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Portfolio Optimization with Transaction Costs

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Abstract

This paper examines the optimal investment and consumption problem in the the market with one risk-free, and one risky asset and presence of proportional transaction costs using shadow price approach. Simply speaking, shadow price process is virtual price that lies within the bid-ask price range in a frictionless market satisfying its maximal expected utilities to be equal to the maximal expected utilities of risky price process in an original proportional transaction costs market.

The procedure of the paper is composed as follows. Firstly, we briefly treat the optimal strategy in frictionless market. Then, we use shadow price process to derive heuristic free boundary problem and verify that candidate solution is really optimal solution for logarithmic utility function. Finally, we extend shadow price approach for all CRRA function to find the optimal strategy and state the explicit condition for finiteness of value function.

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1

Introduction

Merton's portfolio problem is the optimal investment and consumption problem. The objective function of Merton's portfolio problem is

$$\mathbb{E}[\int_0^T e^{-\delta t}U(c(t))dt + \epsilon^p e^{-\rho T}U(V_T)] \quad \text{where } T \in [0,\infty]$$

Here, $U(t)$ is the utility function, c_t is the nonnegative consumption rate, δ >0 denotes the impatient rate and $\epsilon > 0$ stands for the desired level of bequest. With $p < 1$, we consider constant relative risk aversion (CRRA) utility function $U : [0,\infty) \to \mathbb{R}$ which is given by

$$U(c) = \begin{cases} \log(c) & \text{if } p = 0 \text{ and } c > 0 \\ \frac{1}{p}c^p & \text{if } p \neq 0 \text{ and } c > 0 \end{cases} \text{ and } U(0) = \begin{cases} -\infty & \text{if } p \leq 0 \\ 0 & \text{if } p > 0. \end{cases} \tag{1.0.1}$$

Because most people put more value to the present than to the future, it is fairly reasonable to consider the case $p < 1$ to make utility function concave.

Under frictionless market assumption and CRRA utility function, Merton shows that the optimal investment is a constant fraction of value of risky assets over value of risky and risk-free assets. The constant fraction is called Merton's line. The frictionless market assumption has the following properties: same rate of borrowing and lending interest, no transaction cost, no bid-ask spreads, no transaction time, no taxes and no restriction of time and quantity in trade.

Many researchers have tried to remove the assumptions in the frictionless market condition. Among such trials, many works have been done to exclude the transaction cost under the frictionless market condition. Under finite value process and constant rate of transaction cost, [1],[2] and [3] showed optimal investment and consumption. When the value process is finite, Merton's portfolio problem in infinite horizon reduces to

$$\mathbb{E}\int_0^\infty e^{-\delta t}u(c(t))dt \tag{1.0.2}$$

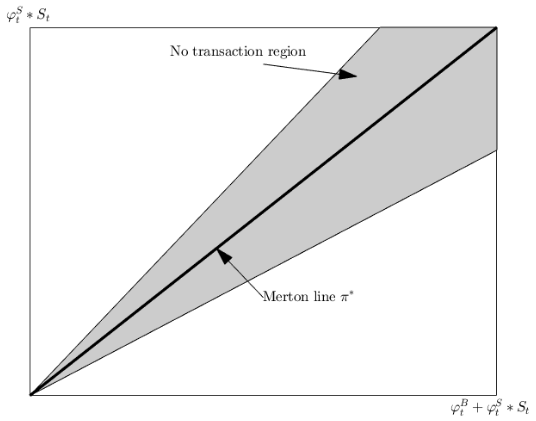

Figure 1-1: Optimal strategy frictionless market and market which have transaction cost. X-axis denotes sum of risky and risk-free asset value, and Y-axis denotes risky asset value

Davis and Norman were the first to show rigorously with mathematics the optimal portfolio and consumption using dynamic programming principle when the value process is $C^{1,2}$. They showed that the optimal portfolio is defined by the two straight lines. Merton's line lies between the two straight lines, and within two straight lines the agents do not trade. Out of the straight lines, agents instantly change the strategy to adjust the portfolio to the nearest of the two lines as illustrated in Figure 1-1.

Later, Shreve and Soner showed that value process is really $C^{1,2}$ using viscosity solution to Hamilton-Jacobi-Bellman (HJB) equations.

Kalllsen and Muhle-Karbe is the first to show the solution without dynamic programming principle when utility function is a logarithmic function. Instead of using the dynamic programming principle, they used shadow price process. In a concise manner, shadow price process $\tilde{S}$ is defined as the price process that lies within the bid-ask price range in a frictionless market satisfying its maximal expected utilities to be equal to the maximal expected utilities of risky price process S in an original constant rate of transaction cost market. Since any price process which lies in the bid-ask price range in a frictionless market has larger maximal utilities than processes in an original market with a transaction cost, shadow price is the lowest price process.

All of the above works under the assumption that value process is finite. Choi, Sirbi and Zitokovic is the first to show the necessary and sufficient condition of a finite value process and find the solution for all CRRA types using shadow price process.

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Mathematical modeling

The paper considers a market which have two assets. One is a risk-free asset whose price process is 1, and the other is a risky asset whose price process S_t is defined by

$$dS_t = S_t(\mu dt + \sigma dB_t), \tag{2.0.1}$$

where μ and σ are positive constants and B is Brownian motion. Market has complete filtered probability space where saturate filtration generated by Brownian motion B. We will assume that the market satisfies frictionless market condition except transaction costs. Throughout the paper, transaction costs are constant fractions of risky asset price. That is, the buying and selling price are represented by $\overline{S}_t = (1+\overline{\lambda})S_t$ and $\underline{S}_t = (1-\underline{\lambda})S_t$, where $\overline{\lambda} > 0$ and $\underline{\lambda} \in [0,1)$. The agent's portfolio and consumption strategy is defined by $(\varphi^B, \varphi^S, c)$ where φ^B and φ^S represent the number of risk-free asset and risky asset, respectively. The process φ^B, φ^S and c are progressively measurable processes. The processes φ^B and φ^S are right-continuous and of finite variation, and c is a nonnegative process which satisfies $\int_0^t c_s ds < \infty$ a.s for all $t \geq 0$.

Definition Admissible strategy (φ, c) which satisfies (i) and (ii).

(i) Agent's wealth is nonnegative for all $t \geq 0$. Moreover, agent's initial wealth is positive. Positive wealth means that the agent can liquidate the wealth to a positive amount of cash. That is, the value process $V(\varphi^B, \varphi^S, \underline{S}_0, \overline{S}_0) > 0$ where

$$V(\varphi^B, \varphi^S, \underline{S}, \overline{S}) = \varphi^B + \underline{S}(\varphi^S)^+ - \overline{S}(\varphi^S)^-. \tag{2.0.2}$$

Let (η^B, η^S) be the initial number of risk-free asset and risky asset for agent. Then, an admissible strategy (φ, c) satisfies

$$V(\eta^B, \eta^S, \underline{S}_0, \overline{S}_0) > 0, \quad V(\varphi^B_t, \varphi^S_t, \underline{S}_t, \overline{S}_t) \geq 0 \quad \text{for all t} > 0.$$

(ii) No funds are added or subtracted to agent, so agent's strategy is self-financing. That is,

$$\varphi_t^B = \eta^B + \int_0^t \underline{S}_s d(\varphi_s^S)^{\downarrow} - \int_0^t \overline{S}_s d(\varphi_s^S)^{\uparrow} - \int_0^t c_s ds. \tag{2.0.3}$$

Here, $d(\varphi^S) = d(\varphi^S)^{\uparrow} - d(\varphi^S)^{\downarrow}$, which is Hahn-decomposition of φ^S.
Given (η_B, η_S) and price process S, the set of admissible strategy is denoted by $\mathcal{A}(S)$. Therefore, the optimal portfolio problem (1.0.2) is rephrased by

$$u(S) := \sup_{(\varphi^B,\varphi^S,c)\in\mathcal{A}(S)} \mathbb{E}[\int_0^{\infty} e^{-\delta t} U(c_t)dt]. \tag{2.0.4}$$

Definition Define $\mathcal{S}$ be a set of Ito process $\tilde{S}$ satisfying

$$\underline{S}_t \leq \tilde{S}_t \leq \overline{S}_t \;\text{ for all } t \geq 0, \text{ a.s.} \tag{2.0.5}$$

$\tilde{S} \in \mathcal{S}$ is called shadow price process if the maximal expected utilities for S with transaction cost and for without transaction cost coincide. That is,

$$u(S) = u(\tilde{S}) = \sup_{(\varphi^B,\varphi^S,c)\in\mathcal{A}(\tilde{S})} \mathbb{E}[\int_0^{\infty} e^{-\delta t} U(c_t)dt]. \tag{2.0.6}$$

Remark Because agents in frictionless market can make more consumption by the difference of price in transaction cost compared to the original transaction market, it is always satisfied that

$$\sup_{(\tilde{\varphi}^B,\tilde{\varphi}^S,\tilde{c})\in\mathcal{A}(\tilde{S})} \mathbb{E}\left(\int_0^{\infty} e^{-\delta t}\log(\tilde{c}_t)dt\right) \geq \sup_{(\varphi^B,\varphi^S,c)\in\mathcal{A}(S)} \mathbb{E}\left(\int_0^{\infty} e^{-\delta t}\log(c_t)dt\right), \tag{2.0.7}$$

for all $\tilde{S} \in \mathcal{S}$. To be more specific, fix $\tilde{S} \in \mathcal{S}$ and $(\varphi^B, \varphi^S, c) \in \mathcal{A}(S)$. Choose $\tilde{c}$ such that $(\varphi^B, \varphi^S, \tilde{c})$ to satisfy self-financing condition in frictionless market with price process $\tilde{S}$. Then,

$$d\varphi_t^B = \underline{S_t} d(\varphi_t^S)^{\downarrow} - \overline{S_t} d(\varphi_t^S)^{\uparrow} - c_t dt = \tilde{S}_t d(\varphi_t^S)^{\downarrow} - \tilde{S}_t d(\varphi_t^S)^{\uparrow} - \tilde{c}_t dt$$
$$(\tilde{c}_t - c_t)dt = (\tilde{S}_t - \underline{S_t})d(\varphi_t^S)^{\downarrow} + (\overline{S}_t - \tilde{S}_t)d(\varphi_t^S)^{\uparrow} \geq 0$$

So, this implies $(\varphi^B, \varphi^S, \tilde{c}) \in \mathcal{A}(\tilde{S})$ and the above statement. For the equality to hold, the transaction of optimal strategy in $A(\tilde{S})$ has to be done only when $\tilde{S}$ reaches the $\underline{S}$ or $\overline{S}$. So, shadow price process can be thought as worst Ito process satisfying $\underline{S_t} \leq \tilde{S}_t \leq \overline{S}_t$ for all t≥ 0, a.s.

3

Frictionless market

In this section, the optimal portfolio/consumption $(\tilde{\varphi}, \tilde{c})$ will be found under no transaction cost. This section refers to [1] for the main contents. Since there is no transaction cost, we introduce π instead of (φ^B, φ^S), where π is defined by

$$\pi = \frac{\varphi^S S}{\varphi^B + \varphi^S S}, \tag{3.0.1}$$

which is a fraction of risky asset over risky and risk-free asset. To prove the theorem simply, a more narrow admissible strategy than $\mathcal{A}(S)$ is needed. Define admissible strategy $\mathcal{A}(\pi) = \{(\pi, c) : (\varphi^B, \varphi^S, c) \in \mathcal{A}(S) \text{ and } |\pi| \text{ is bounded}\}$. Consequently, the following results hold.

Theorem 3.0.1. *Suppose that the following condition holds.*

$$\delta > p(\mu + \frac{\mu^2}{\sigma^2(1-p)}). \tag{3.0.2}$$

Then, the optimal optimal policy $(\tilde{c}, \tilde{\pi}) \in \mathcal{A}(\pi)$ is

$$\tilde{c}_t = CV(\varphi_t^B, \varphi_t^S, S_t, S_t) \text{ and } \tilde{\pi}_t = \frac{\mu}{\sigma^2(1-p)}. \tag{3.0.3}$$

where C is defined by

$$C = \frac{1}{1-p}(\delta - \frac{\mu^2 p}{2\sigma^2(1-p)}). \tag{3.0.4}$$

The procedure of the present section is composed as follows. First, the optimal investment and consumption solution is heuristically derived without rigorous verification. During heuristic derivation, subtle mathematical issues are not cared. Then, it is rigorously verified that it is really a solution by showing that the objective functions are same. Actually, these two steps are used for all following sections. This section is helpful for getting a sense in how to prove the optimal portfolio/consumption problem.

3.1 Heuristic derivation

Throughout this section, we use the notation C in (3.0.4). For simple computation, we abbreviate $V_t = V(\varphi_t^B, \varphi_t^S, S_t, S_t)$. Ito's lemma gives

$$\begin{aligned} dV_t &= d\varphi_t^B + S_t d\varphi_t^S + \varphi_t^S dS_t + <\varphi^S, S>_t \\ &= -c_t dt + \varphi_t^S S_t(\mu dt + \sigma dB_t) \\ &= (\mu\pi_t V_t - c_t)dt + \sigma\pi_t V_t dB_t \end{aligned} \tag{3.1.1}$$

Self-financing condition (2.0.3) is used for the second equality in (3.1.1).

