quad x \to -\infty, \tag{5.51a}$$

$$\mathbf{z}(x,\lambda) \sim (\mathbf{z} \, \mathbf{y}^+)(\lambda) e^{-\mu_1^+(\lambda)x} \mathbf{w}^+(\lambda) \quad \text{as} \quad x \to +\infty. \tag{5.51b}$$

Moreover, if $\mathbf{y}$ *and* $\mathbf{z}$ *are analytic in* $\lambda \in \Omega$ *for each* x*, the limits are uniform in* λ *on any compact subsets of* Ω.

Proof. We fix λ. Let $\widetilde{\mathbf{y}}(x) := e^{-\mu_1^- x}\mathbf{y}(x)$. From (5.39), we obtain the equation (5.47), but now $B(\lambda) := A_\infty^-(\lambda) - \mu_1^- I$ and $R(x,\lambda) := A(x,\lambda) - A_\infty^-(\lambda)$. The eigenvalues of B are composed of 0 and $n-1$ values of strictly positive real parts. We define the linear bounded operator $\mathcal{F}$ on $C_b((-\infty, x_0])$ by

$$\begin{aligned}(\mathcal{F}\widetilde{\mathbf{y}})(x) := &-\int_x^{x_0} e^{B(\lambda)(x-s)} P R(s,\lambda)\widetilde{\mathbf{y}}(s)\, ds \\ &+ \int_{-\infty}^{x} e^{B(\lambda)(x-s)} (I-P) R(s,\lambda)\widetilde{\mathbf{y}}(s)\, ds,\end{aligned} \tag{5.52}$$

where P is a projection operator onto the unstable subspace.[54] Since $\|e^{B(x-s)}P\| \le Ce^{\widetilde{\mu}_*(x-s)}$ for $x - s \le 0$, where $\widetilde{\mu}_* > 0$, and $\|e^{B(x-s)}(I-P)\| \le C$ for $x - s \ge 0$, we have

$$\begin{aligned}|(\mathcal{F}\widetilde{\mathbf{y}})(x)| &\le C \sup_{x \le x_0} |\widetilde{\mathbf{y}}(x)| \left(\int_x^{x_0} |R(s,\lambda)|\, ds + \int_{-\infty}^{x} |R(s,\lambda)|\, ds \right) \\ &\le C \sup_{x \le x_0} |\widetilde{\mathbf{y}}(x)| \int_{-\infty}^{x_0} |R(s,\lambda)|\, ds.\end{aligned}$$

Thus, for $x_0 < 0$ with sufficiently large modulus, $\mathcal{F}$ is a contraction. As before, there is a one-to-one correspondence between bounded solutions of (5.47) and bounded solutions of $\dfrac{d\widetilde{\mathbf{y}}_0}{dx} = B\widetilde{\mathbf{y}}_0$. Moreover, $|\widetilde{\mathbf{y}} - \widetilde{\mathbf{y}}_0| \to 0$ as $x \to -\infty$. Indeed, for given $\varepsilon > 0$, we choose $x_1(\varepsilon) < x_0$ so that

$$C \sup_{x \le x_0} |\widetilde{\mathbf{y}}(x)| \int_{-\infty}^{x_1(\varepsilon)} |R(s,\lambda)|\, ds < \varepsilon/2.$$

Then,

$$\begin{aligned}|(\mathcal{F}\widetilde{\mathbf{y}})(x)| \le &\left| \int_{x_1}^{x_0} e^{B(\lambda)(x-s)} P R(s,\lambda)\widetilde{\mathbf{y}}(s)\, ds \right| + \left| \int_{x}^{x_1} e^{B(\lambda)(x-s)} P R(s,\lambda)\widetilde{\mathbf{y}}(s)\, ds \right| \\ &+ \left| \int_{-\infty}^{x} e^{B(\lambda)(x-s)} (I-P) R(s,\lambda)\widetilde{\mathbf{y}}(s)\, ds \right| \\ \le &\left| \int_{x_1}^{x_0} e^{B(\lambda)(x-s)} P R(s,\lambda)\widetilde{\mathbf{y}}(s)\, ds \right| + \varepsilon/2.\end{aligned}$$

We let $x \to -\infty$ so that

$$|e^{B(\lambda)x}P| \left| \int_{x_1}^{x_0} e^{-sB(\lambda)} R(s,\lambda)\widetilde{\mathbf{y}}(s)\, ds \right| < \varepsilon/2.$$

Any bounded solution (indeed, any solution) of $\dfrac{d\widetilde{\mathbf{y}}_0}{dx} = B\widetilde{\mathbf{y}}_0$ on $(-\infty, x_0]$ satisfies $\lim_{x \to -\infty} \mathbf{y}_0 = c\mathbf{v}^-(\lambda)$ for some constant c since it is a linear combination of $\mathbf{v}^-(\lambda)$ and the other $n-1$ linearly

[54] Direct sum of the generalized eigenspaces corresponding to the eigenvalues with positive real parts.

independent solutions of $\dfrac{d\widetilde{\mathbf{y}}_0}{dx} = B\widetilde{\mathbf{y}}_0$ which tend to zero as $x \to -\infty$.[55] From the previous proposition, we have

$$\mathbf{z}^-\mathbf{y} = e^{\mu_1^- x}\mathbf{z}^-\widetilde{\mathbf{y}} \to c\mathbf{w}^-\mathbf{v}^- = c \quad \text{as} \quad x \to -\infty,$$

which implies that $\mathbf{z}^-\mathbf{y} = c$.

Suppose that $\mathbf{y}$ is analytic in λ. Then, $\mathbf{z}^-(x,\lambda)\mathbf{y}(x,\lambda) = c(\lambda)$ is analytic in λ, and thus $\widetilde{\mathbf{y}}_0 = c(\lambda)\mathbf{v}^-(\lambda)$ is also analytic. Fix a domain Ω_1. From the one-to-one correspondence,

$$\begin{aligned}|\widetilde{\mathbf{y}}(x,\lambda)| &\le \sup_{\lambda\in\Omega_1} |\widetilde{\mathbf{y}}_0(\lambda)| + |(\mathcal{F}\widetilde{\mathbf{y}})(x,\lambda)| \\ &\le \sup_{\lambda\in\Omega_1} |\widetilde{\mathbf{y}}_0(\lambda)| + C(\lambda)\sup_{x\le x_0} |\widetilde{\mathbf{y}}(x,\lambda)| \int_{-\infty}^{x_0} |R(s,\lambda)|\,ds \\ &\le \sup_{\lambda\in\Omega_1} |\widetilde{\mathbf{y}}_0(\lambda)| + \frac{1}{2}\sup_{x\le x_0,\lambda\in\Omega_1} |\widetilde{\mathbf{y}}(x,\lambda)|\end{aligned}$$

for $x_0 < 0$ sufficiently large $|x_0|$ so that $\sup_{\lambda\in\Omega_1}\left(C(\lambda)\int_{-\infty}^{x_0} |R(s,\lambda)|\,ds\right) < \frac{1}{2}$ holds. Thus,

$$\sup_{x\le x_0,\lambda\in\Omega_1} |\widetilde{\mathbf{y}}(x,\lambda)| < 2\sup_{\lambda\in\Omega_1} |\widetilde{\mathbf{y}}_0(\lambda)|.$$

Using this, we can show that $|\widetilde{\mathbf{y}} - \widetilde{\mathbf{y}}_0| \to 0$ as $x \to -\infty$ uniformly in $\lambda \in \Omega_1$. □

Proposition 5.38. *There exist $n-1$ linearly independent solutions $\mathbf{y}_i^-(x,\lambda)$ of (5.39), and $n-1$ linearly independent solutions $\mathbf{z}_i^+(x,\lambda)$ of (5.40) such that the following hold. For $i = 2,\cdots,n$,*

(a) $\mathbf{y}_i^-(x,\lambda)$ and $\mathbf{z}_i^+(x,\lambda)$ are analytic in $\lambda \in \Omega$ for each x;

(b) $\mathbf{y}_i^-(x,\lambda) = O(e^{\mu_*^- x}e^{\delta|x|})$ as $x \to -\infty$ for any $\delta \in (0, \mu_*^+ - \operatorname{Re}\mu_1^+)$, and $\mathbf{z}_i^+(x,\lambda) = O(e^{-\mu_*^+ x}e^{\delta|x|})$ as $x \to +\infty$ for any $\delta \in (0, \mu_*^- - \operatorname{Re}\mu_1^-)$;

(c) Any solution of (5.39) ((5.40)) with $\mathbf{y}(x) = O(e^{\mu_*^- x}e^{\delta|x|})$ as $x \to -\infty$ ($\mathbf{z}(x) = O(e^{-\mu_*^+ x}e^{\delta|x|})$ as $x \to +\infty$) is a linear combination of $\mathbf{y}_i^-(x,\lambda)$ ($\mathbf{z}_i^+(x,\lambda)$ respectively).

Proof. For any fixed $\lambda \in \Omega$, it is classical (see [6]) that that there exist linearly independent solutions $\mathbf{y}_i^-(x)$ of (5.39) with $\mathbf{y}_i^-(x) = O(e^{\delta|x|}e^{\mu_*^- x})$ as $x \to -\infty$ for $i = 2,\cdots,n$. Here we construct such solutions which are analytic in $\lambda \in \Omega$.

