Comptes Rendus

Mathématique

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Partial differential equations / *Equations aux dérivées partielles*

On the uniqueness of linear convection–diffusion equations with integral boundary conditions

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Abstract. We investigate a class of convection–diffusion equations in an expanding domain involving a parameter, where we consider integral boundary conditions that depend non-locally on unknown solutions. Generally, the uniqueness result of this type of equation is unclear. In this work, we obtain a uniqueness result when the domain is sufficiently large or small. This approach has the advantage of transforming the integral boundary conditions into new Dirichlet boundary conditions so that we can obtain refined estimates, and the comparison theorem can be applied to the equations. Furthermore, we show a domain such that under different boundary data, the equation in this domain can have infinitely numerous solutions or no solution. This work may contribute to the first understanding of the domain size's effect on the existence and uniqueness of the linear convection–diffusion equation with integral-type boundary conditions.

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1. Introduction and the statement of the main results

This study examines the existence and uniqueness of the solutions to linear convection–diffusion equations with integral boundary conditions, where the domain has numerous boundary components and a positive parameter.

We first consider a domain with two boundary components for clarity. Define a smooth bounded domain

$$\Omega_\lambda := \left\{ \lambda z : z \in \Omega := \Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}} \right\} \tag{1}$$

which is diffeomorphic to an annulus and expands as λ tends to infinity, where Ω_{in} and Ω_{out} represent bounded simply-connected smooth domains in $\mathbb{R}^N$, $N \geq 2$, such that $0 \in \Omega_{\text{in}} \subsetneq \Omega_{\text{out}}$. This study focuses primarily on the linear convection–diffusion equation

$$-\Delta u_\lambda(x) + \vec{\mathrm{a}}(x) \cdot \nabla u_\lambda(x) + h(x) u_\lambda(x) = 0, \quad x \in \Omega_\lambda, \tag{2}$$

subject to the following integral-type boundary conditions for u_λ:

$$\begin{cases} u_\lambda(x) = b_{\text{i}} + \displaystyle\int_{\Omega_\lambda} g_{\text{i}}(y) u_\lambda(y) \, \mathrm{d}y, & x \in \Gamma_{\lambda,\text{i}} := \left\{ \lambda z : z \in \partial\Omega_{\text{in}} \right\}, \\ u_\lambda(x) = b_{\text{o}} + \displaystyle\int_{\Omega_\lambda} g_{\text{o}}(y) u_\lambda(y) \, \mathrm{d}y, & x \in \Gamma_{\lambda,\text{o}} := \left\{ \lambda z : z \in \partial\Omega_{\text{out}} \right\}. \end{cases} \tag{3}$$

The boundary data of u_λ on $\Gamma_{\lambda,\text{i}} \cup \Gamma_{\lambda,\text{o}}$ satisfies an implicit form with non-local dependence on the unknown u_λ in Ω_λ. Here b_{i} and b_{o} are constants, which are independent of λ, $g_{\text{i}}, g_{\text{o}} \in C(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$, and the variable coefficients $\vec{\mathrm{a}} \in C^\alpha(\mathbb{R}^N \setminus \{0\}; \mathbb{R}^N)$ and $h \in C^\alpha(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ are Hölder continuous with exponent $\alpha \in (0, 1)$. Thus, the restrictions $g_{\text{i}}|_{\overline{\Omega_\lambda}}$, $g_{\text{o}}|_{\overline{\Omega_\lambda}}$, $\vec{\mathrm{a}}|_{\overline{\Omega_\lambda}}$ and $h|_{\overline{\Omega_\lambda}}$ are well defined for any $\lambda > 0$. The conditions of these functions will be assumed specifically in (4) and (7).

Previous research literature [4, 8, 13, 15], in which a one-dimensional (1D) case was numerically examined, inspired the equation that we are studying. We refer the readers to [5, 9] for the details on the relevant qualitative and asymptotic analysis for solutions of singularly perturbed models with different integral-type boundary conditions. Furthermore, several studies have investigated the uniqueness or multiplicity of the solutions to such non-local equations (e.g. see [3] and its references). The primary approach is based on several fixed-point theorems, such as Krasnosel'skii's fixed-point theorem, Schauder fixed-point theorem and the weakly contractive mapping theorem. Different from 1D models, for fixed $\lambda > 0$, studying the uniqueness or multiplicity of the solutions of higher-dimensional equations such as (2)–(3) is a challenge in the analysis because such equations do not have a variational structure.
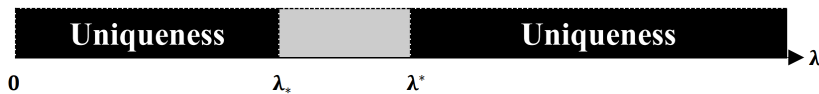

Figure 1. The existence, uniqueness and multiplicity of solutions to equation (2)–(3) may be affected by the domain Ω_λ with respect to the parameter λ. In Theorem 3, we demonstrate that there exist $0 < \lambda_* < \lambda^*$ such that equations (2)–(3) have a unique solution as $\lambda \in (0, \lambda_*) \cup (\lambda^*, \infty)$. However, when $\lambda \in [\lambda_*, \lambda^*]$ (the grey region), the equation may have multiple solutions or no solution. (For clarity, we refer the reader to Example 1 and Proposition 2.)