To use dynamic programming principle, define function $v(x) : [0,\infty) \to [-\infty,\infty)$ given by

$$v(x) = \sup_{(\pi,c)\in\mathcal{A}(\pi)} \mathbb{E}[\int_0^\infty e^{-\delta t} U(c_t)dt], \tag{3.1.2}$$

where $x = \eta_B + \eta_S S_0 = V_0$. Note that $\mathcal{A}(\pi)$ changes when the initial wealth x changes. With the assumption that the value process V_t is finite, it is possible to use dynamic programming principle(see, corollary 4.2 in [2]). The following lemma is an applied version to suit the model.

Lemma 3.1.1. ***(Dynamic programming principle)*** *Let $(\pi, c) \in \mathcal{A}(\pi)$ be given. Then, it satisfies*

$$v(V_0) = \sup_{(\pi,c)\in\mathcal{A}(\pi)} \mathbb{E}[\int_0^t e^{-\delta s} U(c)ds + e^{-\delta t} v(V_t)] \tag{3.1.3}$$

Here, $\mathcal{A}(\pi)$ defined by V_0.

Assume that v is C^2 function. Using Ito's lemma and (3.1.3), the following relation is established

$$\begin{aligned} 0 &= \sup_{(\pi,c)\in\mathcal{A}(\pi)} \mathbb{E}[\int_0^t e^{-\delta s} U(c)ds + e^{-\delta t} v(V_t) - v(V_0)] \\ &= \sup_{(\pi,c)\in\mathcal{A}(\pi)} \mathbb{E}[\int_0^t e^{-\delta s}\Big(U(c) - \delta v(V_s) + (\mu\pi_s V(s) - c_s)v'(V_s) \\ &\qquad + \frac{1}{2}(\sigma^2\pi_s^2 V_s^2)v''(V_s)\Big)ds + \int_0^t e^{-\delta s}\sigma\pi_s V_s v'(V_s)dB_s]. \end{aligned}$$

Assume that $v'(x) = \frac{dv}{dx}$ is bounded. Hence, the last term is zero and Hamilton-Jacobi-Bellman

(HJB) equation is made

$$\begin{cases} \max_{(\pi,c)\in\mathcal{A}(\pi)}(\log(c) + (\mu\pi x - c)v' + \frac{1}{2}x^2\pi^2\sigma^2 v'' - \delta v) = 0 & \text{if } p = 0 \\ \max_{(\pi,c)\in\mathcal{A}(\pi)}(\frac{1}{p}c^p + (\mu\pi x - c)v' + \frac{1}{2}x^2\pi^2\sigma^2 v'' - \delta v) = 0 & \text{if } p \neq 0. \end{cases} \tag{3.1.4}$$

Then the maximum is obtained at $(\tilde{c}, \tilde{\pi})$ defined by

$$\tilde{c} = v^{\frac{1}{p-1}} \quad \text{and} \quad \tilde{\pi} = \frac{-\mu v'}{x\sigma^2 v''}. \tag{3.1.5}$$

So, (3.1.5) changes to

$$\begin{cases} -\frac{\mu^2}{2\sigma^2}\frac{(v')^2}{v''} - \log(v) - 1 - \delta v = 0 & \text{if } p = 0 \\ -\frac{\mu^2}{2\sigma^2}\frac{(v')^2}{v''} + \frac{1-p}{p}(v')^{\frac{p}{p-1}} - \delta v = 0 & \text{if } p \neq 0. \end{cases} \tag{3.1.6}$$

Define $\tilde{v}(x)$

$$\tilde{v}(x) = \begin{cases} \frac{1}{2}(\frac{\mu}{\delta\sigma})^2 + \frac{1}{\delta}(\log(\delta x) - 1) & \text{if } p = 0 \\ \frac{1}{p}C^{p-1}x^p & \text{if } p \neq 0. \end{cases} \tag{3.1.7}$$

Then, $\tilde{v}(x)$ satisfies (3.1.6), and maximizing π and c under $\tilde{v}(x)$ coincide definition of $\tilde{\pi}$ and $\tilde{c}$ in (3.0.3). $\tilde{\pi}$, $\tilde{c}$ and $\tilde{v}$ are used for the candidate solution of optimization problem.

3.2 Verification of the optimal strategy

In this subsection, we will verify that the candidate solution (3.0.3) is an really optimization solution. Throughout this subsection, notations $\tilde{c}$, $\tilde{\pi}$ defined by (3.0.3) and $\tilde{v}$ given by (3.1.7) are used. Since $\tilde{c}$ is nonngeative and $\tilde{\pi}$ is bounded, $(\tilde{\pi}, \tilde{c}) \in \mathcal{A}(\pi)$. Here, it is considered only the case of $p \neq 0$ for the proof. It can be proved in a similar way for $p = 0$. Fix $(\pi, c) \in \mathcal{A}(\pi)$. Then, V_t is given by

$$V_t = e^{\int_0^t(\mu\pi_s - \frac{1}{2}\sigma^2\pi_s^2)ds + \int_0^t \sigma\pi_s dB_s}(V_0 - \int_0^t c_s e^{-\int_0^s(\mu\pi_u - \frac{1}{2}\sigma^2\pi_u^2)du - \int_0^s \pi_u dB_u} ds).$$

From Hölder inequality and boundedness of π_t, V_t has finite moments of all orders. Define process M_t by

$$M_t = \int_0^t e^{-\delta s}U(c_s)ds + e^{-\delta t}\tilde{v}(V_t).$$

Using Ito's lemma, it follows

$$M_t - M_0 = \int_0^t e^{-\delta s}\Big((\mu\pi_s V_s - c_s)\tilde{v}'(V_s) + \frac{\sigma^2\pi_s^2 V_s^2}{2}\tilde{v}''(V_s) + U(c_s) - \delta\tilde{v}(V_s)\Big)ds$$
$$+ \sigma C^{p-1}\int_0^t e^{-\delta s}\pi_s V(s)^p dB_s. \qquad (3.2.1)$$

Since π_t and V_t^p are bounded, expectation of last term in (3.2.1) is zero. Since integrand of first term is nonpositive from (3.1.4), M_t is supermartingale and is martingale when $(c,\pi) = (\tilde{c},\tilde{\pi})$. Therefore, it satisfies

$$\tilde{v}(V_0) = M_0 \geq \mathbb{E}[M_t] = \mathbb{E}[\int_0^t e^{-\delta s}U(c_s)]ds + e^{-\delta t}\mathbb{E}[\tilde{v}(V_t)]. \qquad (3.2.2)$$

From Ito's lemma and (3.1.1), the following equation holds

$$\mathbb{E}[e^{-\delta t}\tilde{v}(V_t)] = \frac{C^{p-1}V_0^p}{p}\mathbb{E}[G_t e^{\int_0^t a(s)ds}], \qquad (3.2.3)$$

where $G_t = e^{-\frac{1}{2}\int_0^t p^2\sigma^2\pi_s^2 ds + \int_0^t p\sigma\pi_s dB_s}$ and $a(s) = p(\mu\pi_s - \frac{c_s}{V_s} - \frac{1}{2}(1-p)\pi_s^2\sigma^2) - \delta$. Since differential dG_t is $dG_t = p\sigma\pi_t dB_t$ and π is bounded, G_t is martingale. Also, $a(s)$ satisfies $a(s) = -C$ when $(c,\pi) = (\tilde{c},\tilde{\pi})$. So, the last term in (3.2.2) goes to 0 as $t \to \infty$. Therefore $\tilde{v}(V_0) = \mathbb{E}[\int_0^\infty e^{-\delta t}U(\tilde{c}_t)dt]$.

Divide the cases $0 < p < 1$ and $p < 0$. For $0 < p < 1$, $a(s)$ has upper bound $(p-1)C$ which is less than 0. Thus, last term in (3.2.2) goes to 0 as $t \to \infty$. For $0 < p < 1$, $\tilde{v}$ is equal to v, and $(\tilde{c},\tilde{\pi})$ is an optimal process. Hence, $(\tilde{c},\tilde{\pi})$ is an optimal process. For $p < 0$, it is required to change the entire proof. Fix $\epsilon > 0$. Define $\tilde{v}_\epsilon(x) = \frac{1}{p}C^{p-1}(x+\epsilon)^p$. Then it satisfies

$$-\frac{\mu^2}{2\sigma^2}\frac{(\tilde{v}_\epsilon')^2}{\tilde{v}_\epsilon''} + \frac{1-p}{p}(\tilde{v}_\epsilon')^{\frac{p}{p-1}} - \delta\tilde{v}_\epsilon = 0. \qquad (3.2.4)$$

Same procedures for $\tilde{v}$ generate $\tilde{v}_\epsilon(V_0) \geq \mathbb{E}\int_0^t e^{-\delta t}U(c_s)ds$ for all $(\pi,c) \in \mathcal{A}(\pi)$. Since $\tilde{v}_\epsilon(V_0) \to \tilde{v}(V_0)$ as $\epsilon \to 0$, $\tilde{v}$ is same to v, and $(\tilde{c},\tilde{\pi})$ is an optimal process.$\square$

4

The Work of Kallsen and Muhle-Karbe

For present section, we will review the paper [4]. Throughout this section, it is assumed that μ and σ satisfy

$$0 < \mu < \sigma^2 \tag{4.0.1}$$

Here, only the logarithmic utility function is considered which is the case when $p = 0$. [4] is the first paper to use shadow price process for solving optimal portfolio problem to my knowledge. Assumption (4.0.1) is used for the existence of explicit formula for shadow price process. This assumption is more restricted than assumption (3.0.2) which requires $\delta > 0$ for existence of solution for $p = 0$. In the next section, the assumption will be modified to coincide with $\delta > 0$.

4.1 Heuristic derivation of free boundary problem

In this subsection, free boundary problem, which is one dimensional second order ODE, will be heuristically derived. This free boundary problem is used for making candidate solution of the problem. Next section, it will be mathematically verified that it is really a solution.

Assume that shadow price $\tilde{S}$ exists and has the form

$$\tilde{S} = Se^{C}. \tag{4.1.1}$$

Here, C is the Ito process with $C \in [\underline{C}, \overline{C}]$ where

$$\underline{C} := \log(1 - \underline{\lambda}), \ \overline{C} := \log(1 - \overline{\lambda}). \tag{4.1.2}$$

The reason for using this form for price process is to use the knowledge of market with no transaction cost in theorem 3.0.1. Because it is derived from the optimal portfolio/consumption $(\tilde{\pi}, \tilde{c})$ in no transaction market when price process follows $S_t = S_0 e^{\left((\mu - \frac{\sigma^2}{2})t + \sigma W_t\right)}$, which is similar form to $\tilde{S}_t = S_t e^{C_t}$.

Since C_t is Ito process, dynamics of the process C_t is given by

$$dC_t = \tilde{\mu}(C_t)dt + \tilde{\sigma}(C_t)dW_t \tag{4.1.3}$$

where $\tilde{\mu}$ and $\tilde{\sigma}$ are deterministic functions. In case of shadow price process, there is no transaction cost. Thus, it is enough to find only $\tilde{\pi}$ and $\tilde{c}$ to determine the optimal portfolio/consumption $(\tilde{\varphi}, \tilde{c})$ where

$$\tilde{\pi} = \frac{\varphi^S \tilde{S}}{\varphi^B + \varphi^S \tilde{S}} \tag{4.1.4}$$

which is the fraction of risky asset and value process with respect to shadow price $\tilde{S}$. Therefore, it remains to find four unknown variable $\tilde{\mu}, \tilde{\sigma}, \tilde{\pi}$ and $\tilde{c}$ to solve the problem. From (3.0.3), $\tilde{\pi}$ and $\tilde{c}$ are given by

$$\tilde{\pi} = \frac{\mu - \frac{\sigma^2}{2} + \tilde{\mu}(C)}{(\sigma + \tilde{\sigma}(C))^2} + \frac{1}{2}, \quad \tilde{c} = \delta \tilde{V}(\varphi) \quad where \quad \tilde{V}(\varphi) = \varphi^B + \varphi^S \tilde{S}. \tag{4.1.5}$$

Because the shadow price process $\tilde{S}$ is

$$\tilde{S}_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2} + \tilde{\mu}(C) + \frac{1}{2}(\sigma + \tilde{\sigma})^2\right)t + (\sigma + \tilde{\sigma})W_t},$$

$\tilde{\pi}$ is defined as (4.1.5). Therefore, the values we have to seek are reduce to two variables $\tilde{\mu}$ and $\tilde{\sigma}$. To solve the problem, we define new variable X which is

$$X := \log(\frac{\tilde{\pi}}{1 - \tilde{\pi}}) = \log(\varphi^S) + \log(\tilde{S}) - \log(\varphi^B) \tag{4.1.6}$$

The reasons for defining X are not only for simple calculation, but it can handle existence of Skorokhod SDE.