Recall that for any solution of (5.39), $f(\mathbf{y}) := \mathbf{z}^-\mathbf{y}$ is independent of x. Indeed, f is a linear mapping from the n-dimensional solution space of (5.39) to a complex number. Thus, $\dim \mathcal{N}(f) = n-1$. From Proposition 5.37, we see that $f(\mathbf{y}_i^-(x)) = 0$. Thus, $\{\mathbf{y}_i^-(x)\}$ is a basis of $\mathcal{N}(f)$, and every element $\mathbf{y} \in \mathcal{N}(f)$ satisfies $\mathbf{y} = O(e^{\delta|x|}e^{\mu_*^- x})$ as $x \to -\infty$.

We fix x_0. Suppose that there exists an analytic $n \times (n-1)$ matrix $V(\lambda)$ with rank $n-1$ such that $\mathbf{z}^-(x_0,\lambda)V(\lambda) = 0$. We solve the initial value problem of a matrix valued ODE

$$\frac{d\mathbf{y}^-}{dx} = A(x,\lambda)\mathbf{y}^-, \quad \mathbf{y}^-(x_0,\lambda) = V(\lambda).$$

[55] We recall that if $a < b$, e^{ax} dominates e^{bx} as $x \to -\infty$ in the sense that $e^{ax}(1 + e^{(b-a)x}) \sim e^{ax}$ as $x \to -\infty$.

Then, the $n-1$ columns of $\mathbf{y}^-(x,\lambda)$ is the desired $\mathbf{y}_i^-(x,\lambda)$ since we have

$$0 = \mathbf{z}^-(x_0,\lambda)\mathbf{y}_i^-(x_0,\lambda) = \mathbf{z}^-(\lambda)\mathbf{y}_i^-(\lambda),$$

which implies that $\mathbf{y}_i^-(x,\lambda) \in \mathcal{N}(f)$.[56]

Let us construct $V(\lambda)$. We define $P(\lambda) := I - \dfrac{\mathbf{z}^t\mathbf{z}}{\mathbf{z}\mathbf{z}^t}$, where $\mathbf{z} = \mathbf{z}^-(x_0,\lambda)$. $P(\lambda)$ is a projection onto the space orthogonal to $\mathbf{z}^-(x_0,\lambda)$. Since $\mathbf{z}^-(x_0,\lambda)$ is analytic, $P(\lambda)$ is also analytic. We fix λ_0. Let $V_0 = (\mathbf{y}_2^-,\cdots,\mathbf{y}_n^-)(x_0,\lambda_0)$. Note that $\mathbf{z}^-(x_0,\lambda_0)\mathbf{y}_i^-(x_0,\lambda_0) = 0$. Thus, $P(\lambda_0)\mathbf{y}_i^-(x_0,\lambda_0) = \mathbf{y}_i^-(x_0,\lambda_0)$. From Lemma 5.35, there exists an analytic $n\times n$ matrix $U(\lambda)$ such that $U(\lambda)P(\lambda_0) = P(\lambda)U(\lambda)$ for all $\lambda \in \Omega$. Let $V(\lambda) := U(\lambda)V_0$. Then,

$$V(\lambda) = U(\lambda)V_0 = U(\lambda)P(\lambda_0)V_0 = P(\lambda)U(\lambda)V_0 = P(\lambda)V(\lambda).$$

Therefore, $\mathbf{z}^-(x_0,\lambda)V(\lambda) = 0$. Recall that $U(\lambda)$ is invertible, and the columns of V_0 are linearly independent. Thus, the columns of $V(\lambda)$ are linearly independent. □

Proposition 5.39 (Characterizations of the asymptotic behaviors of solutions of (5.39))**.** *If* $\mathbf{y}$ *is a solution of* (5.39)*, then* (a)*–*(d) *are equivalent.*

1. *(a)* $\mathbf{y}(x) = o(e^{\mu_1 x})$ *as* $x \to -\infty$*;*

   (b) $\mathbf{z}^-\mathbf{y} = 0$*;*

   (c) $\mathbf{y} = \sum_{i=2}^{n} c_i\mathbf{y}_i^-(x,\lambda)$ *for some* $c_i \in \mathbb{C}$*;*

   (d) $\mathbf{y} = O(e^{\mu_* x}e^{\delta|x|})$ *as* $x\to-\infty$ *for any* $\delta \in (0, \mu_* - \operatorname{Re}\mu_1)$*.*

2. *(a)* $\mathbf{y}(x) = O(e^{\mu_1 x})$ *as* $x \to +\infty$*;*

   (b) $\mathbf{z}_i^+\mathbf{y} = 0$ *for all* i *with* $2 \le i \le n$*;*

   (c) $\mathbf{y} = c\mathbf{y}^+(x,\lambda)$ *for some* $c \in \mathbb{C}$*;*

   (d) $\mathbf{y}(x) = o(e^{\mu_* x}e^{-\delta|x|})$ *as* $x \to +\infty$*.*

Proof. We first prove the first assertion. (a) and (b) are equivalent from Proposition 5.37. (b) and (c) are equivalent since $\{\mathbf{y}_i^-\}$ is a basis of $\ker f$, where $f(\mathbf{y}) = \mathbf{z}^-\mathbf{y}$ is defined in the proof of Proposition 5.38. (c) are (d) are equivalent from Proposition 5.38.

We prove the second assertion. (a) and (c) are equivalent from Proposition 5.37. Since $\mathbf{z}_i^+\mathbf{y}^+ = O(e^{(\delta-\mu_*^+ + \mu_1)x})$ as $x\to+\infty$ for all i and $\delta - \mu_*^+ + \operatorname{Re}\mu_1^+ < 0$, $\mathbf{z}_i^+\mathbf{y}^+ = 0$. Thus, (c) implies (b). Since $\mathbf{z}_i^+$ are linearly independent, the dimension of the solution space of $\mathbf{z}_i^+\mathbf{y} = 0$ $(i = 2,\cdots,n)$ is 1. Since $\mathbf{z}_i^+\mathbf{y}^+ = 0$, (b) implies (c). (a) implies (d) since $e^{(\delta-\mu_*)x}\mathbf{y}(x) = O(e^{(\delta-\mu_*^+ + \mu_1)x})$ as $x\to+\infty$ and $\delta - \mu_*^+ + \operatorname{Re}\mu_1^+ < 0$. (d) implies (b) since $\mathbf{z}_i^+\mathbf{y} = O(e^{\delta x}e^{-\mu_* x})\mathbf{y}$ as $x \to +\infty$. □

[56] By uniqueness, solutions of the linear ODE system with linearly independent initial data are linearly independent. Or one may consider the transposed equation and solve the matrix valued ODE.

Let

$$\mathbf{y}^- := \left[\mathbf{y}_2^-, \cdots, \mathbf{y}_n^-\right] \in \mathbb{C}^{n\times(n-1)} \quad \text{and} \quad \mathbf{z}^+ := \begin{bmatrix} \mathbf{z}_2^+ \\ \vdots \\ \mathbf{z}_n^+ \end{bmatrix} \in \mathbb{C}^{(n-1)\times n}.$$

The proof of Proposition 5.39 implies that for all $\lambda \in \Omega$,

$$(\mathbf{z}^+\mathbf{y}^+)(\lambda) = (0, \cdots, 0)^T \in \mathbb{C}^{n-1} \quad \text{and} \quad (\mathbf{z}^-\mathbf{y}^-)(\lambda) = (0, \cdots, 0) \in \mathbb{C}^{n-1}. \tag{5.53}$$

Remark 26 (Regarding Prop 5.39). When $\mu < \mu_*$, we recall that $e^{\mu_* x}$ dominates $e^{\mu x}$ as $x \to +\infty$ whereas $e^{\mu x}$ dominates $e^{\mu_* x}$ as $x \to -\infty$. Hence, the statements that the big O part implies the small o part is trivial from (5.41). The idea of the converse part is as follows. For simplicity, we suppose that $A_\infty^+ = A_\infty^- \in \mathbb{C}^{3\times 3}$ and $\operatorname{Re}\mu_1 < 0 < \operatorname{Re}\mu_2 < \operatorname{Re}\mu_3$. We forget about analyticity of solutions. For each $\lambda \in \Omega$, we can construct a solution $\mathbf{y}^+$ of (5.39) satisfying $\mathbf{y}^+ \sim e^{\mu_1 x}$ as $x \to +\infty$. Similarly, we can construct linearly independent solutions of (5.39) satisfying

$$\mathbf{y}_1^- \sim e^{\mu_1 x}, \quad \mathbf{y}_2^- \sim e^{\mu_2 x}, \quad \mathbf{y}_3^- \sim e^{\mu_3 x} \quad \text{as } x \to -\infty.$$

Hence $\mathbf{y}^+ = \sum_{i=1}^{3} c_n \mathbf{y}_i^-$ for some functions $c_n(\lambda)$, and we have

$$\mathbf{y}^+ \sim c_1(\lambda) e^{\mu_1 x} \quad \text{as } x \to -\infty.$$

Now it is clear that $c_1(\lambda) = 0$ for some λ if and only if $\mathbf{y}^+$ is a linear combination of $\mathbf{y}_2^-$ and $\mathbf{y}_3^-$, which decays to zero as $x \to -\infty$. The important part is that by introducing the transposed system, one can show that $\mathbf{z}^-\mathbf{y}^+(\lambda) = c_1(\lambda)$. Moreover, $\mathbf{z}^-\mathbf{y}^+(\lambda)$ can be chosen to be analytic in λ.