Ω_λ with $\lambda \to \infty$ is an expanding domain, which is related to the mathematical problems for convection–diffusion models in a reactor of macroscopic length scale (see, e.g. [12, 17]), where the region for a chemical substance to diffuse across is significantly larger compared to that of a reaction process [1, 18]. We focus on convection–diffusion equation (2) satisfying a *weak*

convection effect or *large reaction effect* (cf. [16]). In the framework of our analysis, both effects can be considered as the following mathematical settings:

$$|\vec{a}(x)|^2 < 4h(x),\ x \in \mathbb{R}^N \setminus \{0\}, \quad \text{and} \quad \lim_{\lambda \to \infty} \lambda^\kappa \min_{\overline{\Omega}} \left(4h(\lambda z) - |\vec{a}(\lambda z)|^2 \right) > 0 \text{ for some } \kappa \in (0,2). \quad (4)$$

Equation (2)–(3) with assumption (4) fulfills some well-known convection–diffusion equations. In particular, (4) includes the case $\vec{a}(x) \equiv \vec{0}$ and $h(x) > 0$, which arises in numerous applications such as the homogeneous chemical reactions. (4) is also a case when considering a convection–diffusion equation (2) with a large reaction $h(x)$. For example, $\min_{\overline{\Omega_\lambda}} h$ is sufficiently large so that $\max_{\overline{\Omega_\lambda}} |\vec{a}|^2 < 4 \min_{\overline{\Omega_\lambda}} h$. The latter assumption in (4) includes the case

$$\min_{\overline{\Omega}} \left(4h(\lambda z) - |\vec{a}(\lambda z)|^2 \right) \sim \lambda^{-\widehat{\kappa}} \xrightarrow{\lambda \gg 1} 0 \quad \text{for} \quad 0 < \widehat{\kappa} < \kappa,$$

where the region of κ is optimal in our analysis.

Under (4), equation (2) with the standard Dirichlet boundary conditions on $\partial\Omega_\lambda$ has a unique solution for any $\lambda > 0$. However, the situation becomes quite different for the non-local type boundary condition (3). The issue about the range of λ for the uniqueness of (2)–(3) seems to be a challenge. In a special case when Ω_λ is an annulus, we provide an example to explain that there exists $\lambda = \lambda^* > 0$ such that equation (2)–(3) has infinitely numerous solutions or no solution.

Example 1. We consider the following equation in the domain $\mathbb{A}_{\lambda;2\lambda} := \left\{ x \in \mathbb{R}^2 : \lambda < |x| < 2\lambda \right\}$, i.e. the domain Ω_λ defined in (1) with $\Omega_{\text{in}} = \{x : |x| < 1\}$ and $\Omega_{\text{out}} = \{x : |x| < 2\}$ in $\mathbb{R}^2$:

$$\begin{cases} -\Delta u_\lambda(x) + \dfrac{x}{|x|^2} \cdot \nabla u_\lambda(x) + u_\lambda(x) = 0, \quad x \in \mathbb{A}_{\lambda;2\lambda}, \\[2mm] u_\lambda(x) = b_{\text{i}} + \mathfrak{g} \displaystyle\int_{\mathbb{A}_{\lambda;2\lambda}} u_\lambda(y)\,\mathrm{d}y, \quad \text{for} \quad |x| = \lambda; \qquad u_\lambda(x) = 0, \quad \text{for} \quad |x| = 2\lambda, \end{cases} \quad (5)$$

where $\mathfrak{g} > 0$ is a constant. We shall apply Fourier expansions to demonstrate that all solutions u_λ of (5) are radially symmetric. For $x = r\mathrm{e}^{\sqrt{-1}\theta}$ with $\lambda < r < 2\lambda$ and $0 \le \theta < 2\pi$, we let

$$u_\lambda \left(r\mathrm{e}^{\sqrt{-1}\theta} \right) = \frac{1}{2} u_{1,0}(r) + \sum_{k=1}^{\infty} \left(u_{1,k}(r) \cos(k\theta) + u_{2,k}(r) \sin(k\theta) \right),$$

where

$$u_{1,k}(r) := \frac{1}{\pi} \int_{-\pi}^{\pi} u_\lambda \left(r\mathrm{e}^{\sqrt{-1}\theta} \right) \cos(k\theta)\,\mathrm{d}\theta, \quad u_{2,k}(r) := \frac{1}{\pi} \int_{-\pi}^{\pi} u_\lambda \left(r\mathrm{e}^{\sqrt{-1}\theta} \right) \sin(k\theta)\,\mathrm{d}\theta.$$

Then, one may check that for $i = 1, 2$, and $k \in \mathbb{N}$, $u_{i,k}$ satisfies the equation

$$u_{i,k}''(r) - \left(1 + \frac{k^2}{r^2} \right) u_{i,k}(r) = 0 \text{ in } (\lambda, 2\lambda), \quad u_{i,k}(\lambda) = u_{i,k}(2\lambda) = 0.$$

Such an equation only has a trivial solution $u_{i,k}(r) \equiv 0$, and we obtain that $u_\lambda(x) = \frac{1}{2} u_{1,0}(|x|)$ is radially symmetric. As a consequence, we can solve (5) and verify the non-existence, uniqueness and multiplicity of the solutions u_λ's corresponding to b_{i} and λ. More accurately, we completely classify the solutions of (5) as follows:

(i) When $b_i = 0$ and $\lambda^* := \lambda^*(\mathfrak{g}) > 0$ satisfies[1]

$$\left(\frac{1+\lambda^*}{2} - \frac{1}{4\mathfrak{g}\pi} \right) \frac{e^{2\lambda^*} - 1}{\lambda^*} - 2(e^{\lambda^*} - 1) = 1, \tag{6}$$

for any constant c^*, all $u_{\lambda^*, c^*}(x) = c^*(e^{|x|} - e^{4\lambda^* - |x|})$ are solutions of (5) with $\lambda = \lambda^*$.