To have same utility values, trades happen only when $\tilde{S}$ is same to bid or ask price. Therefore, φ^S is constant on (0,T) where T:= $\inf\{t > 0 : C_t \in \{\underline{C}, \overline{C}\}\}$.

$$d\log(\varphi_t^B) = \frac{1}{\varphi_t^B} d\varphi_t^B = \frac{-c_t}{\varphi_t^B} dt = \frac{-\delta \tilde{V}_t(\varphi)}{\tilde{V}_t(\varphi) - \tilde{\pi}\tilde{V}_t(\varphi)} = \frac{-\delta}{1 - \tilde{\pi}} \quad \text{for } t \in (0, T).$$

So, dX_t is

$$\begin{aligned} dX_t &= d(\log(\varphi^S) + \log(\tilde{S}) - \log(\varphi^B)) \\ &= 0 + d(\log S_t + C_t) - \frac{-\delta}{1-\tilde{\pi}_t} \\ &= (\mu - \frac{\sigma^2}{2} + \tilde{\mu} + \frac{\delta(\sigma+\tilde{\sigma})^2}{\frac{1}{2}(\sigma+\tilde{\sigma})^2 - (\mu - \frac{\sigma^2}{2} + \tilde{\mu})})dt + (\sigma+\tilde{\sigma})dW_t \qquad \text{for t} \in (0,T) \end{aligned} \tag{4.1.7}$$

Since $\tilde{\pi}$ is a function of C, let $X = g(C)$ for some function g. Assume that g is C^2 function. Applying Ito's lemma, differential dX follows

$$dX_t = (g'(C_t)\tilde{\mu}(C_t) + g''(C_t)\frac{\tilde{\sigma}(C_t)^2}{2})dt + (\sigma + \tilde{\sigma}(C_t))dW_t \tag{4.1.8}$$

By (4.1.5), (4.1.7) and (4.1.8), three equations for $g(C), \tilde{\sigma}(C)$ and $\tilde{\mu}(C)$ is obtained with $C \in (\underline{C}, \overline{C})$ which are

$$\frac{1}{1+e^{-g(C)}} = \frac{\mu - \frac{\sigma^2}{2} + \tilde{\mu}(C)}{(\sigma+\tilde{\sigma}(C))^2} + \frac{1}{2} \tag{4.1.9}$$

$$\mu - \frac{\sigma^2}{2} + \tilde{\mu}(C) + \frac{\delta(\sigma+\tilde{\sigma}(C))^2}{\frac{(\sigma+\tilde{\sigma}(C))^2}{2} - (\mu - \frac{\sigma^2}{2} + \tilde{\mu}(C))} = \tilde{\mu}(C)g'(C) + \frac{\tilde{\sigma}(C)^2}{2}g''(C) \tag{4.1.10}$$

$$\sigma + \tilde{\sigma}(C) = \tilde{\sigma}(C)g'(C) \tag{4.1.11}$$

By (4.1.9) and (4.1.11), $\tilde{\sigma}(C)$ and $\tilde{\mu}(C)$ are found for $C \in (\underline{C}, \overline{C})$.

$$\tilde{\sigma}(C) = \frac{\sigma}{g'(C)-1}, \quad \tilde{\mu}(C) = -(\mu - \frac{\sigma^2}{2}) + \frac{\sigma^2}{2}(\frac{g'(C)}{g'(C)-1})^2 \frac{1-e^{-g(C)}}{1+e^{-g(C)}} \tag{4.1.12}$$

Inserting $\tilde{\sigma}$ and $\tilde{\mu}$ to second equation, following second order equation holds

$$\begin{aligned} g''(x) &= \frac{2\delta}{\sigma^2}(1+e^{g(x)}) + \Big(\frac{2\mu}{\sigma^2} - 1 - \frac{4\delta}{\sigma^2}(1+e^{g(x)})\Big)g'(x) \\ &+ \Big(-\frac{4\mu}{\sigma^2} + \frac{2}{1+e^{-g(x)}} + 1 + \frac{2\delta}{\sigma^2}(1+e^{g(x)})\Big)(g'(x))^2 \\ &+ (\frac{2\mu}{\sigma^2} - \frac{2}{1+e^{-g(x)}})(g'(x))^3 \quad \text{for x} \in (\underline{C}, \overline{C}). \end{aligned} \tag{4.1.13}$$

Because there are no boundary conditions, it is needed to make them heuristically to find solution. Since φ^S only changes when $\tilde{S}$ hits $\underline{S}$ or $\overline{S}$, φ^S have singular part. From relation $X = \log(\varphi^S) + \log(\tilde{S}) - \log(\varphi^B)$, X also has singular part which changes only when C reaches the boundary. So, g cannot be $C^2[\underline{C}, \overline{C}]$ function. If g is $C^2[\underline{C}, \overline{C}]$, X is Ito process by Ito's lemma. So, it is impossible to $g \in C^2[\underline{C}, \overline{C}]$. From the relations $X_t = \log(\frac{\varphi_t^S \tilde{S}_t}{\varphi_t^B}) = g(C_t)$ and singular part of φ^S, it is expected that $g'(\underline{C}) = g'(\overline{C}) = -\infty$.

To avoid infinite value in boundary conditions, define function $f : [\underline{\beta}, \overline{\beta}] \to \mathbb{R}$ by $f = g^{-1}$. Then, equation (4.1.13) changes to

$$\begin{aligned} f''(x) = (-\frac{2\mu}{\sigma^2} + \frac{2}{1+e^{-x}}) + \Big(\frac{4\mu}{\sigma^2} - \frac{2}{1+e^{-x}} - 1 - \frac{2\delta}{\sigma^2}(1+e^x)\Big)f'(x) \\ + \Big(- \frac{2\mu}{\sigma^2} + 1 + \frac{4\delta}{\sigma^2}(1+e^x)\Big)(f'(x))^2 - (\frac{2\delta}{\sigma^2}(1+e^x))(f'(x))^3 \end{aligned} \tag{4.1.14}$$

with boundary conditions

$$f(\underline{\beta}) = \overline{C}, \quad f(\overline{\beta}) = \underline{C}, \quad f'(\underline{\beta}) = f'(\overline{\beta}) = 0 \tag{4.1.15}$$

4.2 Verification of the optimal strategy

The procedure of this subsection will be as follows. First, it will be shown that the existence of solution to the free boundary problem derived above. Next, it will be justified that β is the solution to an SDE with instantaneous reflection and $\tilde{S} = Se^C = S^{g(\beta)}$ is really a shadow price process by showing that objective functions are same.

Lemma 4.2.1. *There exist constants $\underline{x} < \overline{x}$ and a strictly decreasing function $f : [\underline{x}, \overline{x}] \to [\underline{C}, \overline{C}]$ satisfying the free boundary problem (4.1.14) and (4.1.15).*

proof) By the assumption $0 < \frac{\mu}{\sigma^2} < 1$, solution for $\frac{2}{1+e^{-x}} - \frac{2\mu}{\sigma^2} = 0$ always exists. Let the solution $x_0 = -\log(\frac{\sigma^2}{\mu} - 1)$. For $\triangle > 0$, define $x_\triangle := x_0 - \triangle$. Then, there exist local solution $f_\triangle$ with $f'_\triangle(\underline{x}_\triangle) = 0$ and $f_\triangle(\underline{x}_\triangle) = \overline{C}$. Let $\overline{x}_\triangle$ be the first of zeros after $\underline{x}_\triangle$. Proof is divided into some steps to show that $f_\triangle(\overline{x}_\triangle) = \underline{C}$ for all $\underline{C} < \overline{C}$ for properly chosen $\triangle$.
claim 1. Existence of $\overline{x}_\triangle$.

Set $M' := \max\left\{ \sqrt[3]{\frac{4(\mu+\sigma^2}{\delta}}, \sqrt{\frac{8\mu}{\delta}}, 8 + \frac{4\mu+2\sigma^2}{\delta} \right\}$. Hence, it establishes

$$\begin{cases} f''_\triangle(x) > 0 & \text{if } f'_\triangle(x) < -M'. \\ f''_\triangle(x) < 0 & \text{if } f'_\triangle(x) > M'. \end{cases}$$

by (4.1.14). So, $f'_\triangle(x) \in [-M', M']$ for all $x \in \mathbb{R}$.

By (4.1.14) and $\triangle > 0$, it implies that $f''_\triangle(x) < 0$ for some neighborhood $\mathcal{U}$ of $\underline{x}_\triangle$. So, $f'_\triangle(x) < 0$ for $x \in \mathcal{U} \cap (\underline{x}_\triangle, \infty)$. If there is no $x \in (\underline{x}_\triangle, \infty)$ such that $f'_\triangle(x) = 0$, right-hand side of (4.1.14) is strictly positive when x is sufficiently large. So, I have more zeros of $f'_\triangle$ after $\underline{x}_\triangle$.
claim 2. $f_\triangle(\overline{\beta}_\triangle) \to \overline{C}$ as $\triangle \to 0$.
If $\underline{x}_\triangle < \overline{x}_\triangle \leq x_0 + \triangle$, then $\overline{x}_\triangle - \underline{x}_\triangle \leq 2\triangle$. So, $\overline{x}_\triangle - \underline{x}_\triangle \to 0$ as $\triangle \to 0$. So, it remains to show case $\overline{x}_\triangle > x_0 + \triangle$.

Suppose that $\overline{x}_\triangle > x_0 + \triangle$. For $|x - x_0| < 1$, (4.1.14) and $f_\triangle(x) \in [-M', M']$ make

$$\begin{aligned}|f''_\triangle(x)| < M'' := \frac{2\mu}{\sigma^2} + 2 + \Big(\frac{4\mu}{\sigma^2} + 3 + \frac{2\delta}{\sigma^2}(1 + e^{x_0+1})\Big)M' \\ + \Big(\frac{2\mu}{\sigma^2} + 1 + \frac{4\delta}{\sigma^2}(1 + e^{x_0+1})\Big)(M')^2 + \frac{2\delta}{\sigma^2}(1 + e^{x_0+1})(M')^3\end{aligned}$$

When $\triangle < 1$, $|f'_\triangle(x)| \le 2M''\triangle$ for $x \in [x_0 - \triangle, x_0 + \triangle]$ by mean value theorem. $\sup_{x\in[x_0-\triangle,x_0+\triangle]} |f'_\triangle(x)| \to 0$ as $\triangle \to 0$ and (4.1.14) imply

$$\sup_{x\in[x_0-\triangle,x_0+\triangle]} |f''_\triangle(x)| \to 0 \qquad \text{as } \triangle \to 0. \tag{4.2.1}$$

For $\triangle$ sufficiently small, $x \in [x_0 + \triangle, x_0 + 1]$,

$$|f'_\triangle| < m_\triangle := \max\left\{\frac{\frac{1}{3}(\frac{-\mu}{\sigma^2} + \frac{1}{1+e^{-(x_0+\triangle)}})}{\frac{4\mu}{\sigma^2} + 3 + \frac{2\delta(1+e^{x_0+1})}{\sigma^2}}, \frac{\frac{1}{3}(\frac{-\mu}{\sigma^2} + \frac{1}{1+e^{-(x_0+\triangle)}})}{\frac{2\mu}{\sigma^2} + 1 + \frac{4\delta(1+e^{x_0+1})}{\sigma^2}}, \sqrt[3]{\frac{\frac{1}{3}(-\frac{\mu}{\sigma^2} + \frac{1}{1+e^{-(x_0+\triangle)}})}{\frac{2\delta}{\sigma^2}(1 + e^{x_0+1})}}\right\},$$

(4.1.14) and Taylor expansion yield

$$f''_\triangle(x) > -\frac{\mu}{\sigma^2} + \frac{1}{1 + e^{(-x_0+\triangle)}} > \triangle\frac{e^{-x_0}}{2(1 + e^{-x_0})^2} > 0. \tag{4.2.2}$$

For sufficiently small $\triangle$, and $\overline{x}_\triangle \le x_0 + 1$ and (4.2.1) makes

$$|f'_\triangle(x_0 + \triangle)| \le 2\triangle \sup_{x\in[x_0-\triangle,x_0+\triangle]} |f''_\triangle(x)| < m_\triangle. \tag{4.2.3}$$

(4.2.1) implies that

$$\begin{aligned}\overline{x}_\triangle - \underline{x}_\triangle &\le 2\triangle + \frac{f'(\overline{x}_\triangle) - f'(x_0 + \triangle)}{f''(c)} \qquad \text{for some c} \in (x_0 + \triangle, \overline{x}_\triangle) \\ &\le 2\triangle + \frac{2\triangle \sup_{[x_0-\triangle,x_0+\triangle]} |f''(x)|}{\triangle e^{-x_0}/(2(1 + e^{-x_0})^2)} \to 0 \qquad \text{as } \triangle \to 0\end{aligned}$$

For second inequality, (4.2.2) is used. Therefore, $f_\triangle(\overline{x}_\triangle) \to \overline{C}$ as $\triangle \to 0$.

Claim 3. $\overline{x}_\triangle \ge x_0$ and $f_\triangle(\overline{x}_\triangle) \to -\infty$ as $\triangle \to \infty$.

Let $x^* < x_0$. Taylor expansion gives $f''_\triangle(x) < -\frac{\mu}{\sigma^2} + \frac{1}{1+e^{x^*}} < 0$ for $x \in [x_\triangle, x^*]$. Since $f''_\triangle(\underline{x}_\triangle) < 0$, $f'_\triangle(x) < 0$ for $x \in [x_\triangle, x^*]$. Therefore $\overline{x} \ge x_0$.

Define m' be

$$m' := \max\left\{ \frac{\frac{1}{3}|-\frac{\mu}{\sigma^2}+\frac{1}{1+e^{-x^*}}|}{\frac{4\mu}{\sigma^2}+3+(1+e^{x^*})\frac{2\delta}{\sigma^2}}, \sqrt{\frac{\frac{1}{3}|-\frac{\mu}{\sigma^2}+\frac{1}{1+e^{-x^*}}|}{\frac{2\mu}{\sigma^2}+1+(1+e^{x^*})\frac{4\delta}{\sigma^2}}}, \sqrt[3]{\frac{\frac{1}{3}|-\frac{\mu}{\sigma^2}+\frac{1}{1+e^{-x^*}}|}{(1+e^{x^*})\frac{2\delta}{\sigma^2}}} \right\}.$$

Then $|f'_\triangle(x)| \geq m'$ for $x \in [x_0 - \triangle + \frac{m'}{\frac{\mu}{\sigma^2}-(1+e^{x^*})^{-1}}, x^*]$. So, mean value theorem gives $f_\triangle(\overline{x}_\triangle) \to -\infty$ as $\triangle \to \infty$.

claim 4. $\overline{x}_\triangle > x_0$.