The proof invokes the relation of the ODE system and its transpose system. The crucial properties are the following:

1. the dot product of the solutions of each system is independent of x;

2. construction of the analytic basis of the kernel of $\mathbf{z}^-\mathbf{y}$ (and $\mathbf{z}_i^+\mathbf{y}$) with the behavior $O(e^{(\mu_*-\delta)x})$ as $x \to -\infty$ (and $O(e^{\mu_1 x})$ as $x \to +\infty$).

The property that small o implies big O means that there are some dichotomies:

1. *as* $x \to +\infty$, *either* $\mathbf{y} = O(e^{\mu_1 x})$ *or* $|\mathbf{y}| \geq Ce^{\mu_* x}$ *for some constant* $C > 0$;

2. *as* $x \to -\infty$, *either* $\mathbf{y} = O(e^{(\mu_*-\delta)x})$ *or* $|\mathbf{y}| \geq Ce^{\mu_1 x}$ *for some constant* $C > 0$.

In particular, this indicates the possible asymptotic behaviors of $\mathbf{y}^+$ (as $x \to -\infty$) and $\mathbf{y}_i^-$ (as $x \to +\infty$).

5.5.3 Definition of the Evans Function

Definition 5.17. For $\lambda \in \Omega$, we define the *Evans function*

$$D(\lambda) := \mathbf{z}^-(x,\lambda)\mathbf{y}^+(x,\lambda).$$

Theorem 5.40. *$D(\lambda)$ is analytic in $\lambda \in \Omega$. $D(\lambda) = 0$ if and only if there exists a non-trivial solution $\mathbf{y}$ of (5.39) satisfying*

$$\mathbf{y}(x) = o(e^{\mu_1^- x}) \quad as \quad x \to -\infty \quad and \quad \mathbf{y}(x) = O(e^{\mu_1^+ x}) \quad as \quad x \to +\infty. \tag{5.54}$$

Proof. Since $\mathbf{z}^-$ and $\mathbf{y}^+$ are analytic in $\lambda \in \Omega$, $D(\lambda)$ is analytic. By the construction (Proposition 5.36), we have $\mathbf{y}^+(x,\lambda) = O(e^{\mu_1^+ x})$ as $x \to +\infty$. From Proposition 5.37, we have

$$\lim_{x \to -\infty} e^{-\mu_1^- x}\mathbf{y}^+(x,\lambda) = (\mathbf{z}^-\,\mathbf{y}^+)\mathbf{v}^- = D(\lambda)\mathbf{v}^-. \tag{5.55}$$

Hence $D(\lambda) = 0$ implies (5.54). Conversely, if $\mathbf{y}(x) = O(e^{\mu_1^+ x})$ as $x \to +\infty$, then $\mathbf{y}$ is a constant multiple of $\mathbf{y}^+$ by Proposition 5.36. Hence, we have $\mathbf{y}^+(x) = o(e^{\mu_1^- x})$ as $x \to -\infty$, which implies that $D(\lambda) = 0$ by (5.55). □

Proposition 5.41. *For $\lambda \in \Omega$, the following statements are equivalent.*

(a) $D(\lambda) := (\mathbf{z}^-\mathbf{y}^+)(x,\lambda) = 0.$ $\quad$ (b) $\det[\mathbf{y}^+\mathbf{y}^-](x,\lambda) = 0.$

(c) $\det(\mathbf{z}^+\mathbf{y}^-)(x,\lambda) = 0.$ $\quad$ (d) $\det \begin{bmatrix} \mathbf{z}^- \\ \mathbf{z}^+ \end{bmatrix} (x,\lambda) = 0.$

Proof. From Proposition 5.39, (a) if and only if $\mathbf{y}^+ = \sum_{i=2}^n c_i \mathbf{y}_i^-$. Without loss of generality, we let $\mathbf{y}_2^- = \mathbf{y}^+ + \sum_{i=3}^n c_i \mathbf{y}_i^-$. Since $\mathbf{z}^+\mathbf{y}^+ = 0$ from (5.53),

$$\begin{aligned} \mathbf{z}^+\mathbf{y}^- &= \left[\mathbf{z}^+\mathbf{y}_2^-, \cdots, \mathbf{z}^+\mathbf{y}_n^-\right] \\ &= \left[\mathbf{z}^+(\mathbf{y}^+ + \textstyle\sum_{i=3}^n c_i \mathbf{y}_i^-), \mathbf{z}^+\mathbf{y}_3^-, \cdots, \mathbf{z}^+\mathbf{y}_n^-\right]. \\ &= \left[\mathbf{z}^+(\textstyle\sum_{i=3}^n c_i \mathbf{y}_i^-), \mathbf{z}^+\mathbf{y}_3^-, \cdots, \mathbf{z}^+\mathbf{y}_n^-\right] \end{aligned}$$

Now it easily follows that (a),(b) and (c) are equivalent. We omit the proof for (d). □

Indeed, the following stronger result is true. We refer to [26], p.64, for the proof.

Proposition 5.42. *Fix $x_0 \in \mathbb{R}$. There exist analytic functions f_1, f_2, f_3 of $\lambda \in \Omega$, having no zeros in Ω, such that*

$$D(\lambda) = f_1 \det[\mathbf{z}^+\mathbf{y}^-](\lambda) = f_2 \det[\mathbf{y}^+\mathbf{y}^-](\lambda) = f_3 \det \begin{bmatrix} \mathbf{z}^- \\ \mathbf{z}^+ \end{bmatrix} (\lambda)$$

Remark 27. On any open connected set of Ω, the zeros of $D(\lambda)$ are isolated points unless $D(\lambda)$ is identically zero.

Remark 28. If $\operatorname{Re}\mu_1 < 0 < \mu_*$, then (5.54) is equivalent to that $\mathbf{y}(x)$ is bounded on $\mathbb{R}$. From Proposition 5.39, (5.54) is equivalent to that

$$\mathbf{y}(x) = o(e^{\mu_1 x}) \quad \text{as} \quad x \to -\infty \quad \text{and} \quad \mathbf{y}(x) = o(e^{(\mu_* - \delta)x}) \quad \text{as} \quad x \to +\infty.$$

Thus, if $\mathbf{y}$ is bounded, then (5.54) holds. On the other hand, (5.54) is equivalent to that

$$\mathbf{y}(x) = O(e^{(\mu_* - \delta)x}) \quad \text{as} \quad x \to -\infty \quad \text{and} \quad \mathbf{y}(x) = O(e^{\mu_1 x}) \quad \text{as} \quad x \to +\infty.$$

Thus, (5.54) implies that $\mathbf{y}$ is bounded. In particular, this shows that (5.54) is equivalent to that $\mathbf{y}(x)$ exponentially decreases as $|x| \to \infty$.

Remark 29. From Proposition 5.39, (5.54) is equivalent to that

$$\mathbf{z}^- \mathbf{y} = 0 \quad \text{and} \quad \mathbf{z}_i^+ \mathbf{y} = 0 \quad \text{for } i = 2, \cdots, n.$$

These can be interpreted as asymptotic conditions of (5.39) on $\mathbb{R}$: one condition $\mathbf{z}^- \mathbf{y} = 0$ for $x \to -\infty$, and $n-1$ conditions $\mathbf{z}_i^+ \mathbf{y} = 0$ for $x \to +\infty$.

Remark 30. Suppose that $A_\infty^+(\lambda) = A_\infty^-(\lambda)$. For a solution $\mathbf{y}^+(x,\lambda) \sim e^{\mu_1 x}\mathbf{v}$ as $x \to +\infty$, we have $\mathbf{y}^+(x,\lambda) \sim D(\lambda) e^{\mu_1 x}\mathbf{v}$ as $x \to -\infty$. If $A(x,\lambda) = A(\lambda)$, we see that $D(\lambda)$ cannot be zero for all $\lambda \in \Omega$ since any solution $\mathbf{y}$ satisfying $\mathbf{y}(x,\lambda) \sim e^{\mu_1 x}\mathbf{v}$ as $x \to +\infty$ must be $e^{\mu_1 x}\mathbf{v}$.

5.5.4 Properties of the Evans Function

Proposition 5.43. *Suppose that $A(x,\lambda)$ is real for all x whenever $\lambda \in \Omega$ is real. Then whenever $\lambda, \overline{\lambda} \in \Omega$, we have $A(x,\lambda) = \overline{A(x,\overline{\lambda})}$, $\mathbf{y}^+(x,\lambda) = \overline{\mathbf{y}^+(x,\overline{\lambda})}$, $\mathbf{y}^-(x,\lambda) = \overline{\mathbf{y}^-(x,\overline{\lambda})}$, $\mathbf{z}^-(x,\lambda) = \overline{\mathbf{z}^-(x,\overline{\lambda})}$, $\mathbf{z}^+(x,\lambda) = \overline{\mathbf{z}^+(x,\overline{\lambda})}$, and $D(\lambda) = \overline{D(\overline{\lambda})}$.*

Proof. $A(x,\lambda) = \overline{A(x,\overline{\lambda})}$ follows from the Schwarz reflection principle. Since we already know that $\mathbf{y}^+$ is analytic in λ, it is enough to show that $\mathbf{y}^+(x,\lambda)$ is real when λ is real. First, we observe that since $A^\pm(\lambda)$ is real when λ is real, $\overline{\mu_1^\pm(\lambda)}$ is also an eigenvalue of $A^\pm(\lambda)$. Hence, $\mu_1^\pm(\lambda)$ must be real due to **H4**.