(ii) When $b_i \neq 0$ and $\lambda = \lambda^*$, (5) has no solution.

(iii) If $\lambda > 0$ and $\lambda \neq \lambda^*$, then for each $b_i \in \mathbb{R}$, (5) has a unique solution.

Simple calculations can be employed to verify this result. We thus omit the details here. For general cases, the issues of the non-existence, uniqueness, and multiplicity of solutions of (2)–(3) become more complicated; see Proposition 2. What we shall point out is that the uniqueness of solutions to (2)–(3) in Ω_λ defined by (1) exactly depends variously on λ, b_i, and b_o.

Additionally, to obtain a more detailed uniqueness result of (2)–(3) with large λ, for g_i and g_o, we made the following precise assumption:

$$\begin{cases} \lim\limits_{\lambda \to \infty} \lambda^{N + \frac{\kappa-2}{2}} \max\limits_{z \in \overline{\Omega}} \left| g(\lambda z) \right| = 0, \\ \lim\limits_{|z| \to 0} |z|^N g(z) = 0, \end{cases} \qquad \text{for } g = g_i, g_o, \tag{7}$$

where $\kappa \in (0, 2)$ has been defined in (4). Assumption (7) allows g to blow up at $z = 0$. An example for (7) is $g(z) = |z|^{-N + \frac{2-\kappa^*}{2}}$ with $\kappa^* \in (\kappa, 2)$. Although (7) implies $|\int_{\Omega_\lambda} g(y)\,dy| = \lambda^N |\int_\Omega g(\lambda z)\,dz| \ll \lambda^{\frac{2-\kappa}{2}}$ as $\lambda \to \infty$, we emphasise that it also includes the case $\lim_{\lambda \to \infty} |\int_{\Omega_\lambda} g(y)\,dy| = \infty$.

1.1. *The main result*

We introduce the equations corresponding to (2) with the standard Dirichlet boundary conditions before stating the main results. Let ϕ_λ be a solution of the equation

$$\begin{cases} -\Delta\phi_\lambda(x) + \vec{a}(x) \cdot \nabla\phi_\lambda(x) + h(x)\phi_\lambda(x) = 0 & \text{in } \Omega_\lambda, \\ \phi_\lambda = 1 \quad \text{on } \Gamma_{\lambda,i}, \qquad\qquad \phi_\lambda = 0 \quad \text{on } \Gamma_{\lambda,o}, \end{cases} \tag{8}$$

and let ψ_λ be a solution of the equation

$$\begin{cases} -\Delta\psi_\lambda(x) + \vec{a}(x) \cdot \nabla\psi_\lambda(x) + h(x)\psi_\lambda(x) = 0 & \text{in } \Omega_\lambda, \\ \psi_\lambda = 0 \quad \text{on } \Gamma_{\lambda,i}, \qquad\qquad \psi_\lambda = 1 \quad \text{on } \Gamma_{\lambda,o}. \end{cases} \tag{9}$$

Note that Ω_λ is a bounded smooth domain for each fixed $\lambda > 0$, and the constraints $\vec{a}|_{\overline{\Omega_\lambda}}$ and $h|_{\overline{\Omega_\lambda}}$ are Hölder continuous with exponent $\alpha \in (0, 1)$. As a consequence, by the standard elliptic regularity theory (cf. [10, Theorem 6.14]), we obtain that the equations (8) and (9) have unique solutions $\phi_\lambda, \psi_\lambda \in C^{2,\alpha}(\overline{\Omega_\lambda})$, where the uniqueness is trivially due to $h|_{\overline{\Omega_\lambda}} > 0$.

Let c_1 and c_2 be constants. Then, by the uniqueness of solutions of (8) and (9), the solution of equation (2) with boundary conditions $u_\lambda = c_1$ on $\Gamma_{\lambda,i}$ and $u_\lambda = c_2$ on $\Gamma_{\lambda,o}$ is uniquely expressed by $u_\lambda = c_1\phi_\lambda + c_2\psi_\lambda$. Accordingly, for each solution u_λ of (2)–(3) (if it exists), we have an implicit representation for u_λ given by

$$u_\lambda(x) = \left(b_i + \int_{\Omega_\lambda} g_i(y) u_\lambda(y)\,dy \right) \phi_\lambda(x) + \left(b_o + \int_{\Omega_\lambda} g_o(y) u_\lambda(y)\,dy \right) \psi_\lambda(x), \quad x \in \overline{\Omega_\lambda}. \tag{10}$$

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[1]Let

$$f(\lambda) = \left(\frac{1+\lambda}{2} - \frac{1}{4\mathfrak{g}\pi} \right) \frac{e^{2\lambda} - 1}{\lambda} - 2\left(e^\lambda - 1\right).$$

Since $\mathfrak{g} > 0$, we have

$$\lim_{\lambda \to 0+} f(\lambda) = 1 - \frac{1}{2\mathfrak{g}\pi} < 1 \quad \text{and} \quad \lim_{\lambda \to \infty} f(\lambda) = \infty.$$

Along with the intermediate value theorem of continuous functions, we verify that (6) has a positive root $\lambda^* = \lambda^*(\mathfrak{g})$.

Here we perform an argument that is different from the fixed point approach. We will show that the two non-local coefficients (depending on unknown u_λ) in the right-hand side of (10) can be explicitly expressed by ϕ_λ and ψ_λ. In Proposition 2, we establish a criterion of the existence and uniqueness for (2)–(3).