It is enough to show that $\overline{x}_\triangle \neq x_0$ by claim 3. Suppose that $\overline{x}_\triangle = x_0$. $f'_\triangle(x_0) = 0$ imply $f''_\triangle(x_0) = 0$. Therefore, $f''_\triangle(x) < 0$ if $x \in (x_0 - \epsilon, x_0)$ for sufficiently small $\epsilon > 0$. For sufficiently small $\epsilon > 0$, first order Taylor expansion of $f''_\triangle$ gives $f'_\triangle(x) > 0$ for $x \in (x_0 - \epsilon, x_0)$. By intermediate theorem, $f'(c) = 0$ where $c \in (\underline{x}_\triangle, x)$ which is a contradiction of claim 3. So, $\overline{x}_\triangle > x_0$.

claim 5. $(f_\triangle, f'_\triangle)$ converges uniformly to $(f_{\triangle_0}, f'_{\triangle_0})$ on compacts as $\triangle \to \triangle_0$.

Let $g^\triangle : R_+ \to R^3$ be the solution of initial problem

$$\frac{d}{dx}(g_1^\triangle, g_2^\triangle, g_3^\triangle)(x) = \Big(1, g_3^\triangle(x), h(g_1^\triangle(x), g_3^\triangle(x))\Big),$$

where function h is defined by

$$\begin{aligned} h(y,z) :=& (-\frac{2\mu}{\sigma^2}+\frac{1}{1+e^{-y}}) + \Big(\frac{4\mu}{\sigma^2} - \frac{2}{1+e^{-y}} - 1 - \frac{2\delta}{\sigma^2}(1+e^y)\Big)z \\ &+ \Big(-\frac{2\mu}{\sigma^2} + 1 + \frac{4\delta}{\sigma^2}(1+e^y)\Big)z^2 - \frac{2\delta(1+e^y)}{\sigma^2}z^3, \end{aligned}$$

and initial values $(g_1^\triangle, g_2^\triangle, g_3^\triangle)(0) = (x_0 - \triangle, \overline{C}, 0)$. It is easy to check that the solution is

$$(g_1^\triangle, g_2^\triangle, g_3^\triangle)(x) = (x + x_0 - \triangle, f_\triangle(x + x_0 - \triangle), f'_\triangle(x + x_0 - \triangle)).$$

It follows that

$$\begin{aligned} |f_\triangle(x) - f_{\triangle_0}(x)| &= |g_2^\triangle(x - x_0 + \triangle) - g_2^{\triangle_0}(x - x_0 + \triangle_0)| \\ &\leq |g_2^\triangle(x - x_0 + \triangle) - g_2^{\triangle_0}(x - x_0 + \triangle)| + M'|\triangle - \triangle_0| \\ |f'_\triangle(x) - f'_{\triangle_0}(x)| &= |g_3^\triangle(x - x_0 + \triangle) - g_3^{\triangle_0}(x - x_0 + \triangle_0)| \\ &\leq |g_3^\triangle(x - x_0 + \triangle) - g_3^{\triangle_0}(x - x_0 + \triangle)| + M'''|\triangle - \triangle_0| \end{aligned}$$

where $M''' = \sup_{[x-x_0+\triangle, x-x_0+\triangle_0]} g_3^{\triangle_0}$ which is bounded as $\triangle \to \triangle_0$. So, it is enough to show

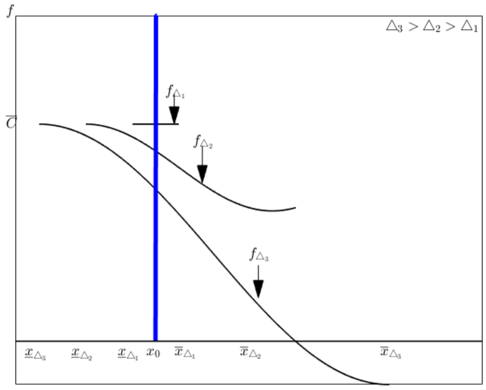

Figure 4-1: Description of f under changing $\triangle$.

that $g^{\triangle}$ uniformly on compacts on initial value $g^{\triangle}(0)$. Locally Lipschitz of h leads globally Lipschitz in y on compacts and in z on $[-M', M']$. By theorem V 3.2 in [5], $g^{\triangle}$ uniformly converge. So, claim is satisfied.

claim 6. $f_{\triangle}(\overline{x}_{\triangle})$ is continuous on $\triangle$.

If $f_{\triangle}(\overline{x}_{\triangle})$ is continuous on $\triangle$, it is always found as a strictly decreasing function satisfying any boundary condition as in Figure 4-2 due to claim 2, claim 3 and intermediate theorem.

Let $\triangle_0 > 0$. From claim 4, $\overline{x}_{\triangle_0} > x_0$. Hence $f''_{\triangle_0} > 0$ in a sufficiently small neighborhood $\mathcal{U}$ of $\overline{x}_{\triangle}$. Suppose that $\triangle$ sufficiently close to $\triangle_0$. $\overline{x}_{\triangle}$ is close to $\overline{x}_{\triangle_0}$ by claim 4.

$$|f_{\triangle}(\overline{x}_{\triangle}) - f_{\triangle_0}(\overline{x}_{\triangle_0})| \leq |f_{\triangle}(\overline{x}_{\triangle}) - f_{\triangle_0}(\overline{x}_{\triangle})| + |f_{\triangle_0}(\overline{x}_{\triangle}) - f_{\triangle_0}(\overline{x}_{\triangle_0})|$$

First term and second term of the right-hand side go to 0 by claim 3 and closeness of $\overline{x}_{\triangle}$ and $\overline{x}_{\triangle_0}$ respectively. $\square$

Lemma 4.2.2. *Let $x \in [\underline{x}, \overline{x}]$, $a(y):=\frac{\sigma^2}{2}(\frac{1-e^{-y}}{1+e^{-y}})(\frac{1}{1-f'(y)})^2 + \delta(\frac{1+e^{-y}}{e^{-y}})$ and $b(y):=\frac{\sigma}{1-f'(y)}$ for $y \in [\underline{x}, \overline{x}]$. Then there exists a solution to the Skorokhod SDE*

$$dX_t^x = a(X_t^x)dt + b(X_t^x)dW_t \tag{4.2.4}$$

with instantaneous reflection at $\underline{x}, \overline{x}$, that is, a continuous, adapted, $[\underline{x}, \overline{x}]$ valued process X^x and nondecreasing adapted process Φ^x, Ψ^x such that Φ^x and Ψ^x increase only on the sets $\{X^x = \underline{x}\}$ and $\{X^x = \overline{x}\}$, respectively, and

$$X_t^x = x + \int_0^t a(X_s^x)ds + \int_0^t b(X_s^x)dW_s + \Phi_t^x - \Psi_t^x \tag{4.2.5}$$

holds for $t \in [0, \infty)$.

proof) a and b are derived from $dX_t^x = \Big(\tilde{\mu}g' + \frac{1}{2}\tilde{\sigma}^2 g''\Big)dt + \tilde{\sigma}g'dW_t$ by inserting values in instead of $\tilde{\sigma}, \tilde{\mu}$. To show that it is really solution of SDE, it is enough to show that a and b are globally Lipschitz on $[\underline{x}, \overline{x}]$ by [6]. To show globally Lipschitz on $[\underline{x}, \overline{x}]$, it remains show that a' and b' are bounded on $(\underline{x}, \overline{x})$ by the mean value theorem. Fix $y \in (\underline{x}, \overline{x})$. Hence $b'(y) = \sigma\frac{g''(y)}{(1-g'(y))^2}$. Since $g'(y) \le 0$ and $g'(y)$ is bounded for $y \in [\underline{x}, \overline{x}]$, b' is bounded. To show boundedness of a', it is enough to show bound of $((\frac{1-e^{-x}}{1+e^{-x}})^2)'$ from nonpositivity of g'. $|((\frac{1-e^{-x}}{1+e^{-x}})^2)'| \le 2$. □

Lemma 4.2.3. *For $x \in [\underline{x}, \overline{x}]$, let X^x be the process from lemma 4.2.2, then $C := f(X^x)$ is a $[\underline{C}, \overline{C}]$-valued Ito process of the form*

$$C_t = f(x) + \int_0^t (-\mu + \frac{\sigma^2}{2} + \frac{\sigma^2}{2}(\frac{1-e^{-X_s^x}}{1+e^{-X_s^x}})(\frac{1}{1-f'(X_s^x)})^2)ds + \int_0^t \frac{\sigma}{1-f'(X_s^x)}dW_s$$

and the Ito process $\tilde{S}^x := Se^C$ satisfies

$$\tilde{S}_t^x = S_0 e^{C_0} exp\Big(\int_0^t \frac{\sigma^2}{2}(\frac{1-e^{-X_s^x}}{1+e^{-X_s^x}})(\frac{1}{1-f'(X_s^x)})^2 ds + \int_0^t \frac{\sigma}{1-f'(X_s^x)}dW_s\Big)$$

proof) It is possible to extend f to C^2 on an open set containing $[\underline{x}, \overline{x}]$ by the definition of f in (4.1.14). $dC_t = df(X_t^x) = (f'(X_t^x)a(X_t^x) + \frac{1}{2}f''(X_t^x)^2)dt + f'(X_t^x)b(X_t^x)dW_t$. Putting values of a and b in lemma 4.2.2, the conclusion is generated. □

It remains to show that $\tilde{S}^x$ is really a shadow price process. First we will find the optimal strategy in market which has a price process $\tilde{S}^x$ and no transaction cost, then the optimal strategy is shown that it is also admissible in market which have price process S and transaction cost. Here, I change the proof slightly in following lemma to be self-contained in contrast with original paper.

Lemma 4.2.4. *Let the function $r : [\underline{x}, \overline{x}] \to \mathbb{R}$ be defined by*

$$r(x) = \frac{\eta_S S_0 e^{f(x)}}{\eta_B + \eta_S S_0 e^{f(x)}} - \frac{1}{1+e^{-x}}. \tag{4.2.6}$$

Set

$$\tilde{x} := \begin{cases} \overline{x} & \text{if } \frac{\eta_S \underline{S_0}}{\eta_B + \eta_S \underline{S_0}} > \frac{1}{1+e^{-\overline{x}}}. \\ \underline{x} & \text{if } \frac{\eta_S \overline{S_0}}{\eta_B + \eta_S \overline{S_0}} < \frac{1}{1+e^{-\underline{x}}} \\ \text{a solution to } r(x) = 0 & \text{otherwise} \end{cases}$$

For processes $X^{\tilde{x}}$ and $\tilde{S}^{\tilde{x}}$ as in lemma 4.2.3, define

$$\tilde{V}_t^{\tilde{x}} := (\eta_B + \eta_S \tilde{S}_0^{\tilde{x}})\mathcal{E}(\int_0 \frac{1}{(1+e^{-X_s^{\tilde{x}}})\tilde{S}_s^{\tilde{x}}} d\tilde{S}_s^{\tilde{x}} - \int_0 \delta ds)_t \tag{4.2.7}$$

$$\tilde{c}_t^{\tilde{x}} := \delta \tilde{V}_t^{\tilde{x}} \tag{4.2.8}$$

$$\tilde{\varphi}_t^{S,\tilde{x}} := \frac{\tilde{V}_t^{\tilde{x}}}{\tilde{S}_t^{\tilde{x}}} \frac{1}{1+e^{-X_t^{\tilde{x}}}}, \quad \tilde{\varphi}_t^{B,\tilde{x}} := \tilde{V}_t^{\tilde{x}} - \tilde{\varphi}_t^{S,\tilde{x}} \tilde{S}_t^{\tilde{x}} \tag{4.2.9}$$

where $\mathcal{E}(Y)_t = 1 + \int_0^t \mathcal{E}(Y)_s dY_s$. Then,

$$\tilde{\varphi}_t^{B,\tilde{x}} = \tilde{\varphi}_0^{B,\tilde{x}} - \int_0^t \tilde{c}_s^{\tilde{x}} ds - \int_0^t \frac{\tilde{V}_s^{\tilde{x}} e^{-X_s^{\tilde{x}}}}{(1+e^{-X_s^{\tilde{x}}})^2} d\Phi_s^{\tilde{x}} + \int_0^t \frac{\tilde{V}_s^{\tilde{x}} e^{-X_s^{\tilde{x}}}}{(1+e^{-X_s^{\tilde{x}}})^2} d\Psi_s^{\tilde{x}} \tag{4.2.10}$$

$$\tilde{\varphi}_t^{S,\tilde{x}} = \tilde{\varphi}_0^{S,\tilde{x}} + \int_0^t \frac{\tilde{\varphi}_s^{S,\tilde{x}} e^{-X_s^{\tilde{x}}}}{(1+e^{-X_s^{\tilde{x}}})} d\Phi_s^{\tilde{x}} - \int_0^t \frac{\tilde{\varphi}_s^{S,\tilde{x}} e^{-X_s^{\tilde{x}}}}{(1+e^{-X_s^{\tilde{x}}})} d\Psi_s^{\tilde{x}} \tag{4.2.11}$$

and $(\tilde{\varphi}^{\tilde{x}}, \tilde{c}^{\tilde{x}})$ is an optimal portfolio/consumption pair with value process $\tilde{V}$ for initial wealth $\eta_B + \eta_S \tilde{S}_0^{\tilde{x}}$ in the frictionless market with price process $\tilde{S}^{\tilde{x}}$.