We claim that $\mathbf{v}^\pm(\lambda)$ is real when λ is real. To do this, we recall the construction of $\mathbf{v}^\pm(\lambda)$. Recall that we chose $\mathbf{v}(\lambda) := U(\lambda)\mathbf{v}_0(\lambda_0)$, where $U(\lambda)$ is the solution of

$$U'(\lambda) = (P'P - PP')(\lambda)U(\lambda)$$

with $U(\lambda_0) = I$. It is enough to show that $P(\lambda)$ is pure imaginary valued when λ is real. Then, $P'(\lambda)$ is also pure imaginary valued by the definition of analytic function. Then, we choose real-valued $\mathbf{v}_0(\lambda_0)$ for a fixed real λ_0, which is possible since $A(\lambda)$ is real.

We observe that if ν is not an eigenvalue of $A_\infty(\lambda)$, then

$$\overline{(A_\infty(\lambda) - \nu I)^{-1}} = (A_\infty(\lambda) - \overline{\nu} I)^{-1}.$$

Choose a contour $\Gamma(\theta) := \mu_1(\lambda) + \varepsilon e^{i\theta}$ for sufficiently small $\varepsilon > 0$. Then, for λ real,

$$\begin{aligned}\overline{P(\lambda)} &= \overline{\int_\Gamma (A_\infty(\lambda) - \nu I)^{-1}\, d\nu} \\ &= \int_0^{2\pi} (A_\infty(\lambda) - (\mu_1(\lambda) + \varepsilon e^{-i\theta})I)^{-1}(-\varepsilon i e^{-i\theta})\, d\theta \\ &= \int_0^{-2\pi} (A_\infty(\lambda) - (\mu_1(\lambda) + \varepsilon e^{i\theta'})I)^{-1}(\varepsilon i e^{i\theta'})\, d\theta' \\ &= -P(\lambda).\end{aligned}$$

Now, since $\overline{\mathbf{y}^+(x,\lambda)}$ satisfies (5.39) and has the same asympotic behavior as $\mathbf{y}^+(x,\lambda)$ from Proposition 5.36, we see that $\mathbf{y}^+(x,\lambda) = \overline{\mathbf{y}^+(x,\lambda)}$ for λ real. For $\mathbf{y}_j^-(x,\lambda) = \mathbf{y}_j^-(x,\overline{\lambda})$, we choose V_0 to be real for real λ_0 (See the construction of $\mathbf{y}_j^-$ in the proof of Proposition 5.38.). □

Proposition 5.44 (Resolvent formula)**.** *For $\lambda \in \Omega$ with $D(\lambda) \neq 0$, let*

$$G(x,s;\lambda) := \begin{cases} \dfrac{1}{D(\lambda)}\mathbf{y}^+(x,\lambda)\mathbf{z}^-(s,\lambda), & x > s, \\ -\mathbf{y}^-(x,\lambda)(\mathbf{z}^+\mathbf{y}^-)^{-1}(\lambda)\mathbf{z}^+(s,\lambda), & x < s. \end{cases}$$

Then G satisfies

$$\begin{cases} \dfrac{dG}{dx}(x,s;\lambda) = A(x,\lambda)G(x,s;\lambda), \quad (x \neq s), \\ \lim\limits_{x\to s^+} G(x,s) - \lim\limits_{x\to s^-} G(x,s) = I. \end{cases} \tag{5.56}$$

Proof. We note that for any $x_0 \in \mathbb{R}$,

$$\left(\begin{bmatrix} \mathbf{z}^- \\ \mathbf{z}^+ \end{bmatrix} \begin{bmatrix} \mathbf{y}^+ \mathbf{y}^- \end{bmatrix} \right)(x_0,\lambda) = \begin{bmatrix} D(\lambda) & 0 \\ 0 & (\mathbf{z}^+\mathbf{y}^-)(\lambda) \end{bmatrix}.$$

Since $D(\lambda) \neq 0$, we have

$$I = \begin{bmatrix} \mathbf{z}^- \\ \mathbf{z}^+ \end{bmatrix}^{-1} \begin{bmatrix} D & 0 \\ 0 & \mathbf{z}^+\mathbf{y}^- \end{bmatrix} \begin{bmatrix} \mathbf{y}^+ \mathbf{y}^- \end{bmatrix}^{-1},$$

and thus,

$$\begin{aligned} I &= \begin{bmatrix} \mathbf{y}^+ \mathbf{y}^- \end{bmatrix} \begin{bmatrix} D^{-1} & 0 \\ 0 & (\mathbf{z}^+\mathbf{y}^-)^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{z}^- \\ \mathbf{z}^+ \end{bmatrix} \\ &= \frac{\mathbf{y}^+(x_0,\lambda)\mathbf{z}^-(x_0,\lambda)}{D(\lambda)} + \mathbf{y}^-(x_0,\lambda)(\mathbf{z}^+\mathbf{y}^-)^{-1}(\lambda)\mathbf{z}^+(x_0,\lambda) \end{aligned}$$

Let

$$P(\lambda) := \frac{\mathbf{y}^+(x_0,\lambda)\mathbf{z}^-(x_0,\lambda)}{D(\lambda)}$$

and $\Phi(x,\lambda)$ be the fundamental matrix such that

$$\frac{d}{dx}\Phi(x,\lambda) = A(x,\lambda)\Phi(x,\lambda), \quad \Phi(x_0,\lambda) = I.$$

Then we get

$$\begin{aligned}\frac{1}{D(\lambda)}\mathbf{y}^+(x,\lambda)\mathbf{z}^-(s,\lambda) &= \frac{1}{D(\lambda)}\Phi(x;)\mathbf{y}^+(x_0;)\mathbf{z}^-(x_0;)\Phi^{-1}(s;) \\ &= \Phi(x;)P(\lambda)\Phi^{-1}(s;)\end{aligned}$$

and

$$-\mathbf{y}^-(x,\lambda)(\mathbf{z}^+\mathbf{y}^-)^{-1}(\lambda)\mathbf{z}^+(s,\lambda) = -\Phi(x;)(I-P(\lambda))\Phi^{-1}(s;).$$

One may check that

$$G(x,s;\lambda) := \begin{cases} \Phi(x;)P(\lambda)\Phi^{-1}(s;) & x > s, \\ -\Phi(x;)(I-P)(\lambda)\Phi^{-1}(s;) & x < s. \end{cases}$$

satisfies (5.56). □

Theorem 5.45 (Derivatives of $D(\lambda)$). *Let*

$$\mu(x,\lambda) := \begin{cases} \mu_1^-(\lambda) & \textit{for} \quad x < 0, \\ \mu_1^+(\lambda) & \textit{for} \quad x > 0. \end{cases}$$

Then for all $\lambda \in \Omega$,

$$\begin{aligned} D'(\lambda) = &-\int_{-\infty}^{\infty} \mathbf{z}^-(x,\lambda)\left[A_\lambda(x,\lambda) - \mu_\lambda(x,\lambda)I\right]\mathbf{y}^+(x,\lambda)\,dx \\ &+ D(\lambda)\left[\mathbf{w}_\lambda^-(\lambda)\cdot\mathbf{v}^-(\lambda) + \mathbf{w}^+(\lambda)\cdot\mathbf{v}_\lambda^+(\lambda)\right], \end{aligned} \tag{5.57}$$

in the sense of an improper integral. Also for $k \geq 2$, $\partial_\lambda^k D$ *is given by taking the derivatives of* (5.57).

Proof. $D'(\lambda) = \mathbf{z}_\lambda^-\mathbf{y}^+ + \mathbf{z}^-\mathbf{y}_\lambda^+$. On the other hand, we have (see remark below)

$$\frac{d\mathbf{z}_\lambda^-}{dx} = -\mathbf{z}^-A_\lambda - \mathbf{z}_\lambda^-A, \quad \frac{d\mathbf{y}_\lambda^+}{dx} = A_\lambda\mathbf{y}^+ + A\mathbf{y}_\lambda^+. \tag{5.58}$$

Hence, from (5.39) and (5.40),

$$\begin{aligned}\frac{d}{dx}(\mathbf{z}_\lambda^-\mathbf{y}^+) &= \frac{d\mathbf{z}_\lambda^-}{dx}\mathbf{y}^+ + \mathbf{z}_\lambda^-\frac{d\mathbf{y}^+}{dx} \\ &= -\mathbf{z}^-A_\lambda\mathbf{y}^+ \\ &= -\frac{d}{dx}(\mathbf{z}^-\mathbf{y}_\lambda^+).\end{aligned} \tag{5.59}$$

For $R, S > 0$, we have

$$\mathbf{z}_\lambda^-\mathbf{y}^+(0,\lambda) - \mathbf{z}_\lambda^-\mathbf{y}^+(-R,\lambda) = \int_{-R}^{0} -\mathbf{z}^-A_\lambda\mathbf{y}^+\,dx \tag{5.60}$$

$$\mathbf{z}^-\mathbf{y}_\lambda^+(S,\lambda) - \mathbf{z}^-\mathbf{y}_\lambda^+(0,\lambda) = \int_{0}^{S} \mathbf{z}^-A_\lambda\mathbf{y}^+\,dx \tag{5.61}$$