Proposition 2. *Let Ω_λ be defined by* (1). *Assume that b_i and b_o are constants independent of λ, $g_\mathrm{i}, g_\mathrm{o} \in \mathrm{C}(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ satisfy* (7), *and $\vec{\mathrm{a}} \in \mathrm{C}^\alpha(\mathbb{R}^N \setminus \{0\}; \mathbb{R}^N)$ and $h \in \mathrm{C}^\alpha(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ satisfy* (4), *where $\alpha \in (0,1)$. Then we have the following:*

(i) *If λ satisfies $\det(\mathscr{I} - \mathscr{R}_\lambda) \neq 0$, then* (2)–(3) *has a unique solution satisfying*

$$u_\lambda(x) = \det\big((\mathscr{I} - \mathscr{R}_\lambda)^{-1}\mathscr{C}_{\psi_\lambda}\big)\phi_\lambda(x) + \det\big((\mathscr{I} - \mathscr{R}_\lambda)^{-1}\mathscr{C}_{\phi_\lambda}\big)\psi_\lambda(x), \tag{11}$$

where

$$\mathscr{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathscr{R}_\lambda = \begin{bmatrix} \displaystyle\int_{\Omega_\lambda} g_\mathrm{i}(y)\phi_\lambda(y)\,\mathrm{d}y & \displaystyle\int_{\Omega_\lambda} g_\mathrm{i}(y)\psi_\lambda(y)\,\mathrm{d}y \\ \displaystyle\int_{\Omega_\lambda} g_\mathrm{o}(y)\phi_\lambda(y)\,\mathrm{d}y & \displaystyle\int_{\Omega_\lambda} g_\mathrm{o}(y)\psi_\lambda(y)\,\mathrm{d}y \end{bmatrix} \tag{12}$$

and

$$\mathscr{C}_{\psi_\lambda} = \begin{bmatrix} b_\mathrm{i} & -\displaystyle\int_{\Omega_\lambda} g_\mathrm{i}(y)\psi_\lambda(y)\,\mathrm{d}y \\ b_\mathrm{o} & 1 - \displaystyle\int_{\Omega_\lambda} g_\mathrm{o}(y)\psi_\lambda(y)\,\mathrm{d}y \end{bmatrix}, \quad \mathscr{C}_{\phi_\lambda} = \begin{bmatrix} 1 - \displaystyle\int_{\Omega_\lambda} g_\mathrm{i}(y)\phi_\lambda(y)\,\mathrm{d}y & b_\mathrm{i} \\ -\displaystyle\int_{\Omega_\lambda} g_\mathrm{o}(y)\phi_\lambda(y)\,\mathrm{d}y & b_\mathrm{o} \end{bmatrix}. \tag{13}$$

(ii) *If λ, b_i, and b_o satisfy $\det(\mathscr{I} - \mathscr{R}_\lambda) = 0$ and $\det\mathscr{C}_{\psi_\lambda} = 0$, then* (2)–(3) *has infinitely many solutions.*

(iii) *If λ, b_i, and b_o satisfy $\det(\mathscr{I} - \mathscr{R}_\lambda) = 0$ and $\det\mathscr{C}_{\psi_\lambda} \neq 0$, then* (2)–(3) *has no solution.*

Having Proposition 2 at hand, we are in a position to state the main result.

Theorem 3. *Under the same assumptions as in Proposition 2, there exist $0 < \lambda_* < \lambda^*$ such that for each $\lambda \in (0,\lambda_*) \cup (\lambda^*,\infty)$, we have $\det(\mathscr{I} - \mathscr{R}_\lambda) > 0$, and* (2)–(3) *has a unique solution u_λ satisfying* (11). *Moreover, as $\lambda \gg 1$, there exist positive constants C^* and M^* independent of λ such that*

$$|u_\lambda(x)| \leq C^* \exp\left(-M^* \lambda^{-\frac{\kappa}{2}} \mathrm{dist}(x, \partial\Omega_\lambda)\right), \quad \text{for all} \quad x \in \overline{\Omega_\lambda}. \tag{14}$$

Figure 1 is a fundamental understanding of Theorem 3. In Section 3, we will prove Proposition 2 and Theorem 3.

Remark 4. (14) shows that for an interior point $x \in \Omega_\lambda$ satisfying

$$\lim_{\lambda \to \infty} \mathrm{dist}\left(\frac{x}{\lambda}, \partial\Omega\right) > 0,$$

there holds $\lim_{\lambda \to \infty} |u_\lambda(x)| = 0$ because for $\kappa \in (0,2)$,

$$|u_\lambda(x)| \leq C^* \exp\left(-M^* \lambda^{-\frac{\kappa}{2}} \mathrm{dist}(x, \partial\Omega_\lambda)\right) = C^* \exp\left(-M^* \lambda^{1-\frac{\kappa}{2}} \mathrm{dist}\left(\frac{x}{\lambda}, \partial\Omega\right)\right) \ll 1.$$