proof) It is easy to check that $\tilde{x}$ is well-defined. For simplicity of calculation, we will omit the $\tilde{x}$ from the superscript of notations. From definition, $\tilde{\varphi}^S, \tilde{\varphi}^B$ are right continuous, and $\log(\tilde{\varphi}_t^S) = -\log(1+e^{-X_t}) + \log(\tilde{V}_t) - (\mu - \frac{\sigma^2}{2})t - \sigma W_t - C_t$. Use the property that when Y_t is continuous semimartingale,

$$\mathcal{E}(Y)_t = e^{Y_t - Y_0 - \frac{1}{2}<Y,Y>_t}.$$

So, $\tilde{V}_t$ follows $\tilde{V}_t = (\eta_B + \eta_S \tilde{S}_0) e^{Y_t - \frac{1}{2}<Y,Y>_t}$, where $Y_t = \int_0^t \frac{1}{(1+e^{-X_s})\tilde{S}_s} d\tilde{S}_s - \int_0^t \delta ds$. Therefore, it remains to calculate $d\tilde{S}_t$ to find $d\log(\tilde{V}_t)$. By Ito product rule,

$$\begin{aligned}
d\tilde{S}_t &= S_t de^{C_t} + e^{C_t} dS_t + d < S, e^C >_t \\
&= \frac{\tilde{S}_t}{(1+e^{-X_t})} (\frac{\sigma}{1-f'(X_t)})^2 dt + \tilde{S}_t (\frac{\sigma}{1-f'(X_t)}) dW_t, \\
d\log(\tilde{V}_t) &= \frac{1}{(1+e^{-X_t})\tilde{S}_t} d\tilde{S}_t - \delta dt - \frac{1}{2(1+e^{-X_t})^2} \frac{1}{\tilde{S}_t^2} d < \tilde{S}, \tilde{S} >_t \\
&= \Big(\frac{1}{2(1+e^{-X_t})^2} (\frac{\sigma}{1-f'(X_t)})^2 - \delta\Big) dt + (\frac{1}{1+e^{-X_t}})(\frac{\sigma}{1-f'(X_t)}) dW_t, \\
-dC_t &= \Big(\mu - \frac{\sigma^2}{2} - \frac{\sigma^2}{2} (\frac{1-e^{-X_t}}{1+e^{-X_t}})(\frac{1}{1-f'(X_t)})^2\Big) dt - \frac{\sigma}{1-f'(X_t)} dW_t,
\end{aligned}$$

$$
\begin{aligned}
-d\log(1+e^{-X_t}) &= \frac{e^{-X_t}}{1+e^{-X_t}}dX_t - \frac{e^{-X_t}}{2(1+e^{-X_t})^2} < X, X >_t \\
&= \Big((\frac{e^{-X_t}}{1+e^{-X_t}}) \Big(\frac{\sigma^2}{2}(\frac{1-e^{-X_t}}{1+e^{-X_t}})(\frac{1}{1-f'(X_t)})^2 + \delta(\frac{1+e^{-X_t}}{e^{-X_t}}) \Big) \\
&- \frac{e^{-X_t}}{2(1+e^{-X_t})^2}(\frac{\sigma}{1-f'(X_t)})^2 \Big) dt \\
&+ \frac{e^{-X_t}}{1+e^{-X_t}}d\Phi_t - \frac{e^{-X_t}}{1+e^{-X_t}}d\Psi_t + (\frac{e^{-X_t}}{1+e^{-X_t}})(\frac{\sigma}{1-f'(X_t)})dW_t.
\end{aligned}
$$

Hence, $d\log(\tilde{\varphi}_t^S)$ is $d\log(\tilde{\varphi}_t^S) = \frac{e^{-X_t}}{1+e^{-X_t}}d\Phi_t - \frac{e^{-X_t}}{1+e^{-X_t}}d\Psi_t$, and $\tilde{\varphi}_t^S$ follows expression (4.2.11), Since Φ and Ψ are finite variation processes, $\tilde{\varphi}^S$ is also finite variation process.

Since $\tilde{V}$ is value process of $\tilde{\varphi}$ in frictionless market whose price process is $\tilde{S}$, differential $d\tilde{V}_t$ is $d\tilde{V}_t = \tilde{\varphi}_t^S d\tilde{S}_t - \tilde{c}_t dt$. Then, Ito lemma gives

$$
d\tilde{\varphi}_t^B = d(\tilde{V}_t - \tilde{\varphi}_t^S \tilde{S}_t) = -\tilde{c}_t dt - \tilde{S}_t d\tilde{\varphi}_t^S.
$$

Therefore, $\tilde{\varphi}_t^B$ follows (4.2.10) and satisfies self financing condition (2.0.3). $(\tilde{\varphi}, \tilde{c}) \in \mathcal{A}(\tilde{S})$. Define $Z_t = \frac{1}{\delta e^{\delta t}\tilde{V}_t}$. Then Z_t is positive local martingale is due to

$$
dZ_t = \frac{1}{\delta e^{\delta t}\tilde{V}_t}\Big(-\frac{1}{(1+e^{-X_t})}\frac{\sigma}{(1-f')}\Big)dW_t.
$$

Let $\{\tau_n\}_{n=1}^{\infty}$ be localizing sequence of Z_t. Since $\mathbb{E}[\delta\tilde{V}_0 Z_{t\wedge\tau_n}] = \delta\tilde{V}_0 Z_0 = 1$, $\delta\tilde{V}_0 Z_{t\wedge\tau_n}$ is Radon-Nikodým derivative process. Define $d\bar{\mathbb{P}} = \delta\tilde{V}_0 Z_{t\wedge\tau_n} d\mathbb{P}$, then $\bar{\mathbb{P}}$ is probability measure. Fix $(\varphi, c) \in \mathcal{A}(\tilde{S})$. For all $n \in \mathbb{N}$,

$$
\begin{aligned}
\mathbb{E}[\int_0^n Z_{t\wedge\tau_n} c_t e^{-\delta t} dt] &= \frac{1}{\delta\tilde{V}_0}\mathbb{E}[\int_0^n \delta\tilde{V}_0 Z_{t\wedge\tau_n} c_t e^{-\delta t}dt] = \frac{1}{\delta\tilde{V}_0}\bar{\mathbb{E}}[\int_0^n c_t e^{-\delta t}dt] \\
&= -\frac{1}{\delta\tilde{V}_0}\bar{\mathbb{E}}[\int_0^n e^{-\delta t}(d\varphi_t^B + \tilde{S}_t d\varphi_t^S)]] = -\frac{1}{\delta\tilde{V}_0}\bar{\mathbb{E}}[\int_0^n e^{-\delta t} d\tilde{V}_t] \\
&= -\frac{1}{\delta\tilde{V}_0}\bar{\mathbb{E}}[\int_0^n d(\tilde{V}_t e^{-\delta t}) + \int_0^n \delta e^{-\delta t}\tilde{V}_t dt] \\
&\leq \frac{1}{\delta\tilde{V}_0}(\tilde{V}_0 - \bar{\mathbb{E}}[\int_0^n \delta e^{-\delta t}\tilde{V}_t dt]) \\
&\leq \frac{1}{\delta}.
\end{aligned} \tag{4.2.12}
$$

Combined with monotone convergence theorem, (4.2.12) generates

$$
\mathbb{E}[\int_0^\infty Z_t c_t e^{-\delta t} dt] \leq \frac{1}{\delta} \tag{4.2.13}
$$

So, following results are made

$$\begin{aligned}\mathbb{E}[\int_0^\infty \log(c_t e^{\frac{\delta t}{2}})e^{-\delta t}dt] &\leq \mathbb{E}[\int_0^\infty \Big(Z_t c_t - \log(Z_t e^{-\frac{\delta t}{2}}) - 1\Big)e^{-\delta t}dt] \\ &\leq -\mathbb{E}[\int_0^\infty \log(Z_t e^{-\frac{\delta t}{2}})e^{-\delta t}dt] \\ &= \mathbb{E}[\int_0^\infty \log(e^{\frac{\delta t}{2}}\tilde{c}_t)e^{-\delta t}dt]\end{aligned}$$

For first inequality, we use the relation that $\log(x)+1 \leq x$ for $x > 0$. With $\log(Z_t c_t)+1 \leq Z_t c_t$, $\log(c_t e^{\frac{\delta t}{2}}) \leq Z_t c_t - \log(Z_t e^{-\frac{\delta t}{2}}) - 1$. For second inequality, (4.2.13) is used. Consequently, $(\tilde{\varphi}, \tilde{c})$ is really an optimal investment/consumption strategy with $\tilde{V}$ for initial wealth $\eta_B + \eta_S \tilde{S}_0$ in the no transaction market with price process $\tilde{S}$ □

Remark (i) By definition of $\tilde{x}$ above lemma, $(\tilde{\varphi}_0^{B,\tilde{x}}, \tilde{\varphi}_0^{S,\tilde{x}}) = (\eta_B, \eta_S)$ for $r(\tilde{x}) = 0$. Because following relation satisfies

$$r(\tilde{x}) = 0 \Leftrightarrow \frac{\eta_S \tilde{S}_0^{\tilde{x}}}{\tilde{V}_0^{\tilde{x}}} = \frac{1}{1+e^{-\tilde{x}}} \Leftrightarrow \eta_S = \tilde{\varphi}_0^{S,\tilde{x}}.$$

But $(\tilde{\varphi}_0^{B,\tilde{x}}, \tilde{\varphi}_0^{S,\tilde{x}}) \neq (\eta_B, \eta_S)$ if $\frac{\eta_S \overline{S}_0}{\eta_B + \eta_S \overline{S}_0} > \frac{1}{1+e^{-\underline{x}}}$ or $\frac{\eta_S \underline{S}_0}{\eta_B + \eta_S \underline{S}_0} < \frac{1}{1+e^{-\overline{x}}}$ holds. Because $\eta_S > (\frac{\eta_B + \eta_S \overline{S}_0}{\overline{S}_0})(\frac{1}{1+e^{-\underline{x}}}) \geq \tilde{\varphi}_0^{S,\tilde{x}}$ for first case and, $\eta_S < (\frac{\eta_B + \eta_S \underline{S}_0}{\underline{S}_0})(\frac{1}{1+e^{-\overline{x}}}) \leq \tilde{\varphi}_0^{S,\tilde{x}}$ for second case. This happen when initial position out of no trade region as Figure 4-2. This case, it can be modified $(\tilde{\varphi}_0^{B,\tilde{x}}, \tilde{\varphi}_0^{S,\tilde{x}}) = (\eta_B, \eta_S)$ without changing of initial wealth.

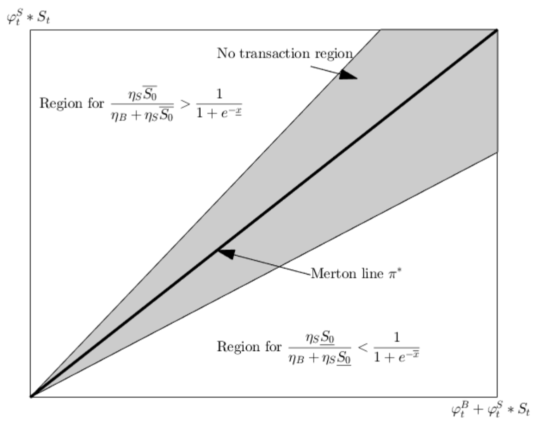

Figure 4-2: Illustration of $\tilde{x}$ in lemma 4.2.4

(ii) (4.2.10) and (4.2.11) show that $\varphi^{S,\tilde{x}}$ changes when $\tilde{S}^{\tilde{x}}$ hit the boundary. To have same maximal expected utilities for S and $\tilde{S}^{\tilde{x}}$, risky asset have to be sold only at $\tilde{S}^{\tilde{x}} = \underline{S}$ and have to be bought only at $\tilde{S}^{\tilde{x}} = \overline{S}$. Since $(\tilde{\varphi}^{\tilde{x}}, \tilde{c}^{\tilde{x}})$ which is defined above lemma satisfies this, It remains to show that $(\tilde{\varphi}^{x}, \tilde{c}^{\tilde{x}})$ is admissible portfolio/consumption in original market which have price process S with transaction cost.