Thus,

$$\begin{aligned} D'(\lambda) &= \mathbf{z}_\lambda^- \mathbf{y}^+(0,\lambda) + \mathbf{z}^- \mathbf{y}_\lambda^+(0,\lambda) \\ &= \int_{-R}^{S} -\mathbf{z}^- A_\lambda \mathbf{y}^+ \, dx + \mathbf{z}_\lambda^- \mathbf{y}^+(-R,\lambda) + \mathbf{z}^- \mathbf{y}_\lambda^+(S,\lambda). \end{aligned} \tag{5.62}$$

Let $\widetilde{\mathbf{y}} = e^{-\mu x}\mathbf{y}^+$ and $\widetilde{\mathbf{z}} = e^{\mu x}\mathbf{z}^-$. Since $\widetilde{\mathbf{y}}_\lambda = -\mu_\lambda x e^{-\mu x}\mathbf{y}^+ + e^{-\mu x}\mathbf{y}_\lambda^+$ and $\widetilde{\mathbf{z}}_\lambda = \mu_\lambda x e^{\mu x}\mathbf{z}^- + e^{\mu x}\mathbf{z}_\lambda^-$, we have

$$\begin{aligned} D'(\lambda) &= \int_{-R}^{S} -\mathbf{z}^- A_\lambda \mathbf{y}^+ \, dx + (\widetilde{\mathbf{z}}_\lambda e^{-\mu x}\mathbf{y}^+ - \mu_\lambda x \mathbf{z}^- \mathbf{y}^+)(-R,\lambda) \\ &\quad + (\mathbf{z}^- e^{\mu x}\widetilde{\mathbf{y}}_\lambda + \mu_\lambda x \mathbf{z}^- \mathbf{y}^+)(S,\lambda) \\ &= -\int_{-R}^{S} \mathbf{z}^-(A_\lambda - \mu_\lambda I)\mathbf{y}^+ \, dx + (\widetilde{\mathbf{z}}_\lambda \widetilde{\mathbf{y}})(-R,\lambda) \\ &\quad + (\widetilde{\mathbf{z}}\widetilde{\mathbf{y}}_\lambda)(S,\lambda), \end{aligned} \tag{5.63}$$

where we have used that $(\mathbf{z}^-\mathbf{y}^+)_x = 0$ in the second line. Since

$$\widetilde{\mathbf{y}} \to \mathbf{v}^+ \quad (x \to +\infty), \quad \widetilde{\mathbf{y}} \to D(\lambda)\mathbf{v}^- \quad (x \to -\infty), \tag{5.64a}$$

$$\widetilde{\mathbf{z}} \to \mathbf{w}^- \quad (x \to -\infty), \quad \widetilde{\mathbf{z}} \to D(\lambda)\mathbf{w}^+ \quad (x \to +\infty) \tag{5.64b}$$

uniformly in λ, we have $\widetilde{\mathbf{y}}_\lambda \to \mathbf{v}_\lambda^+$ as $x \to +\infty$ and $\widetilde{\mathbf{z}}_\lambda \to \mathbf{w}_\lambda^-$ as $x \to -\infty$. Letting $R, S \to +\infty$, we obtain (5.57). The higher order derivatives are obtained by taking derivatives of (5.63) and (5.64) in λ. □

Remark 31 (Interchanging the order of derivative (5.58)). We have

$$\frac{d\mathbf{y}^+}{dx} = A(x,\lambda)\mathbf{y}^+, \quad \frac{\partial}{\partial\lambda}\frac{d\mathbf{y}^+}{dx} = A_\lambda \mathbf{y}^+ + A\mathbf{y}_\lambda^+.$$

$\mathbf{y}_x^+(x,\lambda)$ is analytic in λ for each fixed x from the ODE. Thus, from the Cauchy integral formula, we have

$$\frac{\partial}{\partial\lambda}\frac{d\mathbf{y}^+}{dx}(x,\lambda) = \frac{1}{2\pi i}\int_\Gamma \frac{\mathbf{y}_x^+(x,z)}{(z-\lambda)^2}\, dz.$$

Since $\mathbf{y}^+(x,\lambda)$ is analytic in λ, by the Cauchy integral formula, we have

$$\frac{\partial}{\partial\lambda}\mathbf{y}^+(x,\lambda) = \frac{1}{2\pi i}\int_\Gamma \frac{\mathbf{y}^+(x,z)}{(z-\lambda)^2}\, dz.$$

Since $A(x,\lambda)$ and $\mathbf{y}^+(x,\lambda)$ are jointly continuous[57], $\mathbf{y}_x^+(x,\lambda)$ is also jointly continuous from the ODE.

Thus, by the bounded convergence theorem, (or by other useful theorem) $\partial_x\partial_\lambda\mathbf{y}^+(x,\lambda)$ exists and

$$\frac{d\mathbf{y}_\lambda^+}{dx} = \frac{\partial}{\partial\lambda}\frac{d\mathbf{y}^+}{dx}(x,\lambda).$$

[57]Since we know that $\mathbf{y}^+(x_0,\lambda)$ is continuous in λ and $A(x,\lambda)$ is jointly continuous, the proof can be done by the standard iteration argument on any compact set of $\mathbb{R}\times\Omega$. For instance, fixed point of $y(x,\lambda) = y(x_0,\lambda) + \int_{x_0}^{x} A(s,\lambda)y(s,\lambda)\, ds$.

If the parameter is real, we cannot use this argument. In this case, we show that $\mathbf{y}_\lambda$ exists and it a solution of inhomogeneous linear ODE $\partial_x y = Ay + A_\lambda \mathbf{y}$. Due to this fact, it turns out that $\partial_x\partial_\lambda \mathbf{y}^+ = \partial_\lambda\partial_x \mathbf{y}^+$. See [18], Chapter 5.

The following proposition relates the order of zeros of $D(\lambda)$ and the decay rates of $\partial_\lambda^j \mathbf{y}^+$. This is useful when one calculate the algebraic multiplicity of the eigenvalues, equivalently, the longest possible length of the Jordan chain.

Proposition 5.46. *Suppose that $\lambda \in \Omega$ is a zero of the Evans function D of order $k \geq 1$, that is, $0 = D^{(j)}(\lambda) \neq D^{(k)}(\lambda)$ for $0 \leq j \leq k-1$. Then, for $0 \leq j \leq k-1$ and any sufficiently small $\delta > 0$,*

$$\partial_\lambda^j \mathbf{y}^+(x,\lambda) = O(e^{\mu_*^- x} e^{\delta|x|}) \quad as \quad x \to -\infty, \tag{5.65}$$

$$D^{(k)}(\lambda) = \lim_{x\to-\infty} \mathbf{z}^+(x,\lambda)\partial_\lambda^k \mathbf{y}^+(x,\lambda). \tag{5.66}$$

Remark 32. Proposition 5.46 says that the order of $\partial_\lambda^k \mathbf{y}^+$ as $x \to -\infty$ is exactly $e^{\mu_1^- x}$. Let $\mu^- = \mu_1^-$. Suppose that we have $0 = D(0) \neq D'(0)$. We recall that the limit

$$\lim_{x\to-\infty} e^{-\mu^- x}\mathbf{y}^+ = D(\lambda)\mathbf{v}(\lambda)$$

is uniform in λ. By taking the derivative in λ, we obtain

$$\lim_{x\to-\infty} \left(e^{-\mu^- x}\mathbf{y}_\lambda^+ - \mu_\lambda^- x e^{-\mu^- x}\mathbf{y}^+\right) = D'(0)\mathbf{v}(0) + D(0)\mathbf{v}_\lambda(0).$$

at $\lambda = 0$. Since $D(0) = 0$, we have $\mathbf{y}^+ = o(e^{\mu^- x})$ as $x \to -\infty$, equivalently, $\mathbf{y}^+ = O(e^{\mu_*^- x}e^{\delta|x|})$ as $x \to -\infty$ from Proposition 5.39. Hence from $\mu^- < \mu_*^-$, we have

$$\lim_{x\to-\infty} x e^{-\mu^- x}\mathbf{y}^+ = 0.$$

Therefore, we obtain

$$\lim_{x\to-\infty} e^{-\mu^- x}\mathbf{y}_\lambda^+ = D'(0)\mathbf{v}(0) \neq \mathbf{0}.$$

Indeed, Proposition 5.46 can be proved by induction based on this observation. We refer to [26].

The next two propositions are useful for studying the asymptotic behavior of the Evans function for large $|\lambda|$. In many cases, the asymptotic matrix is diagonalizable with distinct matrix eigenvalues for large $|\lambda|$. One may apply some perturbation arguments such as Proposition 5.47 to investigate the asymptotic behavior of the matrix eigenvalues.