Remark 5 (The H^1-capacity of Ω_λ). Inspired by (11), which presents a linear combination of ϕ_λ and ψ_λ for solution u_λ, the H^1-capacity of Ω_λ is naturally considered since it is a relevant measure of the "size" for Ω_λ supporting Dirichlet boundary conditions (cf. [2, 14]). (Whereas notice the boundary conditions of ϕ_λ and ψ_λ in (8) and (9).) Recall the H^1-capacity of Ω_λ:

$$\mathrm{cap}(\Omega_\lambda) = \min\left\{\int_{\Omega_\lambda} |\nabla U|^2 \,\mathrm{d}x : U \in H^1(\Omega_\lambda), \quad U = 1 \text{ on } \Gamma_{\lambda,\mathrm{i}}, \quad U = 0 \text{ on } \Gamma_{\lambda,\mathrm{o}}\right\}, \tag{15}$$

where the minimum is achieved when U is harmonic in Ω_λ. In particular, Ω_λ has the property

$$\mathrm{cap}(\Omega_\lambda) = \lambda^{N-2}\mathrm{cap}(\Omega).$$

When $N = 2$, $\mathrm{cap}(\Omega_\lambda)$ is independent of λ. As a consequence, the effect of $\mathrm{cap}(\Omega_\lambda)$ on the existence and uniqueness of the solutions of the equation (2)–(3) appears insignificant. However, when $N \geq 3$, Theorem 3 can be reinterpreted that there exist $0 < L_* < L^*$ depending primarily on $\mathrm{cap}(\Omega)$ and dimension N such that if $\mathrm{cap}(\Omega_\lambda) \in (0, L_*) \cup (L^*, \infty)$, the equation (2)–(3) has a unique solution. For Ω_λ replaced with the general domains, the *domain capacity effect* on the uniqueness issue of (2)–(3) seems more subtle. The main difficulty, in our perspective, lies in obtaining the refined estimates (with respect to λ) of the non-local terms in (3). The general theory for this problem is highly nontrivial, which is a direction of our research in the future.

Theorem 3 focuses primarily on the uniqueness result of equation (2)–(3) in the annular-like domains Ω_λ defined by (1). We shall stress that the argument can straightforwardly be generalized to the same equation as (2) in a bounded domain with several boundary components (see Theorem 10 in Section 4).

1.2. *An example contrasting to Theorem 3*

In this section, we will emphasise that the property of the domain Ω_λ plays a critical role in the uniqueness result presented in Theorem 3.

Proposition 2 presents that the equation (2)–(3) has a unique solution if and only if $\mathscr{I} - \mathscr{R}_\lambda$ is invertible, where all elements in $\mathscr{R}_\lambda$ are associated with the integral-type boundary condition (3), which involves the domain Ω_λ. Note that the result of Proposition 2 still holds for general expanding domains because the property of the domain Ω_λ is not required in the proof (cf. Section 3.1). We emphasise that Theorem 3 holds for the domain Ω_λ defined by (1), but it may not hold for the general expanding domains. We would provide an example to show that Theorem 3 may not hold when the expanding domain has a fixed inner boundary that is independent of λ.

For a more detailed explanation, let us recall Example 1, where all domains $\mathbb{A}_{\lambda;2\lambda} = \{x \in \mathbb{R}^2 : \lambda < |x| < 2\lambda\}$ have the H^1-capacity $\mathrm{cap}(\mathbb{A}_{\lambda;2\lambda}) = \frac{2\pi}{\ln 2}$, which is independent of $\lambda > 0$ (see (15) and the footnote[2]). In this case, $\lambda = \lambda^*$ (see (6)) for the non-existence/non-uniqueness of the solutions of equation (5) depends mainly on the coefficient $\mathfrak{g}$ of the integral-type boundary condition. In the following Example 6, we adopt the same equation as in Example 1, but the domain $\mathbb{A}_{\lambda;2\lambda}$ is replaced by the annular domain $\mathbb{A}_{1;\lambda} := \{x \in \mathbb{R}^2 : 1 < |x| < \lambda\}$ with an inner boundary independent of λ. A difference between $\mathbb{A}_{\lambda;2\lambda}$ and $\mathbb{A}_{1;\lambda}$ with λ varying comes from the fact that $\mathrm{cap}(\mathbb{A}_{1;\lambda}) = \frac{2\pi}{\ln \lambda}$ depends on $\lambda > 1$. Because the inner boundary of $\mathbb{A}_{1;\lambda}$ is fixed, assumption (7) is not effective for $g = g_\mathrm{i}$ on $|x| = 1$. This observation prompts us to consider the boundary condition $u_\lambda = b_\mathrm{i} + \int_{\mathbb{A}_{1;\lambda}} g_\mathrm{i}(y) u_\lambda(y)\, \mathrm{d}y$ with a precise g_i (see (17)) on the inner boundary $|x| = 1$. We will show in **Case 2** (II) of Example 6 that when $b_\mathrm{i} = 0$ ($b_\mathrm{i} \neq 0$, respectively), there exists a strictly increasing sequence $\{\lambda_k\}_{k \in \mathbb{N}}$ with $\lambda_k \xrightarrow{k \to \infty} \infty$ such that the equation in the domain $\mathbb{A}_{1;\lambda_k}$ has infinitely many solutions (no solution, respectively).