Theorem 4.2.1. *The portfolio/consumption pair* $(\tilde{\varphi}^{\tilde{x}}, \tilde{c}^{\tilde{x}})$ *and* $\tilde{x}$ *defined in lemma 4.2.4 is also optimal in the market with price process* S *and proportional transaction costs* $\overline{\lambda}, \underline{\lambda}$*. In particular,* $\tilde{S}^{\tilde{x}}$ *is a shadow price process in this market.*

proof) From (2.0.7), maximal expected utilities satisfy following relation

$$\sup_{(\tilde{\varphi}^{B,\tilde{x}}, \tilde{\varphi}^{S,\tilde{x}}, \tilde{c}^{\tilde{x}}) \in \mathcal{A}(\tilde{S}^{\tilde{x}})} \mathbb{E}\Big(\int_0^\infty e^{-\delta t} \log(\tilde{c}_t^{\tilde{x}}) dt \Big) \geq \sup_{(\varphi^B, \varphi^S, c) \in \mathcal{A}(S)} \mathbb{E}\Big(\int_0^\infty e^{-\delta t} \log(c_t) dt \Big).$$

So, it is enough to prove that $(\tilde{\varphi}^{\tilde{x}}, \tilde{c}^{\tilde{x}}) \in \mathcal{A}(S)$. For $(\tilde{\varphi}_0^{B,\tilde{x}}, \tilde{\varphi}_0^{S,\tilde{x}}) \neq (\eta^B, \eta^S)$, it is possible to change initial position to satisfy $(\tilde{\varphi}_0^{B,\tilde{x}}, \tilde{\varphi}_0^{S,\tilde{x}}) = (\eta^B, \eta^S)$ without changing initial wealth as mentioned remark. So, it is enough to show the case $(\tilde{\varphi}_0^{B,\tilde{x}}, \tilde{\varphi}_0^{S,\tilde{x}}) = (\eta^B, \eta^S)$. Note that $\Phi^{\tilde{x}}$ and $\Psi^{\tilde{x}}$ increase only on the sets $\{\tilde{S}^{\tilde{x}} = \overline{S}\}$ and $\{\tilde{S}^{\tilde{x}} = \underline{S}\}$, respectively. So, the self-financing condition (2.0.3) for $(\tilde{\varphi}^{\tilde{x}}, \tilde{c}^{\tilde{x}})$ in frictionless market yields

$$\begin{aligned} \tilde{\varphi}_t^{B,\tilde{x}} &= \tilde{\varphi}_0^{B,\tilde{x}} + \int_0^t \tilde{S}_s d(\tilde{\varphi}_s^{S,\tilde{x}})^{\downarrow} - \int_0^t \tilde{S}_s d(\tilde{\varphi}_u^{S,\tilde{x}})^{\uparrow} - \int_0^t \tilde{c}_s^{\tilde{x}} ds \\ &= \tilde{\varphi}_0^{B,\tilde{x}} + \int_0^t \underline{S}_s d(\tilde{\varphi}_u^{S,\tilde{x}})^{\downarrow} - \int_0^t \overline{S}_s d(\tilde{\varphi}_s^{S,\tilde{x}})^{\uparrow} - \int_0^t \tilde{c}_s^{\tilde{x}} ds. \end{aligned}$$

So, $(\tilde{\varphi}^{\tilde{x}}, \tilde{c}^{\tilde{x}})$ satisfies self-financing condition in the original market which have price process S with transaction costs. Hence, $(\tilde{\varphi}^{\tilde{x}}, \tilde{c}^{\tilde{x}}) \in \mathcal{A}(S)$ and we obtain the claim.□

5

The Work of Choi, Sirbu and Zitkovic

In this section, we review the work of Choi, Sirbu and Zitkovic [3]. Significant work of [3] is that it firstly finds the necessary and sufficient condition for the value function to be finite. Also, this paper extends shadow price method for all CRRA types of utility functions. The following theorem states the conditions.

Theorem 5.0.1. *Define*

$$G(\sigma,p,\delta)=\sqrt{\frac{2\delta(1-p)\sigma^2}{p}} \quad \text{and } A(\sigma,p,\delta)=\frac{\delta}{p}+\frac{(1-p)\sigma^2}{2}. \tag{5.0.1}$$

The following statements are equivalent:
(1) The problem is well posed, i.e,

$$-\infty < u < \infty$$

(2) The parameters of the model satisfy one of the following three conditions:
(i) $p \leq 0$
(ii) $0 < p < 1$ *and* $\mu < G$
(iii) $0 < p < 1$, $G \leq \mu \leq A$ *and* $C(\mu,\sigma,p,\delta) < log(\frac{1+\bar{\lambda}}{1-\underline{\lambda}})$ *where the function* C *is given by*

$$C=\int_0^K k\Big(\frac{X_1'(k)}{k-X_1(k)}-\frac{X_2'(k)}{k-X_2(k)}\Big)dk$$

Here, X_1 *and* X_2 *are ordered solutions of* $a(k)X^2-b(k)X+c(k)=0$ *where* $a(\cdot)$, $b(\cdot)$ *and* $c(\cdot)$

are defined by

$$a(k) = 2p\delta(1+k),$$
$$b(k) = \Big(2\delta + p(1-p)(2\mu - \sigma^2)\Big)k + 2p(1-p)\mu,$$
$$c(k) = (1-p)\Big(2\mu + (p^2-1)\sigma^2\Big)k + (1-p)^3\sigma^2.$$

Note that theorem 5.0.1 improves the condition for existence of solution of logarithmic utility function compared to [4] which requires the condition $0 < \mu < \sigma^2$.

5.1 Heuristic derivation of free boundary problem

Free boundary problem will be heuristically derived, which is one dimensional first order ODE. Firstly, we introduce some term to derive free boundary problem.

$$q = \frac{p}{1-p}, \quad \alpha_q(\Sigma,\theta) = \theta\mu - \mu - \Sigma(\frac{1}{2}\Sigma + \sigma - \theta(1+q)),$$
$$\beta(\theta) = (1+q)(\delta - \frac{1}{2}q\theta^2), \quad \textit{and} \quad \gamma(\theta) = \begin{cases} \frac{1}{2}\theta^2 & \text{if } p = 0. \\ sgn(p) & \text{if } p \neq 0\ . \end{cases} \tag{5.1.1}$$

Definition (5.1.1) will be used throughout the following chapters of the paper.

Rephrase $\inf_{\tilde{S}\in\mathcal{S}} u(\tilde{S})$ as

$$\inf_{\tilde{S}\in\mathcal{S}} u(\tilde{S}) = \inf_{s_0\in[(1-\underline{\lambda})S_0,(1+\overline{\lambda})S_0]} \inf_{\tilde{S}\in\mathcal{S},\tilde{S}_0=s_0} u(\tilde{S}). \tag{5.1.2}$$

Here, $\mathcal{S}$ is defined in (2.0.5). Since $\tilde{S}\in\mathcal{S}$ is Ito process, it is possible to write $\tilde{S}$ to

$$d\tilde{S}_t = \tilde{S}_t(\sigma+\Sigma_t)(dB_t + \theta_t dt) \quad \text{and} \quad \tilde{S}_0 = s_0,$$

where $\Sigma_t = \Sigma(\tilde{S}_t)$ and $\theta_t = \theta(\tilde{S}_t)$ are processes. Since $\tilde{S}$ and S is positive process, define $Y_t = \log(\frac{\tilde{S}_t}{S_t})$. Then $Y_t \in [\underline{y},\overline{y}]$ where $\underline{y} = \log(1-\underline{\lambda}), \log(1+\overline{\lambda})$. By Ito formula, differential dY_t is given by

$$dY_t = \alpha_0(\Sigma_t,\theta_t)dt + \Sigma_t dB_t, \tag{5.1.3}$$

where α_0 is defined in (5.1.1). Parametrize $\mathcal{S}$ by $\mathcal{P}$ which it is defined by

$$\mathcal{P} = \{(y,\Sigma,\theta) : y\in[\underline{y},\overline{y}], (\Sigma,\theta)\in\mathcal{P}(y)\},$$

where $\mathcal{P}(y) = \{(\Sigma,\theta) : dY_t = \alpha_0(\Sigma_t,\theta_t)dt + \Sigma_t dB_t \text{ and } Y_0 = y\}$. Since any process $\tilde{S}\in\mathcal{S}$

have no transaction costs, it is enough to consider initial value process $V(\eta_B, \eta_S, S_0e^y, S_0e^y) = \eta_B + S_0e^y\eta_S$ instead of (η_B, η_S). Complete market duality theory (see, Theorem 9.11 in [7]) yields

$$u(\tilde{S}) = \inf_{z>0}\Big((\eta_B + S_0e^y\eta_s)z + \mathcal{V}(z\mathcal{E}(-\theta\cdot B))\Big), \tag{5.1.4}$$

where $\mathcal{V}$ defined by

$$\mathcal{V}(Z) = \mathbb{E}[\int_0^\infty e^{-\delta t}V(e^{\delta t}Z_t)dt], \quad \text{where } V(z) = \sup_{c>0}(U(c) - cz). \tag{5.1.5}$$

Since $\sup_{c>0}(U(c) - cz)$ is obtained when $c = z^{\frac{1}{p-1}}$, following relation is obtained

$$V(z) = \begin{cases} \frac{1}{q}z^{-q} & \text{if } p = 0 \\ -1 - \log(z) & \text{if } p \neq 0 \ . \end{cases} \tag{5.1.6}$$

Here, q is defined in (5.1.1). By (5.1.2) and (5.1.4), solving the existence of shadow price process changes over to solving the optimal problem over $\mathcal{P}(y)$, y and z. That is,

$$\inf_{\tilde{S}\in\mathcal{S}} u(\tilde{S}) = \inf_{(y,z)\in[\underline{y},\overline{y}]\times(0,\infty)}\Big((\eta_B + S_0e^y\eta_S)z + \inf_{(\Sigma,\theta)\in\mathcal{P}(y)}\mathcal{V}(z)\mathcal{E}(-\theta\cdot B)).\Big)$$

Also, Equation (5.1.6) makes

$$\mathcal{V}(z\mathcal{E}(-\theta\cdot B)) = \begin{cases} -\frac{1+\log(z)}{\delta} + \mathbb{E}[\int_0^\infty e^{-\delta t}\Big(-\log(\mathcal{E}(-\theta\cdot B)_t)\Big)dt] & \text{if } p = 0 \\ \frac{z^{-q}}{q}\mathbb{E}[\int_0^\infty e^{-\tilde{\delta}t}\mathcal{E}(-\theta\cdot B)_t^{-q}dt] & \text{if } p \neq 0 \ . \end{cases}$$

Here, $\tilde{\delta}$ is defined by $\delta(1+q)$ and it is used throughout the paper. So, $\inf_{\tilde{S}\in\mathcal{S}} u(\tilde{S})$ can be written by

$$\inf_{\tilde{S}\in\mathcal{S}} u(\tilde{S}) = \inf_{y\in[\underline{y},\overline{y}]}\begin{cases} \frac{1}{\delta}\Big(-1 + \log(\delta(\eta_B + S_0e^y\eta_S)) + w(y)\Big) & \text{if } p = 0 \\ \frac{(\eta_B+S_0e^y\eta_S)^p}{p}|w(y)|^{1-p} & \text{if } p \neq 0. \end{cases} \tag{5.1.7}$$

where

$$w(y) = \inf_{(\sum,\theta)\in\mathcal{P}(y)}\begin{cases} \mathbb{E}[\int_0^\infty \delta e^{-\delta t}\Big(-\log(\mathcal{E}(-\theta\cdot B)_t)\Big)dt] & \text{if } p = 0 \\ sgn(p)\mathbb{E}[\int_0^\infty e^{-\tilde{\delta}t}\mathcal{E}(-\theta\cdot B)_t^{-q}dt] & \text{if } p \neq 0. \end{cases}$$

Assume that processes $\theta\cdot B$ and $\mathcal{E}(q\theta\cdot B)$ are martingales. Since $\theta\cdot B$ is continuous martingale, $\mathcal{E}(q\theta\cdot B)$ satisfies $\mathcal{E}(q\theta\cdot B)_t = e^{q\theta_t\cdot B_t - q\theta_0\cdot B_0 - \frac{1}{2}<q\theta\cdot B, q\theta\cdot B>_t}$. Therefore, process $\mathcal{E}(-\theta\cdot B)_t^{-q}$ is represented by

$$\mathcal{E}(-\theta \cdot B)_t^{-q} = \mathcal{E}(q\theta \cdot B)_t e^{\frac{1}{2}q(1+q)\int_0^t \theta_s^2 ds} \tag{5.1.8}$$

Using (5.1.8) and martingale property of $\mathcal{E}(q\theta \cdot B)$ and assumption that $\lim_{t\to\infty} e^{-\delta t}\int_0^t \theta_s^2 ds = 0$ for $p = 0$, $w(y)$ can be written by

$$w(y) = \inf_{(\sum,\theta)\in\mathcal{P}(y)} \begin{cases} \frac{1}{2}\mathbb{E}[\int_0^\infty e^{-\delta t}\theta_t^2 dt] & \text{if } p = 0 \\ sgn(p)\bar{\mathbb{E}}[\int_0^\infty e^{\frac{1}{2}q(1+q)\int_0^t \theta_s^2 ds} dt] & \text{if } p \neq 0 \end{cases} \tag{5.1.9}$$

where $\bar{\mathbb{E}}$ is expectation under probability measure $\overline{\mathbb{P}}$. Note that dimensionality of w is reduced. Since $\mathcal{E}(q\theta \cdot B)$ is positive martingale with expectation 1, Girsanov's theorem yields probability measure $\bar{\mathbb{P}}_t$ and Brownian motion under $\bar{\mathbb{P}}_t$ which are given by

$$d\bar{\mathbb{P}}_t = \mathcal{E}(q\theta \cdot B)_t d\mathbb{P} \text{ and } \bar{B}_t = B_t - \int_0^t q\theta_s ds. \tag{5.1.10}$$

Hence, differential of the process Y can be represented by

$$dY_t = a_q(\Sigma_t, \theta_t)dt + \Sigma_t d\bar{B}_t.$$

From (5.1.9), HJB equation is written as

$$\inf_{(\sum,\theta)\in\mathcal{P}(y)} (\frac{1}{2}\Sigma^2 w''(y) + \alpha_q(\Sigma,\theta)w'(y) - \beta(\theta)w(y) + \gamma(\theta)) = 0, \quad y \in (\underline{y}, \overline{y}) \tag{5.1.11}$$

where β and γ are defined in (5.1.1).