Proposition 5.47. *Suppose that $\widetilde{P}(\mu;\lambda)$ and $L(\mu;\lambda)$ are analytic functions in μ, where λ is a parameter. Suppose that $\widetilde{P}$ has a simple zero $\tilde{\mu} = \tilde{\mu}(\lambda)$ as $|\lambda| \to 0$ and that there is a positive function $\rho(\lambda)$ and a constant $\rho_0 > 1$ such that $\rho(\lambda) \to 0$ as $|\lambda| \to \infty$, and $\rho \geq \rho_0 \dfrac{|L(\tilde{\mu})|}{|\widetilde{P}'(\tilde{\mu})|}$ on $|\mu - \tilde{\mu}| = \rho$. Then, $P := \widetilde{P} + L$ has exactly one zero $\mu_0 = \mu_0(\lambda)$ satisfying $|\mu_0 - \tilde{\mu}| \leq \rho$.*

Proof. By Taylor's theorem, $\widetilde{P}(\mu) = \widetilde{P}'(\tilde{\mu})(\mu-\tilde{\mu})(1+O_{\widetilde{P}}(|\mu-\tilde{\mu}|))$ and $L(\mu) = L(\tilde{\mu})(1+O_L(|\mu-\tilde{\mu}|))$. On $|\mu-\tilde{\mu}| = \rho$,

$$\begin{aligned} |\widetilde{P}(\mu)| &= |\widetilde{P}'(\tilde{\mu})||\mu-\tilde{\mu}||1+O_{\widetilde{P}}(|\mu-\tilde{\mu}|)| = \rho|\widetilde{P}'(\tilde{\mu})||1+O_{\widetilde{P}}(|\mu-\tilde{\mu}|)| \\ &> \rho_0|L(\tilde{\mu})||1+O_{\widetilde{P}}(|\mu-\tilde{\mu}|)| \\ &> |L(\tilde{\mu})||1+O_L(|\mu-\tilde{\mu}|)| = |L(\mu)|. \end{aligned}$$

The proof is finished by applying Rouché's theorem. □

Proposition 5.48. *We assume that for a matrix $A(x,\lambda)$ with $\lim_{x\to\pm\infty} A(x,\lambda) = A_\infty(\lambda)$, the system* (5.39) *satisfies the hypotheses **H1**–**H4**. We further assume that $A_\infty(\lambda)$ is diagonalizable such that for the matrices W and V defined by*

$$W := \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_n \end{bmatrix}, \quad V := \begin{bmatrix} \mathbf{v}_1, \cdots, \mathbf{v}_n \end{bmatrix},$$

where $\mathbf{w}_i$ and $\mathbf{v}_i$ are the left and right eigenvectors of $A_\infty(\lambda)$ associated with μ_i, we have

$$WA_\infty(\lambda)V = \mathrm{diag}\{\mu_j\}, \quad WV = I.$$

Let $R(x,\lambda) := A(x,\lambda) - A_\infty(\lambda)$. Then, there exists $0 < \delta_0 < 1$ such that if $\int_{-\infty}^{\infty} |WR(x,\lambda)V|\,dx \le \delta_0$, then

$$|D(\lambda) - 1| \le C\int_{-\infty}^{\infty} |WR(x,\lambda)V|\,dx. \tag{5.67}$$

Proof. By the construction of $\mathbf{y}^+$ (see Proposition 5.36), we have $\lim_{x\to+\infty} e^{-\mu_1 x}W\mathbf{y}^+(x) = \mathbf{e}_1$. We define $\mathbf{v}^+(x) := e^{-\mu_1 x}W\mathbf{y}^+(x) - \mathbf{e}_1$. Then $\mathbf{v}^+(x) \to 0$ as $x \to +\infty$, and

$$\frac{d\mathbf{v}^+}{dx} = B\mathbf{v}^+(x) + F(x)(\mathbf{e}_1 + \mathbf{v}^+(x)), \tag{5.68}$$

where

$$\begin{aligned} B(\lambda) &:= W(A_\infty(\lambda) - \mu_1(\lambda)I)V = \mathrm{diag}\{\mu_j - \mu_1\}, \\ F(x) &:= W(A(x,\lambda) - A_\infty(\lambda))V. \end{aligned}$$

We multiply by e^{-Bx} and integrate it on $[x, x_1]$, then we have

$$e^{-Bx_1}\mathbf{v}^+(x_1) - e^{-Bx}\mathbf{v}^+(x) = \int_x^{x_1} e^{-Bs}F(s)(\mathbf{e}_1 + \mathbf{v}^+(s))\,ds. \tag{5.70}$$

Multiplying by e^{Bx}, we have

$$e^{B(x-x_1)}\mathbf{v}^+(x_1) - \mathbf{v}^+(x) = \int_x^{x_1} e^{B(x-s)}F(s)(\mathbf{e}_1 + \mathbf{v}^+(s))\,ds. \tag{5.71}$$

Since B is diagonal and the smallest value of real part of the eigenvalues is 0, we have $\|e^{Bx}\| \le 1$ for $x \le 0$, and thus the first term tends to 0 as $x_1 \to +\infty$. Moreover,

$$\int_x^{+\infty} e^{B(x-s)}F(s)(\mathbf{e}_1 + \mathbf{v}^+(s))\,ds < \infty.$$

Hence, we have

$$\mathbf{v}^+(x) = -\int_x^\infty e^{B(x-s)} F(s)(\mathbf{e}_1 + \mathbf{v}^+(s))\, ds.$$

And

$$|\mathbf{v}^+(x)| \leq \int_0^\infty |F(s)|\, ds(1 + \sup_{s\in[0,\infty)} |\mathbf{v}^+(s)|). \tag{5.72}$$

Thus, if $\int_0^\infty |F(s)|\, ds < \delta_0 < 1$, then

$$\sup_{x\in[0,\infty)} |\mathbf{v}^+(x)| \leq C \int_0^\infty |F(s)|\, ds. \tag{5.73}$$

In a similar fashion, let $\mathbf{w}^-(x) := \mathbf{z}^-(x)e^{\mu_1 x}V - \mathbf{e}_1^t$. Then

$$\frac{d\mathbf{w}^-}{dx} = -\mathbf{w}^-(x)B - (\mathbf{e}_1^t + \mathbf{w}^-(x))F(x). \tag{5.74}$$

Since $e^{\mu_1 x}\mathbf{z}^-(x) \to \mathbf{w}_1$ as $x \to -\infty$, we have $\mathbf{w}^-(x) \to 0$ as $x \to -\infty$. Multiplying (5.74) by e^{Bx}, and then integrating the resultant over $[x_1, x]$, we get

$$\mathbf{w}^-(x) = \mathbf{w}^-(x_1)e^{B(x_1-x)} - \int_{x_1}^x (\mathbf{e}_1^t + \mathbf{w}^-(s))F(s)e^{B(s-x)}\, ds. \tag{5.75}$$

The first term tends to zero as $x_1 \to -\infty$. Thus,

$$\mathbf{w}^-(x) = -\int_{-\infty}^x (\mathbf{e}_1^t + \mathbf{w}^-(s))F(s)e^{B(s-x)}\, ds. \tag{5.76}$$

Thus, we have if $\int_0^\infty |F(s)|\, ds < \delta_0$, then

$$\sup_{x\in(-\infty,0]} |\mathbf{w}^-(x)| \leq C \int_{-\infty}^0 |F(s)|\, ds. \tag{5.77}$$

Since $D(\lambda, \varepsilon) = \mathbf{z}^-\mathbf{y}^+ = (\mathbf{w}^- + \mathbf{e}_1^t)(\mathbf{v}^+ + \mathbf{e}_1)$ (recall that $WV = I$), we arrive at

$$|D(\lambda, \varepsilon) - 1| \leq C \int_{-\infty}^\infty |F(s)|\, ds. \tag{5.78}$$

□

5.5.5 Application: Linear Instability of Nonlinear Waves

We consider the generalized KdV equation

$$\partial_t u + \partial_s f(u) + \partial_s^3 u = 0,$$

where $f(u) = u^{p+1}/(p+1)$ and $p \geq 1$. Considering the change of variable $x = s - ct$, one can show via a phase plane analysis that the gKdV equation has a solitary wave solution $u_c(s - ct)$ traveling with the speed $c > 0$, and it satisfies

$$-c\partial_x u_c + \partial_x f(u_c) + \partial_x^3 u_c = 0. \tag{5.79}$$

Indeed, following the calculation given in Appendix, one can obtain the explicit form of u_c,

$$u_c(x) = \left(\frac{1}{2}c(p+1)(p+2)\right)^{1/p} \operatorname{sech}^{2/p}\left(\frac{p\sqrt{c}\,x}{2}\right), \quad (c > 0). \tag{5.80}$$

We consider the linearized gKdV equation around u_c:

$$\partial_t v = \partial_x L_c v,$$

where $L_c := -\partial_x^2 + c - f'(u_c)$. We consider the operator $\partial_x L_c : H^3(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and the eigenvalue problem

$$\lambda v = \partial_x L_c v. \tag{5.81}$$

We study the eigenvalue problem (5.81). By applying the Evans function, we show that there is an unstable eigenvalue of the operator $\partial_x L_c$. This result is due to [26]. We recall that the Evans function can be constructed such that $D(\lambda) \in \mathbb{R}$ for $\lambda \in \mathbb{R}$.

Reformulation of the eigenvalue problem We let

$$\mathbf{y} = (v, \partial_x v, \partial_x^2 v)^T.$$

Then we obtain from (5.81) that

$$\frac{d\mathbf{y}}{dx} = A(x,\lambda)\mathbf{y} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda - \partial_x(f'(u_c)) & c - f'(u_c) & 0 \end{pmatrix} \mathbf{y}. \tag{5.82}$$

We consider the transpose equation $d\mathbf{z}/dx = -\mathbf{z}A(x,\lambda)$. Each component of $\mathbf{z}$, say z_j, satisfies

$$z_1' = -z_3(-\lambda - \partial_x(f'(u_c))), \quad z_2' = -z_1 - z_3(c - f'(u_c)), \quad z_3' = -z_2. \tag{5.83}$$

In particular, z_3 satisfies the transpose equation of the eigenvalue problem (5.81)[58]:

$$\lambda z_3 = -L_c \partial_x z_3. \tag{5.84}$$

The asymptotic matrix of $A(x,\lambda)$ is

$$A^\infty(\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda & c & 0 \end{pmatrix},$$

and the characteristic polynomial is

$$d(\mu;\lambda) = \mu^3 - c\mu + \lambda. \tag{5.85}$$

[58] A similar observation also holds for the general n-th order differential operators.