Example 6. Let $N = 2$. For $\lambda > 1$, we consider the equation

$$\begin{cases} -\Delta u_\lambda(x) + \dfrac{x}{|x|^2} \cdot \nabla u_\lambda(x) + u_\lambda(x) = 0, & x \in \mathbb{A}_{1;\lambda}, \\[2mm] u_\lambda(x) = b_\mathrm{i} + \displaystyle\int_{\mathbb{A}_{1;\lambda}} g_\mathrm{i}(y) u_\lambda(y)\, \mathrm{d}y, & \text{for } |x| = 1; \qquad u_\lambda(x) = 0, \quad \text{for } |x| = \lambda, \end{cases} \tag{16}$$

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[2]We refer the readers to [2] (see also, (15) with $N = 2$) for the H^1-capacity of annular domains in $\mathbb{R}^2$. Let $\mathbb{A}_{R_1;R_2} := \{x \in \mathbb{R}^2 : R_1 < |x| < R_2\}$. Then, the H^1-capacity of $\mathbb{A}_{R_1;R_2}$ is $\mathrm{cap}(\mathbb{A}_{R_1;R_2}) = \frac{2\pi}{\ln(R_2/R_1)}$ which can be obtained when

$$U(x) = \frac{\ln R_2 - \ln |x|}{\ln R_2 - \ln R_1}, x \in \overline{\mathbb{A}_{R_1;R_2}}.$$

where

$$g_{\mathrm{i}}(y) = C_0 \frac{\mathrm{e}^{-|y|}}{|y|} \sin(|y|) \tag{17}$$

obeys the assumption (7), and C_0 is a constant.

We will apply Proposition 2 to equation (16) since its consequence still holds for equation (2)–(3) when the domain Ω_λ is replaced with $\mathbb{A}_{1;\lambda}$. First, one solves the corresponding equations (8) and (9) with $\vec{\mathrm{a}}(x) = \frac{x}{|x|^2}$ and $h(x) = 1$:

$$\phi_\lambda(x) = \frac{\mathrm{e}^{-|x|} - \mathrm{e}^{-2\lambda+|x|}}{\mathrm{e}^{-1} - \mathrm{e}^{-2\lambda+1}}, \quad \psi_\lambda(x) = \frac{\mathrm{e}^{|x|} - \mathrm{e}^{2-|x|}}{\mathrm{e}^\lambda - \mathrm{e}^{2-\lambda}}. \tag{18}$$

Next, we shall determine a constant C_0 in (17) so that Proposition 2 (ii) or (iii) occurs for "infinitely many" λ. By (18), a simple calculation yields

$$
\begin{aligned}
\int_{\mathbb{A}_{1;\lambda}} & g_{\mathrm{i}}(y)\phi_\lambda(y)\,\mathrm{d}y \\
&= \frac{2\pi C_0}{\mathrm{e}^{-1} - \mathrm{e}^{-2\lambda+1}} \int_1^\lambda \left(\mathrm{e}^{-r} - \mathrm{e}^{-2\lambda+r}\right)\mathrm{e}^{-r}\sin r\,\mathrm{d}r \\
&= \frac{2\pi C_0}{\mathrm{e}^{-1} - \mathrm{e}^{-2\lambda+1}} \left[\frac{\mathrm{e}^{-2}}{5}(2\sin 1 + \cos 1) - \mathrm{e}^{-2\lambda}\left(\cos 1 - \cos\lambda + \frac{1}{5}(2\sin\lambda + \cos\lambda)\right)\right],
\end{aligned}
\tag{19}
$$

which implies

$$\lim_{\lambda\to\infty}\int_{\mathbb{A}_{1;\lambda}} g_{\mathrm{i}}(y)\phi_\lambda(y)\,\mathrm{d}y = \frac{2\pi C_0}{5\mathrm{e}}(2\sin 1 + \cos 1). \tag{20}$$

On the other hand, by (12) with $\Omega_\lambda = \mathbb{A}_{1;\lambda}$ and $g_{\mathrm{o}} = 0$, we have

$$\det(\mathscr{I} - \mathscr{R}_\lambda) = 1 - \int_{\mathbb{A}_{1;\lambda}} g_{\mathrm{i}}(y)\phi_\lambda(y)\,\mathrm{d}y. \tag{21}$$

As a consequence, we have the following results:

(**Case 1**) If $C_0 \neq \frac{5\mathrm{e}}{2\pi(2\sin 1+\cos 1)}$, then by (20)–(21), we have $\det(\mathscr{I} - \mathscr{R}_\lambda) \neq 0$ as $\lambda \gg 1$. Applying Proposition 2 (i) to this case, we verify that as $\lambda \gg 1$, (16) has a unique solution.

(**Case 2**) If $C_0 = \frac{5\mathrm{e}}{2\pi(2\sin 1+\cos 1)}$, then by (19) and (21), we obtain that (see the Appendix A):

$$\left\{\lambda > 1 : \det(\mathscr{I} - \mathscr{R}_\lambda) = 0\right\} = \left\{2k\pi + 1, (2k-1)\pi + 2\theta_0 - 1 : k \in \mathbb{N}\right\} := \mathsf{S}_0, \tag{22}$$

where $\theta_0 = \arcsin\frac{2}{\sqrt{5}} \in (1, \frac{\pi}{2})$. Thus, by Proposition 2 (i)–(iii) and (21)–(22), we have the following results:

   (I) If $\lambda \notin \mathsf{S}_0$, then $\mathscr{I} - \mathscr{R}_\lambda$ is invertible. Therefore, equation (16) has a unique solution.
   (II) If $\lambda \in \mathsf{S}_0$, then for $b_{\mathrm{i}} \neq 0$, (16) has no solution; for $b_{\mathrm{i}} = 0$, (16) has infinitely many solutions $u_\lambda = c_*\phi_\lambda$, where ϕ_λ is defined by (18) and $c_* \in \mathbb{R}$ is arbitrary.

The rest of this paper is structured as follows. In Section 2, we establish estimates of ϕ_λ and ψ_λ in $\overline{\Omega_\lambda}$ for $\lambda \gg 1$, which is crucial for the proof of Theorem 3. In Section 3, we state the proof of Proposition 2 and Theorem 3 and provide a remark for the uniqueness under an approach of the maximum principle. Next, as a generalization of Theorem 3, in Section 4, we consider equation (2) in an expanding domain with numerous boundary components. Under the corresponding integral-type boundary conditions, we establish the uniqueness result which will be introduced in Theorem 10. We make the concluding remarks and future problems in Section 5. Finally, in the Appendix we state the proof of (22) for the sake of clarity and completeness.