Because there is no boundary conditions, it is required to make them heuristically. To reflect at the boundary, Σ satisfies $\Sigma(\overline{y}) = \Sigma(\underline{y}) = 0$. From the definition α_q, it is expected that

$$w''(\overline{y}) = w''(\underline{y}) = \infty. \tag{5.1.12}$$

To avoid infinite value for boundary condition, we introduce $g : [\underline{x}, \overline{x}] \to R$ by $g(-w'(y)) = w(y)$. Then equation (5.1.11) changes to

$$\inf_{(\sum,\theta)\in\mathcal{P}(y)} \Big(\frac{1}{2}\Sigma^2 \frac{x}{g'(x)} - \alpha_q(\Sigma,\theta)x - \beta(\theta)g(x) + \gamma(\theta)\Big) = 0, \quad y \in (\underline{y}, \overline{y}). \tag{5.1.13}$$

(5.1.12) yields $g'(\underline{x}) = g'(\overline{x}) = 0$. Furthermore, direct calculation gives $\int_{\underline{x}}^{\overline{x}} \frac{g'(x)}{x} dx = \overline{y} - \underline{y}$. So, the boundary conditions

$$g'(\underline{x}) = g'(\overline{x}) = 0 \quad \text{and} \quad \int_{\underline{x}}^{\overline{x}} \frac{g'(x)}{x} dx = \overline{y} - \underline{y} \tag{5.1.14}$$

are obtained. Note that changing variable from w' to g gives order reduction and an addition of boundary condition.

5.2 Verification of the optimal strategy

The contents of this subsection are composed as follows. First, we begin by stating two lemma 5.2.1 and 5.2.2 without proof. Proof of conditions that when the assumption of lemma 5.2.1 is satisfied i.e theorem 5.0.1, is provided in [3].

Lemma 5.2.1 shows that existence of a solution of the free boundary problem for the certain condition and, Lemma 5.2.2 is used for simple calculation in verification of the optimal strategy. With these assumptions, we will prove the optimal investment and consumption strategy in frictionless market with the price process $\tilde{S} = Se^{f(X_t)}$ for some function f and state variable X. Note that it is the form of shadow price process in previous section. Then, we will verify that $\tilde{S}$ is shadow price process.

Lemma 5.2.1. *Assume that well-posed condition of theorem 5.0.1 hold. There exist constants* $\underline{x}, \overline{x}$ *with* $0 < \underline{x} < \overline{x}$ *and a strictly increasing function* $g \in C^2[\underline{x}, \overline{x}]$ *satisfying (5.1.13) and (5.1.14).*

Lemma 5.2.2. *Assume that well-posed condition of theorem 5.0.1 hold. The function h:* $[\underline{x}, \overline{x}] \to \mathbb{R}$*, defined by*

$$h(x) = \begin{cases} (1-x)g'(x) + 1 & \text{if } p = 0. \\ qg(x)(g'(x)+1) - x(q+1)g'(x) & \text{if } p \neq 0 \end{cases} \tag{5.2.1}$$

admits no zeros on $[\underline{x}, \overline{x}]$.

Fix $(\underline{x}, \overline{x}, g)$ among satisfying (5.1.13) and (5.1.14). It is possible due to lemma 5.2.1. Let $\tilde{\theta} : [\underline{x}, \overline{x}] \to \mathbb{R}$, $\tilde{\Sigma} : [\underline{x}, \overline{x}] \to \mathbb{R}$ be optimizer of (5.1.13) and $\tilde{\alpha}_q, \tilde{\alpha}_0, \tilde{\beta}, \tilde{\gamma} : [\underline{x}, \overline{x}] \to \mathbb{R}$ be the functions when $\theta = \tilde{\theta}$ and $\Sigma = \tilde{\Sigma}$. Optimizer $\tilde{\theta}$ and $\tilde{\Sigma}$ can be expressed by

$$\begin{cases} \tilde{\theta}(x) = \frac{\sigma x}{h(x)}, \quad \tilde{\Sigma}(x) = -\frac{\sigma(1-x)g'(x)}{h(x)} & \text{if } p = 0. \\ \tilde{\theta}(x) = -\frac{\sigma(1-p)x(qg'(x)-1)}{h(x)}, \quad \tilde{\Sigma}(x) = -\frac{\sigma(qg(x)-x)g'(x)}{h(x)} & \text{if } p \neq 0 \end{cases} \tag{5.2.2}$$

where h is defined in (5.2.1). Also, direct calculations yield equation

$$\frac{1}{2}\tilde{\Sigma}^2(x)\frac{d}{dx}(\frac{x}{g'(x)}) - \tilde{\alpha}_q(x) - g'(x)\tilde{\beta}(x) = 0 \text{ for all } x \in (\underline{x}, \overline{x}) \text{ with } g'(x) \neq 0. \tag{5.2.3}$$

Next, define state process X_t^x which is the solution of Skorokhod SDE

$$\begin{aligned} dX_t^x &= \Big(X_t^x\tilde{\beta}(X_t^x) - q\tilde{\theta}(X_t^x)\tilde{\Gamma}(X_t^x)\Big)dt + \tilde{\Gamma}(X_t^x)dB_t + d\Phi_t^x - d\Psi_t^x \\ X_0^x &= x \in [\underline{x}, \overline{x}] \end{aligned} \tag{5.2.4}$$

where $\tilde{\Gamma}(x) = -\frac{x}{g'(x)}\tilde{\Sigma}(x)$ and Φ^x, Ψ^x are nondecreasing process such that increase only the sets $\{x = \underline{x}\}, \{x = \overline{x}\}$, respectively. To show that solution X_t^x exists, it is enough to show that $(X_t^x\tilde{\beta}(X_t^x) - q\tilde{\theta}(X_t^x)\tilde{\Gamma}(X_t^x))$ is globally Lipschitz on $[\underline{x}, \overline{x}]$ by [6]. Since h is globally Lipschitz and nonzero on $[\underline{x}, \overline{x}]$, and g and g' are bounded on $[\underline{x}, \overline{x}]$, Γ is also globally Lipschitz on $[\underline{x}, \overline{x}]$. Therefore, $(X_t^x\tilde{\beta}(X_t^x) - q\tilde{\theta}(X_t^x)\tilde{\Gamma}(X_t^x))$ is globally Lipschitz on $[\underline{x}, \overline{x}]$.

Define W_t^x

$$W_t^x = \begin{cases} -\delta \log(\mathcal{E}(-\tilde{\theta}(X^x)\cdot B)_t) & \text{if } p = 0, \\ sgn(p)\mathcal{E}(-\tilde{\theta}(X^x)\cdot B)_t^{-q} & \text{if } p \neq 0. \end{cases}$$

From following lemma, function g is simply represented by W_t^x.

Lemma 5.2.3. *Assume that lemma 5.2.1 and 5.2.2 hold. For $x \in [\underline{x}, \overline{x}]$, I have*

$$g(x) = \mathbb{E}[\int_0^\infty e^{-\tilde{\delta}t}W_t^x dt]. \tag{5.2.5}$$

proof) Using Ito's lemma, differential dg is given by

$$\begin{aligned} dg(X_t^x) &= g'(X_t^x)\Big(\big(X_t^x\tilde{\beta}(X_t) - q\tilde{\theta}\tilde{\Gamma}(X_t^x)\big)dt + \tilde{\Gamma}(X_t^x)dB_t + d\Phi_t^x - d\Psi_t^x\Big) + \frac{g''(X_t^x)}{2}\tilde{\Gamma}(X_t^x)^2 dt \\ &= \Big(g'(X_t^x)X_t^x\tilde{\beta}(X_t) + \frac{g''(X_t^x)}{2}\tilde{\Gamma}(X_t^x)^2\Big)dt + g'(X_t^x)\tilde{\Gamma}(X_t^x)d\overline{B}_t \\ &= \Big(\tilde{\beta}(X_t^x)g(X_t^x) - \tilde{\gamma}(X_t^x)\Big)dt + g'(X_t^x)\tilde{\Gamma}(X_t^x)d\overline{B}_t \end{aligned} \tag{5.2.6}$$

where $\overline{B}$ is Brownian motion on $[0, t]$ under $\overline{\mathbb{P}}_t$ defined in (5.1.10) with $\theta = \tilde{\theta}(X^x)$. It is possible due to boundedness of $\tilde{\theta}$. Because g' is zero whenever $d\Phi_t^x$ and $d\Psi_t^x$ are nonzeros, second equality hold. Considering two equation which are (5.1.13) and multiplying x at (5.2.3), third equality is made.

Let $\rho_t^x = e^{-\int_0^t \tilde{\beta}(X_s^x)ds}$ and $H_t^x = \int_0^t \rho_s^x\tilde{\gamma}(X_s^x)ds + \rho_t^x g(X_t^x)$. Using Ito's lemma, (5.2.3) and finite variation property of Φ and Ψ, differential dH is generated

$$dH_t^x = \rho_t^x g'(X_t^x)\tilde{\Gamma}(X_t^x)d\bar{B}_t$$

Since X_t^x, ρ^x, g' and $\tilde{\gamma}$ are bounded, H is martingale under $\bar{\mathbb{P}}_t$. So, following relation satisfies

$$\begin{aligned} g(x) = \bar{\mathbb{E}}[H_0^x] = \bar{\mathbb{E}}[H_t^x] &= \bar{\mathbb{E}}[\rho_t^x g(X_t^x) + \int_0^t \rho_s^x \tilde{\gamma}(X_s^x) ds] \\ &= \bar{\mathbb{E}}[\rho_t^x g(X_t^x)] + \mathbb{E}[\int_0^t e^{-\tilde{\delta} s} W_s^x ds] + \begin{cases} \frac{1}{2} e^{-\delta t} \mathbb{E}[\int_0^t \tilde{\theta}(X_s^x)^2 ds] & \text{if } p = 0 \\ 0 & \text{if } p \neq 0 \end{cases} \end{aligned} \tag{5.2.7}$$

For last equality, (5.1.8) and definition of W_t^x is used. (5.2.7) is considered for $p = 0$, $p > 0$ and $p < 0$. For $p = 0$, (5.2.5) is obtained from

$$\bar{\mathbb{E}}[\rho_t^x g(X_t^x)] = \mathbb{E}[e^{-\delta t} g(X_t^x)] \to 0 \quad \text{and} \quad e^{-\delta t} \mathbb{E}[\int_0^t \tilde{\theta}(X_s^x)^2 ds] \to 0 \quad \text{as t} \to \infty$$

which are proved by boundedness of $\tilde{\theta}$ and g. For $p > 0$, nonnegativity of g, W_t^x and ρ_t^x imply

$$\int_0^\infty e^{-\tilde{\delta} t} \mathbb{E}[W_t^x] dt < \infty. \tag{5.2.8}$$

Boundedness of g, (5.2.8) and continuity of $e^{-\tilde{\delta} t}\mathbb{E}[W_t^x]$ implies

$$\bar{\mathbb{E}}[\rho_t^x g(X_t^x)] \leq \sup_{x \in [\underline{x}, \overline{x}]} |g(x)| \bar{\mathbb{E}}[\rho_t^x] = \sup_{x \in [\underline{x}, \overline{x}]} |g(x)| e^{-\tilde{\delta} t} \mathbb{E}[W_t^x] \to 0 \quad \text{as t} \to \infty$$

Hence, (5.2.5) holds for $p > 0$.

For $p < 0$, $\rho_t^x \leq e^{-\tilde{\delta} t}$ implies

$$\bar{\mathbb{E}}[\rho_t^x |g(X_t^x)| \leq \sup_{x \in [\underline{x}, \overline{x}]} |g(x)| e^{-\tilde{\delta} t} \to 0 \quad \text{as t} \to \infty$$

So, it yields (5.2.5) □

Finally, it is time to prove that shadow price process $\tilde{S}$. We will prove shadow price process $\tilde{S}_t$ follows form $\tilde{S}_t^x = S_t e^{f(X_t^x)}$ which is similar form in section 4. Here, $f : [\underline{x}, \overline{x}] \to \mathbb{R}$ is defined by $f(x) = \underline{y} + \int_{\underline{x}}^{\overline{x}} \frac{g'(s)}{s} ds$. From (5.1.14), f can be thought inverse of $-w'$.

Lemma 5.2.4. *Assume that lemma 5.2.1 and 5.2.2 hold. For $x \in [\underline{x}, \overline{x}]$, I have*

1. $f(X_t^x) \in [\underline{y}, \overline{y}]$ *for all* $t \geq 0$, *a.s*
2. $f(X_0^x) = f(x)$ *and* $df(X_t^x) = \tilde{\alpha}_0(X_t^x)dt + \tilde{\Sigma}(X_t^x)dB_t$.

proof) (1) is proved by definition of f.