The right and left eigenvectors, $\mathbf{v}_j$ and $\mathbf{w}_j$, of A^∞ corresponding the matrix eigenvalue μ_j are given by

$$\mathbf{v}_j = (1, \mu_j, \mu_j^2)^t, \quad \mathbf{w}_j = \frac{1}{3\mu_j^2 - c}(\mu_j^2 - c, \mu_j, 1)$$

so that $\mathbf{w}_j\mathbf{v}_j = 1$ holds. We note that $3\mu_j^2 - c = \partial_\mu d(\mu_j) \neq 0$ if μ_j is simple. By a simple calculation, one can check that there is a simply connected domain Ω containing the closed right-half plane such that for $\lambda \in \Omega$,

$$\operatorname{Re}\mu_1 < \operatorname{Re}\mu_2, \operatorname{Re}\mu_3,$$

which implies that the assumption **H4** holds on Ω. **H1-H3** also hold on Ω since $u_c(x)$ exponentially decreases to zero as $|x| \to +\infty$. Hence the Evans function $D(\lambda)$ is defined on Ω. In particular, for $\lambda \in \Omega$ with $\operatorname{Re}\lambda > 0$,

$$\operatorname{Re}\mu_1 < 0 < \operatorname{Re}\mu_2, \operatorname{Re}\mu_3$$

holds. Therefore, the zeros of $D(\lambda)$ lies on the open left-half corresponds to the isolated eigenvalues of $\partial_x L_c$ in $L^2(\mathbb{R})$.

Asymptotic behavior of eigenvalues as $|\lambda| \to \infty$ Let $\widetilde{P} = \mu^3 + \lambda$ and $L(\mu) = -c\mu$. $\widetilde{P}$ has three simple roots $\tilde{\mu}_i = (-\lambda)^{1/3}$ for $\lambda \neq 0$. For μ sufficiently close to $\tilde{\mu}$,

$$\frac{|L(\tilde{\mu})|}{|\widetilde{P}'(\tilde{\mu})|} = \frac{c}{3|\lambda|^{1/3}}, \quad L(\mu) = -c\tilde{\mu}(1 + O(|\mu - \tilde{\mu}|), \quad \widetilde{P}'(\mu) = 3\tilde{\mu}^2(1 + O(|\mu - \tilde{\mu}|)).$$

We let $\rho(\lambda) = \rho_0 \dfrac{c}{3|\lambda|^{1/3}}$ for any $\rho_0 > 1$. Applying Proposition 5.47, we conclude that the characteristic polynomial $d(\mu; \lambda)$ in (5.85) has three simple zeros μ_i satisfying $|\mu_i - \tilde{\mu}_i| \leq \rho$.

Asymptotic behavior of $D(\lambda)$ as $|\lambda| \to \infty$ Let us define 3×3 matrices

$$W := \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \end{bmatrix} \quad \text{and} \quad V := \begin{bmatrix} \mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \end{bmatrix}.$$

By a direct computation, we obtain

$$\begin{aligned}
&(W(A - A^\infty)V)_{jk} \\
&= \left(W \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\partial_x(f'(u_c)) - f'(u_c)\mu_1 & -\partial_x(f'(u_c)) - f'(u_c)\mu_2 & -\partial_x(f'(u_c)) - f'(u_c)\mu_3 \end{bmatrix} \right)_{jk} \\
&= -\frac{\partial_x(f'(u_c)) + f'(u_c)\mu_k}{3\mu_j^2 - c},
\end{aligned}$$

where $j = 1, 2, 3$ is the row index and $k = 1, 2, 3$ is the column index. Using the above result for asymptotic behavior of the matrix eigenvalues, we obtain

$$\int_{-\infty}^{\infty} W(A - A^\infty)V\, dx \to 0 \quad \text{as} \quad |\lambda| \to \infty.$$

Applying Proposition 5.48, we conclude that $D(\lambda) \to 1$ as $|\lambda| \to \infty$.

Derivatives of $D(\lambda)$ at $\lambda = 0$ At $\lambda = 0$, we have $-\sqrt{c} = \mu_1 < \mu_2 = 0 < \mu_3 = \sqrt{c}$. We let $y^+ = y_1^+$, $z^- = z_3^-$, and $\mu = \mu_1$ for simplicity. At $\lambda = 0$, the following hold:

$$\begin{aligned} &y^+ \sim e^{\mu x} \quad (x \to +\infty), \\ &y^+ \sim D(\lambda) e^{\mu x} \quad (x \to -\infty), \\ &z^- \sim e^{-\mu x}(2c)^{-1} \quad (x \to -\infty), \\ &0 = \partial_x L_c y^+, \quad 0 = -L_c \partial_x z^-. \end{aligned}$$

Recall that the solitary wave solution $u_c(x)$ satisfies

$$-\partial_x^2 u_c + c u_c - f(u_c) = 0. \tag{5.87}$$

Differentiating (5.87) in x, we see that

$$0 = L_c \partial_x u_c, \quad 0 = \partial_x L_c \partial_x u_c, \tag{5.88}$$

that is, $\mathbf{y}_c := (\partial_x u_c, \partial_x^2 u_c, \partial_x^3 u_c)^T$ is a solution of (5.82) with $\lambda = 0$, and $\mathbf{z}_c$ defined by the relation (5.83) with $z_3 = u_c$ is a solution of the transpose ODE. Since $\mathbf{y}_c$ and $\mathbf{z}_c$ decays exponentially, $\mathbf{y}_c = o(e^{-\delta x})$ as $x \to +\infty$ and $\mathbf{z}_c = o(e^{+\delta x})$ as $x \to -\infty$ for sufficiently small $\delta > 0$. Thus, by Proposition 5.39, there exist non-zero constants β_1 and β_1' satisfying

$$y^+ = \beta_1 \partial_x u_c, \quad z^- = \beta_1' u_c.$$

Since y^+ decays exponentially (or bounded) as $x \to -\infty$, we conclude that $D(\lambda) = 0$. Since we know that $D(0) = 0$, the formula (5.57) is simplified, and we have[59]

$$D'(0) = \int_{-\infty}^{\infty} z^- y^+ \, dx = \beta_1 \beta_1' \int_{-\infty}^{\infty} u_c \partial_x u_c \, dx = 0. \tag{5.89}$$

Since $D(0) = D'(0) = 0$, we have

$$D''(0) = \int_{-\infty}^{\infty} z_\lambda^- y^+ + z^- y_\lambda^+ \, dx. \tag{5.90}$$

Here we need to determine exact values of β_1 and β_1'. From (5.80), we see that there is a constant $\beta > 0$ such that[60]

$$(u_c, \partial_x u_c) e^{-\mu x} \to \beta(1, \mu) \quad \text{as} \quad x \to +\infty.$$

Thus, we have

$$y^+ = (\beta\mu)^{-1} \partial_x u_c, \quad z^- = (2c\beta)^{-1} u_c. \tag{5.91}$$

Differentiating (5.81) and (5.84) in λ, we see that at $\lambda = 0$,

$$y^+ = \partial_x L_c y_\lambda^+, \quad z^- = -L_c \partial_x z_\lambda^-,$$

[59] Observe that $\int u^2 \, dx$ is constant of motion of the gKdV which is conserved.

[60] It is also possible to derive this from the equation (5.87). See [6] p.104.

where y^+_λ and z^-_λ decays to zero exponentially. By differentiating (5.87) in c,

$$L_c\partial_c u_c = -u_c, \quad \partial_x L_c \partial_c u_c = -\partial_x u_c, \quad L_c\partial_x \int_{-\infty}^{x} \partial_c u_c\, dx = -u_c. \tag{5.92}$$

Since

$$\partial_x L_c y^+_\lambda = y^+ = (\beta\mu)^{-1}\partial_x u_c = -(\beta\mu)^{-1}\partial_x L_c \partial_c u_c, \tag{5.93}$$

we have

$$\partial_x L_c(y^+_\lambda + (\beta\mu)^{-1}\partial_c u_c) = 0. \tag{5.94}$$

Similarly, we have

$$L_c\partial_x z^-_\lambda = -z^- = -(2c\beta)^{-1}u_c = (2c\beta)^{-1}L_c\partial_x \int_{-\infty}^{x} \partial_c u_c\, dx,$$

and thus,

$$L_c\partial_x(z^-_\lambda - (2c\beta)^{-1}\int_{-\infty}^{x} \partial_c u_c\, dx) = 0.$$

Therefore, there exist constants $\beta_2, \beta_2' \neq 0$ such that $\beta_2 y^+ = y^+_\lambda + (\beta\mu)^{-1}\partial_c u_c$ and $\beta_2' z^- = z^-_\lambda - (2c\beta)^{-1}\int_{-\infty}^{x} \partial_c u_c\, dx$.