2. Preliminary estimates of ϕ_λ and ψ_λ

In this section, we first establish the required estimates of ϕ_λ and ψ_λ for $\lambda \gg 1$, where ϕ_λ and ψ_λ are solutions of equations (8) and (9), respectively. Let $(\Phi_\lambda(z), \Psi_\lambda(z)) = (\phi_\lambda(x), \psi_\lambda(x))$ with $z = \lambda^{-1} x \in \Omega$. Then $\Phi_\lambda(z)$ and $\Psi_\lambda(z)$ satisfy

$$\begin{cases} -\dfrac{1}{\lambda^2}\Delta_z \Phi_\lambda(z) + \dfrac{1}{\lambda}\vec{\mathfrak{a}}(\lambda z)\cdot \nabla_z \Phi_\lambda(z) + h(\lambda z)\Phi_\lambda(z) = 0 & \text{in} \quad \Omega, \\ \Phi_\lambda = 1 \quad \text{on} \quad \partial\Omega_{\text{in}}, \qquad\qquad\qquad\quad \Phi_\lambda = 0 \quad \text{on} \quad \partial\Omega_{\text{out}}, \end{cases} \tag{23}$$

and

$$\begin{cases} -\dfrac{1}{\lambda^2}\Delta_z \Psi_\lambda(z) + \dfrac{1}{\lambda}\vec{\mathfrak{a}}(\lambda z)\cdot \nabla_z \Psi_\lambda(z) + h(\lambda z)\Psi_\lambda(z) = 0 & \text{in} \quad \Omega, \\ \Psi_\lambda = 0 \quad \text{on} \quad \partial\Omega_{\text{in}}, \qquad\qquad\qquad\quad \Psi_\lambda = 1 \quad \text{on} \quad \partial\Omega_{\text{out}}, \end{cases} \tag{24}$$

respectively. Since $h > 0$ (by (4)), the maximum principle implies $0 \le \Phi_\lambda, \Psi_\lambda \le 1$ in Ω.

We first establish an interior estimate for Φ_λ. Multiplying both sides of the equation in (23) by Φ_λ, one may check from (4) that, if $\lambda > 0$ is sufficiently large, there exists a positive constant M independent of λ such that

$$\frac{1}{\lambda^2}\Delta_z \Phi_\lambda^2 = \frac{2}{\lambda^2}|\nabla_z \Phi_\lambda|^2 + \frac{2}{\lambda}\vec{\mathfrak{a}}(\lambda z)\Phi_\lambda \cdot \nabla_z \Phi_\lambda + 2h(\lambda z)\Phi_\lambda^2$$

$$\ge 2\left(h(\lambda z) - \frac{|\vec{\mathfrak{a}}(\lambda z)|^2}{4}\right)\Phi_\lambda^2 \ge \lambda^{-\kappa}M^2 \Phi_\lambda^2,$$

i.e.,

$$\Delta_z \Phi_\lambda^2 \ge \lambda^{2-\kappa}M^2 \Phi_\lambda^2 \qquad \text{in} \quad \Omega, \tag{25}$$

where we notice $\lambda^{2-\kappa} \gg 1$ since $0 < \kappa < 2$.

Let $\mu_1 > 0$ be the principal eigenvalue of $-\Delta_z$ in $H_0^1(\Omega)$, and let φ_1 be the corresponding positive eigenfunction with $\|\varphi_1\|_{L^\infty(\Omega)} = 1$. Consider the following auxiliary function:

$$\mathcal{V}_\lambda(z) := \exp\left(-M_1 \lambda^{1-\frac{\kappa}{2}} \varphi_1(z)\right) \quad \text{in} \quad \overline{\Omega}. \tag{26}$$

We will determine a positive constant M_1 such that (26) is a superposition of (25). First, it is easy to check that $\mathcal{V}_\lambda$ satisfies

$$\Delta_z \mathcal{V}_\lambda = \lambda^{2-\kappa}\left(M_1 \lambda^{\frac{\kappa}{2}-1}\mu_1 \varphi_1 + M_1^2 |\nabla\varphi_1|^2\right)\mathcal{V}_\lambda.$$

Since $0 < \lambda^{\frac{\kappa}{2}-1} \ll 1$, and $|\nabla\varphi_1|$ is bounded in $\overline{\Omega}$ (cf. [10]), one can choose a suitable $M_1 > 0$ such that $\Delta_z \mathcal{V}_\lambda \le \lambda^{2-\kappa}M^2 \mathcal{V}_\lambda$ in Ω for sufficiently large λ. As a consequence, by combining this differential inequality with (25), we arrive at

$$\Delta_z\left(\Phi_\lambda^2 - \mathcal{V}_\lambda\right) \ge \lambda^{2-\kappa}M^2\left(\Phi_\lambda^2 - \mathcal{V}_\lambda\right) \quad \text{in} \quad \Omega. \tag{27}$$

Note also that $\Phi_\lambda^2 \le 1 = \mathcal{V}_\lambda$ on $\partial\Omega$, i.e.

$$\Phi_\lambda^2 - \mathcal{V}_\lambda \le 0 \quad \text{on} \quad \partial\Omega. \tag{28}$$