(2) Using Ito's lemma, (5.2.3) and the fact that $g' = 0$ when $d\Phi$ or $d\Psi$ is nonzero, differential df satisfies $df(X_t^x) = \tilde{\alpha}_0(X_t^x)dt + \tilde{\Sigma}(X_t^x)dB_t$.□

Using second property in lemma 5.2.4 and Ito formula, the differential $d\tilde{S}_t^x$ satisfies following equation

$$d\tilde{S}_t^x = \tilde{S}_t^x\Big(\sigma + \tilde{\Sigma}(X_t^x)\Big)\Big(\tilde{\theta}(X_t^x)dt + dB_t\Big), \quad \tilde{S}_0^x = S_0 e^{f(x)} \quad \text{for any } x \in [\underline{x}, \overline{x}]. \tag{5.2.9}$$

Before the proof of main lemma which is optimal strategy in frictionless market, introduce some notation for simple calculation.

$$\xi(x) = \eta_B + S_0 e^{f(x)}\eta_S,\ \Pi(x) = \begin{cases} x & \text{if } p = 0 \\ \frac{x}{qg(x)} & \text{if } p \neq 0 \end{cases}, \quad K(x) = \begin{cases} \delta & \text{if } p = 0 \\ \frac{1}{|g(x)|} & \text{if } p \neq 0 \end{cases} \tag{5.2.10}$$

Lemma 5.2.5. *Assume that lemma 5.2.1 and 5.2.2 hold. For the function $r : [\underline{x}, \overline{x}] \to \mathbb{R}$, given by $r(x) = \eta_S S_0 e^{f(x)}(1 - \Pi(x)) - \eta_B \Pi(x)$, let the constant $\tilde{x} \in [\underline{x}, \overline{x}]$ be defined by*

$$\tilde{x} = \begin{cases} \overline{x} & r(x)>0 \text{ for all } x \in [\underline{x}, \overline{x}] \\ \underline{x} & r(x)<0 \text{ for all } x \in [\underline{x}, \overline{x}] \\ \text{a solution to } r(x) = 0 & \text{otherwise.} \end{cases} \tag{5.2.11}$$

For $x \in [\underline{x}, \overline{x}]$ and the initial position (η_B, η_S) with $\eta_B + S_0 e^{f(x)}\eta_S \geq 0$ in frictionless market with price process $\tilde{S}_t^{\tilde{x}}$, I have

$$\tilde{u}(\eta_B, \eta_S; \tilde{x}) = \begin{cases} \frac{1}{\delta}\Big(-1 + \log(\delta\xi(\tilde{x})) + g(\tilde{x})\Big) & \text{if } p = 0, \\ \frac{1}{p}\xi(\tilde{x})^p |g(\tilde{x})|^{1-p} & \text{if } p \neq 0. \end{cases} \tag{5.2.12}$$

Moreover, with the processes $\{\tilde{\pi}_t^{\tilde{x}}\}_{t\in[0,\infty)}$, $\{\tilde{\kappa}_t^{\tilde{x}}\}_{t\in[0,\infty)}$, and $\{\tilde{V}_t^{\tilde{x}}\}_{t\in[0,\infty)}$ defined by

$$\tilde{\pi}_t^{\tilde{x}} = \Pi(X_t^{\tilde{x}}),\ \tilde{\kappa}_t^{\tilde{x}} = K(X_t^{\tilde{x}}),\ \text{and } \tilde{V}_t^{\tilde{x}} = \xi(\tilde{x})\mathcal{E}\Big(\int_0 \frac{\tilde{\pi}_s^{\tilde{x}}}{\tilde{S}_s^{\tilde{x}}} d\tilde{S}_s^{\tilde{x}} - \int_0 \tilde{\kappa}_s^{\tilde{x}} ds\Big)_t, \tag{5.2.13}$$

the optimal strategy $(\varphi^{\tilde{x}}, c^{\tilde{x}})$ is given for $t \geq 0$ by

$$\tilde{\varphi}_t^{B,\tilde{x}} = \tilde{V}_t^{\tilde{x}}(1 - \tilde{\pi}_t^{\tilde{x}}), \quad \tilde{\varphi}_t^{S,\tilde{x}} = \frac{\tilde{V}_t^{\tilde{x}}\tilde{\pi}_t^{\tilde{x}}}{\tilde{S}_t^{\tilde{x}}}, \quad \text{and } \tilde{c}_t^{\tilde{x}} = \tilde{V}_t^{\tilde{x}}\tilde{\kappa}_t^{\tilde{x}}. \tag{5.2.14}$$

proof) It is easy to check that $\tilde{x}$ is well defined. We will omit the $\tilde{x}$ from the superscript of notations for simplicity. Since $\tilde{V}_0 = \xi(x)$ and $\tilde{V}_t = \tilde{\varphi}_t^B + \tilde{\varphi}_t^S \tilde{S}_t$, $\tilde{V}_0$ is value process for price process $\tilde{S}_t$. Combining (5.1.7) and (5.2.5) yield (5.2.12). Define the processes $\tilde{\pi}_t$ as

$$\tilde{\pi}_t = \frac{\tilde{\theta}(X_t)}{(1-p)(\sigma + \tilde{\Sigma}(X_t))} + \begin{cases} 0 & \text{if } p = 0 \\ -\frac{X_t^x}{g(X_t)} \frac{\tilde{\Sigma}(X_t)}{\sigma + \Sigma(\tilde{X}_t)} & \text{if } p \neq 0 \end{cases} \tag{5.2.15}$$

Let the processes $\tilde{\kappa}_t$ and $\tilde{V}_t$ be as (5.2.13). Then direct calculation shows that $\tilde{\pi}_t$ coincide definition (5.2.13) and make optimal strategy. I will only solve the case for $p = 0$. $p \neq 0$ can be solved exactly the same way. Since h has no zeros on $[\underline{x}, \overline{x}]$,

$$\tilde{\pi}_t = \frac{\tilde{\theta}(X_t)}{(\sigma + \tilde{\Sigma}(X_t))} = \frac{\frac{\sigma X_t}{h(X_t)}}{\sigma - \frac{\sigma(1-X_t)g'(X_t)}{h(X_t)}} = \frac{\frac{\sigma X_t}{h(X_t)}}{\sigma - \frac{\sigma h(X_t) - \sigma}{h(X_t)}} = X_t.$$

Also, (5.2.15) generates

$$\begin{aligned} \mathbb{E}[\int_0^\infty e^{-\delta t} U(\tilde{c}_t) dt] &= \mathbb{E}[\int_0^\infty e^{-\delta t} \Big(\log(\delta \xi(\tilde{x})) + \frac{\theta^2 (\tilde{X_t})}{2} \Big) dt] + \mathbb{E}[\int_0^\infty e^{-\delta t} \tilde{\theta}(X_t) dB_t \Big)] \\ &= \frac{1}{\delta} \Big(-1 + \log(\delta \xi(\tilde{x})) + g(\tilde{x}) \Big). \end{aligned}$$

(5.2.5) is used for second equality.

dlog $\varphi_t^S = \log \tilde{V}_t + \log \tilde{\pi}_t - \log \tilde{S}_t$. Property 2 in Lemma 5.2.5 and Ito's lemma give

$$\begin{aligned} d\tilde{S}_t &= S_t d e^{f(X_t)} + e^{f(X_t)} dS_t - d < S, e^{f(X)} > t \\ &= \tilde{S}_t \Big(\tilde{\alpha_0}(X_t) + \frac{\tilde{\Sigma}^2(X_t)}{2} + \mu + \sigma \tilde{\Sigma}(X_t) \Big) dt + \tilde{S}_t (\tilde{\Sigma}(X_t) + \sigma) dB_t. \\ &= \tilde{S}_t \Big(\tilde{\Sigma}(X_t) \tilde{\theta}(X_t) + \sigma \tilde{\theta}(X_t) \Big) dt + \tilde{S}_t (\tilde{\Sigma}(X_t) + \sigma) dB_t. \end{aligned} \tag{5.2.16}$$

Using (5.2.16), we can calculate $-d \log \tilde{S}_t$ and $d \log \tilde{V}_t$ which are

$$\begin{aligned} -d \log \tilde{S}_t &= -\frac{1}{\tilde{S}_t} d\tilde{S}_t + \frac{1}{\tilde{S}_t^2} d < \tilde{S}, \tilde{S} >_t \\ &= \Big(\tilde{\Sigma}(X_t)^2 + 2\tilde{\Sigma}(X_t)\sigma + \sigma^2 - \tilde{\Sigma}(X_t)\tilde{\theta}(X_t) - \sigma \tilde{\theta}(X_t) \Big) dt - (\tilde{\Sigma}(X_t) + \sigma) dB_t \\ &= \big(\frac{\sigma^2 (1 - X_t)}{h(X_t)^2} \big) dt - \frac{\sigma}{h(X_t)} dB_t, \\ d \log \tilde{V}_t &= \frac{\tilde{\pi}_t}{\tilde{S}_t} d\tilde{S}_t - \tilde{\kappa}_t dt - \frac{1}{2} \big(\frac{\tilde{\pi}_t}{\tilde{S}_t} \big)^2 < \tilde{S}, \tilde{S} >_t \\ &= \Big((\tilde{\Sigma}_t \tilde{\theta}_t + \sigma \tilde{\theta}_t) X_t - \delta - \frac{(\sigma + \tilde{\Sigma})^2}{2} X_t^2 \Big) dt + (\tilde{\Sigma} + \sigma) X_t dB_t \\ &= \Big(\frac{\sigma^2 X_t^2}{2h(X_t)^2} - \delta \Big) dt + \frac{\sigma X_t}{h(X_t)} dB_t. \end{aligned}$$

From (5.2.4), $d\log\pi_t$ yields

$$
\begin{aligned}
d\log\pi_t = d\log X_t &= \frac{1}{X_t}dX_t - \frac{1}{2X_t^2} < X, X >_t \\
&= \Big(\tilde\beta(X_t) - \frac{1}{2}(\frac{\tilde\Sigma(X_t)}{g'(X_t)})^2\Big)dt - \frac{\tilde\Sigma(X_t)}{g'(X_t)}dB_t + \frac{1}{X_t}(d\Psi_t - d\Phi_t) \\
&= (\delta - \frac{1}{2}(\frac{\sigma(1-X_t)}{h(X_t)})^2)dt + \frac{\sigma(1-X_t)}{h(X_t)}dB_t + \frac{1}{X_t}(d\Psi_t - d\Phi_t).
\end{aligned}
$$

Summing up all the terms, $d\log\tilde\varphi_t^S = \frac{1}{X_t}(d\Psi_t - d\Phi_t)$. Therefore, Ito's lemma makes $d\tilde\varphi_t^S = (\frac{\tilde\varphi_t^S}{X_t})(d\Phi_t - d\Psi_t)$. Because Φ and Ψ are nondecreasing finite variations increase only the the $X = \underline{x}$ and $X = \overline{x}$, respectively. $(\tilde\varphi^S, \tilde\varphi^B) \in \mathcal{A}(\tilde S^x)$.□

Remark $r(\tilde x) = 0$ implies $(\tilde\varphi_0^{B,\tilde x}, \tilde\varphi_0^{S,\tilde x}) = (\eta_B, \eta_S)$. Because it yields relations

$$
\frac{\eta_S S_0 e^{f(\tilde x)}}{\eta_B + \eta_S S_0 e^{f(\tilde x)}} = \pi(\tilde x) \Leftrightarrow \frac{\eta_S \tilde S_0^{\tilde x}}{\tilde V_0^{\tilde x}} = \pi(\tilde x) \Leftrightarrow \eta_S = \tilde\varphi_0^{S,\tilde x}. \tag{5.2.17}
$$

Since $\tilde V_0^{\tilde x}$ is value process for price process S_0, $(\tilde\varphi_0^{B,\tilde x}, \tilde\varphi_0^{S,\tilde x})$ is equal to (η_B, η_S). Using same procedure, the following facts are generated

$$
\begin{aligned}
\eta_S > \tilde\varphi_0^{S,\tilde x} \qquad & \text{if } r(x) > 0 \text{ for all } x \in [\underline{x}, \overline{x}] \\
\eta_S < \tilde\varphi_0^{S,\tilde x} \qquad & \text{if } r(x) < 0 \text{ for all } x \in [\underline{x}, \overline{x}].
\end{aligned}
$$

So, $(\tilde\varphi_0^{B,\tilde x}, \tilde\varphi_0^{S,\tilde x}) \neq (\eta_B, \eta_S)$ if there is no solution to $r(x) = 0$ for $x \in [\underline{x}, \overline{x}]$. Then, It can be change to satisfy $(\tilde\varphi_0^{B,\tilde x}, \tilde\varphi_0^{S,\tilde x}) = (\eta_B, \eta_S)$. The method is that the y-axis value is modified to enter the no transaction region for a predetermined x value in Figure 5-1.

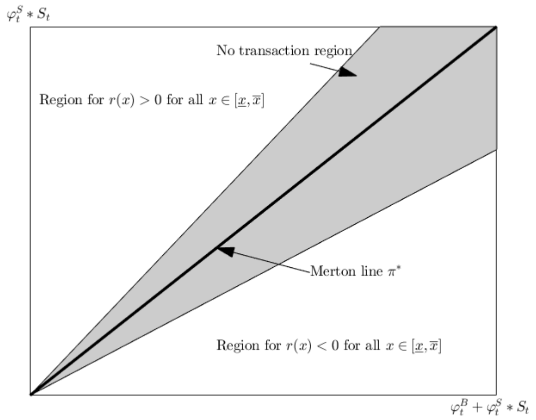

Figure 5-1: Illustration of $\tilde x$ in lemma 5.2.5

Theorem 5.2.1. *Assume that lemma 5.2.1 and 5.2.2 hold. Then $\tilde S = \tilde S^{\tilde x}$ is a shadow price.*

proof) Same procedure of proof in Theorem 4.2.1 shows that $(\varphi^{B,\tilde x}, \varphi^{S,\tilde x}, c^{\tilde x}) \in \mathcal{A}(S)$ □.

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