From (5.91),

$$y^+_\lambda = \beta_2(\beta\mu)^{-1}\partial_x u_c - (\beta\mu)^{-1}\partial_c u_c \tag{5.95a}$$

$$z^-_\lambda = \beta_2'(2c\beta)^{-1}u_c + (2c\beta)^{-1}\int_{-\infty}^{x} \partial_c u_c\, dx. \tag{5.95b}$$

From (5.90), (5.91), and (5.95), we have

$$\begin{aligned} D''(0) &= (2c\beta^2\mu)^{-1}\int_{-\infty}^{\infty} \partial_x u_c \int_{-\infty}^{x} \partial_c u_c\, dx - u_c\partial_c u_c\, dx \\ &= -(c\beta^2\mu)^{-1}\int_{-\infty}^{\infty} u_c\partial_c u_c\, dx \\ &= -(c\beta^2\mu)^{-1}\partial_c\int_{-\infty}^{\infty} \frac{u_c^2}{2}\, dx. \end{aligned}$$

Since $\mu < 0$, we have $\operatorname{sgn} D''(0) = \operatorname{sgn}\partial_c \int_{-\infty}^{\infty} \frac{u_c^2}{2}\, dx$. We let $Q[u_c] := \frac{1}{2}\int_{-\infty}^{\infty} u_c^2\, dx$.

Sign of $\partial_c Q[u_c]$ Let $\alpha = \left(\frac{1}{2}c(p+1)(p+2)\right)^{1/p}$ and $\gamma = \frac{p\sqrt{c}}{2}$. Then, we have

$$Q[u_c] = \frac{\alpha^2}{2}\int \operatorname{sech}^{4/p}(\gamma x)\, dx = \frac{\alpha^2}{2\gamma}\int \operatorname{sech}^{4/p}(x)\, dx.$$

Thus,

$$
\begin{aligned}
Q[u_c]^{-1}\partial_c Q[u_c] &= \partial_c \ln Q[u_c] \\
&= \partial_c \left(2\ln\alpha + \ln \int \operatorname{sech}^{4/p}(x)\,dx - \ln 2 - \ln\gamma \right) \\
&= \partial_c \left(2\ln\alpha - \ln\gamma \right) \\
&= \partial_c \left(\frac{2}{p} \ln\left(\frac{c}{2}(p+1)(p+2) \right) - \ln p\sqrt{c} + \ln 2 \right) \\
&= \frac{2}{pc} - \frac{1}{2c} \\
&= \frac{4-p}{2cp}.
\end{aligned}
$$

Thus, we conclude that

$$\partial_c Q[u_c] > 0 \quad \text{if } p < 4 \tag{5.96a}$$

$$\partial_c Q[u_c] < 0 \quad \text{if } p > 4. \tag{5.96b}$$

References

[1] Bae, J., Kwon, B.: Small amplitude limit of solitary waves for the Euler-Poisson system. J. Differential Equations **266**, 3450-3478 (2019)

[2] Benjamin, T.B.: The stability of solitary waves. Proc. R. Soc. Lond. A, **328**, 153-183 (1972)

[3] Bona, J. L.: On the stability of solitary waves. Proc. Roy. Soc. London Ser. A **344**, 363-374 (1975)

[4] Boussinesq, J.: Théorie de l'intumescence liquide, applelée onde solitaire ou de translation, se propageant dans un canal rectangulaire. Comptes Rendus, **72**, 755–759(1871)

[5] Chen, F.F.: Introduction to plasma physics and controlled fusion. 2nd edition, Springer (1984)

[6] Coppel, W.A. : Stability and asymptotic behavior of differential equations. Boston: D.C. Heath and company. (1965)

[7] Coppel, W.A. : Dichotomies in stability theory. Lecture Notes in Math., Vol. 629, Springer, Berlin (1978)

[8] Cordier, S., Degond, P., Markowich, P, Schmeiser, C.: Travelling wave analysis of an isothermal Euler-Poisson model. Ann. Fac. Sci. Toulouse Math. **5**, 599-643 (1996)

[9] Davidson, R.C.: Methods in nonlinear plasma theory. 1st edition, Academic Press (1972)

[10] Engel, K. and Nagel, R.: One-Parameter Semigroups for Linear Evolution Equations. Springer-Verlag, New York (2000)

[11] Evans, J.W.: Nerve axon equations. I. Linear approximation. Indiana Univ. Math. J. **21**, 877-955 (1972)

[12] Evans, J.W.: Nerve axon equations. II. Stability at rest. Indiana Univ. Math. J. **22**, 75-90 (1972)

[13] Evans, J.W.: Nerve axon equations. III. Stability of the nerve impulse. Indiana Univ. Math. J. **22**, 577-594 (1972)

[14] Evans, J.W.: Nerve axon equations. IV. The stable and unstable impulse. Indiana Univ. Math. J. **24**, 1169-1190 (1975)

[15] Gardner, C. S., Morikawa, G. K.: Similarity in the asymptotic behavior of collision-free hydromagnetic waves and water waves. New York Univ., Courant Inst. Math. Sci., Res. Rep. NYO-9082 (1960)

[16] Guo, Y., Pu, X.: KdV limit of the Euler–Poisson system. Arch. Ration. Mech. Anal. **211**, 673-710 (2014)

[17] Haragus, M., Scheel, A. : Linear stability and instability of ion-acoustic plasma solitary waves. Physica D **170**, 13-30 (2002)

[18] Hartman, P.: Ordinary Differential Equations. John Wiley, New York (1964)

[19] Ikezi, H., Taylor, R.J., Baker, D.R.: Formation and interaction of ion-acoustic solitons. Phys. Rev. Lett. **25**(1), 11-14 (1970)

[20] Kato, T.: Perturbation theory for linear operators. Springer-Verlag, Berlin, (1980)

[21] Korteweg, D.J., de Vries, G.: On the change of form of long waves advancing in a rectangular channel, and on a new type of long stationary waves. Phil. Mag. (5) **39** 422-443 (1895)

[22] Li, Y., Sattinger, D.H. : Soliton Collisions in the ion acoustic plasma equations. J. Math. Fluid Mech. **1**, 117-130 (1999)

[23] Palmer, K.J.: Exponential dichotomies and transversal homoclinic points. J. Differential Equations **55**, 225-256 (1984)

[24] Palmer, K.J.: Exponential dichotomies and Fredholm operators. Proc. Amer. Math. Soc. **104**, 149-156 (1988)

[25] Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer-Verlag, New York (1983)

[26] Pego, R.L., Weinstein, M.I.: Eigenvalues, and instabilities of solitary waves. Philos. Trans. Roy. Soc. London Ser. A **340**, 47-94 (1992)

[27] Pego, R.L., Weinstein, M.I.: Asymptotic stability of solitary waves. Commun. Math. Phys. **164**, 305-349 (1994)

[28] Pego, R.L., Weinstein, M.I.: Convective linear stability of solitary waves for Boussinesq equations. Stud. Appl. Math. **99**, 311-375 (1997)

[29] Prüss, J.: On the spectrum of C_0-semigroups. Trans. Am. Math. Soc. **284**(2), 847–857 (1984).

[30] Russell, J.S.: Report on waves, Rep. 14th Meet. Brit. Assoc. Adv. Sci., York, 311-390 (1844)

[31] Sagdeev, R.Z.: Cooperative phenomena and shock waves in collisionless plasmas. “Reviews of Plasma Physics” (M. A. Leontoich, ed.), Vol.IV, Consultants Bureau, New York, 23-91 (1966)

[32] Sandstede, B.: Stability of travelling waves, in Handbook of Dynamical Systems. Vol. 2, B. Fielder, ed., North-Holland, Amsterdam, 983–1055 (2002)

[33] Sattinger, D.H.: On the stability of waves of nonlinear parabolic systems. Adv. in Math, **22**, 312-355 (1976)

[34] Washimi, H., Taniuti, T.: Propagation of ion-acoustic waves of small amplitude. Phys. Rev. Lett. **17**(9), 996–998 (1966)

[35] Kapitula, T., Promislow, K.: Spectral and Dynamical Stability of Nonlinear waves. Springer, New York (2013)

Acknowledgements

Firstly, I would like to express my sincere appreciation to my advisor Prof. Bongsuk Kwon for the continuous support of my Ph.D study and related research. His kind guidance helped me in all the time of research and writing of this thesis.

I was working at UNIST before I started my Ph.D study, and I had a chance to attend a reading seminar led by him on fluid equations for a few months. He introduced me a variety of aspects of the fluid equations and their physical origins. I did not know that such interesting phenomena and structures would exist in the field. This encounter with the fluid equations motivated me to decide to pursue my doctoral degree in this field. His enthusiasm, motivation, and immense knowledge convinced me that continuing my education with him would afford me the best possible training and opportunity. Now I cannot imagine having a better advisor and mentor for my Ph.D study and future research.

I would like to thank the rest of my thesis committee members: Prof. Chang Yeol Jung, Prof. Hantaek Bae, Prof. Jinmyoung Seok, and Prof. Injo Hur for their valuable and insightful comments. I also would like to thank to Prof. Kyudong Choi for his strong encouragement.

My sincere thanks also goes to Prof. Soyeon Jung and Prof. Jungjin Lee who provided me precious support for my research, and my labmates for the stimulating discussions.

Last but not the least, I would like to thank my family for supporting me spiritually throughout writing this thesis and my life.