Applying the standard comparison theorem to (27)–(28), we obtain

$$\Phi_\lambda^2(z) \le \mathcal{V}_\lambda(z), \quad \text{for all} \quad z \in \overline{\Omega}. \tag{29}$$

Now we claim that there exists a positive constant M^* independent of λ such that

$$\mathcal{V}_\lambda(z) \le \exp\left(-M^* \lambda^{1-\frac{\kappa}{2}} \text{dist}(z, \partial\Omega)\right), \quad \text{for} \quad z \in \Omega. \tag{30}$$

Since $\partial\Omega$ is smooth, the eigenfunction φ_1 is both bounded above and below by strictly positive multiples of $\text{dist}(x, \partial\Omega)$ (see, e.g. [6, Section 3]). Here, for the sake of completeness, we offer an alternative proof for a required lower bound of φ_1 as follows. Since φ_1 is smooth and strictly positive in Ω and $\varphi_1 = 0$ on $\partial\Omega$, by the Hopf's lemma, the outward normal derivative of φ_1 on $\partial\Omega$

is strictly negative, implying that $\varphi_1(x) \geq M_2 \operatorname{dist}(x, \partial\Omega)$ in $\overline{\Omega_{d_2}} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \leq d_2\}$, for a small $d_2 > 0$ and a positive constant M_2 depending on d_2,

$$\|\varphi_1\|_{C^2\left(\overline{\Omega_{d_2}}\right)} \quad \text{and} \quad \min_{\partial\Omega}\left|\partial_{\vec{n}}\varphi_1\right|.$$

As a consequence,

$$\inf_{z\in\Omega} \frac{\varphi_1(z)}{\operatorname{dist}(z, \partial\Omega)} \geq \min_{\Omega\setminus\Omega_{d_2}}\left\{M_2, \frac{\varphi_1}{\operatorname{diam}(\Omega)}\right\} > 0.$$

Along with (26), we arrive at (30) with

$$M^* = \frac{1}{2}M_1 \min_{\Omega\setminus\Omega_{d_2}}\left\{M_2, \frac{\varphi_1}{\operatorname{diam}(\Omega)}\right\}.$$

By (29) and (30), we have

$$0 \leq \Phi_\lambda(z) \leq \exp\left(-M^*\lambda^{1-\frac{\kappa}{2}}\operatorname{dist}(z, \partial\Omega)\right), \ \forall\ z \in \overline{\Omega}.$$

That is,

$$0 \leq \phi_\lambda(x) \leq \exp\left(-M^*\lambda^{-\frac{\kappa}{2}}\operatorname{dist}(x, \partial\Omega_\lambda)\right), \quad \text{for all } x \in \overline{\Omega_\lambda}. \tag{31}$$

Similarly, by (4), (24) and the fact that $0 \leq \psi_\lambda \leq 1$ on the boundary, we also have

$$0 \leq \psi_\lambda(x) \leq \exp\left(-M^*\lambda^{-\frac{\kappa}{2}}\operatorname{dist}(x, \partial\Omega_\lambda)\right), \quad \text{for all } x \in \overline{\Omega_\lambda}. \tag{32}$$

Estimates (31) and (32) play a crucial role in the proof of Theorem 3.

Remark 7. An alternative proof of (31) and (32) can be found in [11, Proposition 2]; see also [11, (2.30)].

3. Proofs of the main results

3.1. *Proof of Proposition 2*

Proof. For the sake of simplicity, we let $B_{\lambda,i} = b_i + \int_{\Omega_\lambda} g_i(y)u_\lambda(y)\,dy$ and $B_{\lambda,o} = b_o + \int_{\Omega_\lambda} g_o(y)u_\lambda(y)\,dy$. By (3) and (10), we have

$$\begin{cases} B_{\lambda,i} = b_i + \displaystyle\int_{\Omega_\lambda} g_i(y)\left(B_{\lambda,i}\phi_\lambda(y) + B_{\lambda,o}\psi_\lambda(y)\right)dy, \\ B_{\lambda,o} = b_o + \displaystyle\int_{\Omega_\lambda} g_o(y)\left(B_{\lambda,i}\phi_\lambda(y) + B_{\lambda,o}\psi_\lambda(y)\right)dy, \end{cases} \tag{33}$$

which is equivalent to

$$(\mathscr{I} - \mathscr{R}_\lambda)\begin{bmatrix} B_{\lambda,i} \\ B_{\lambda,o} \end{bmatrix} = \begin{bmatrix} b_i \\ b_o \end{bmatrix}, \tag{34}$$

where $\mathscr{I}$ and $\mathscr{R}_\lambda$ have been defined by (12).

Accordingly, the existence, uniqueness, and multiplicity of the solutions to (2)–(3) are determined by that of $(B_{\lambda,i}, B_{\lambda,o})$ to (34). If $\det(\mathscr{I} - \mathscr{R}_\lambda) \neq 0$, then by applying Cramer's rule to (34), we get a unique solution

$$B_{\lambda,i} = \det\left((\mathscr{I} - \mathscr{R}_\lambda)^{-1}\mathscr{C}_{\psi_\lambda}\right), \qquad B_{\lambda,o} = \det\left((\mathscr{I} - \mathscr{R}_\lambda)^{-1}\mathscr{C}_{\phi_\lambda}\right), \tag{35}$$

where $\mathscr{C}_{\psi_\lambda}$ and $\mathscr{C}_{\phi_\lambda}$ have been defined by (13). Thus, $\det(\mathscr{I} - \mathscr{R}_\lambda) \neq 0$ implies that (2)–(3) has a unique solution u_λ satisfying (11). This completes the proof of (i). Similarly, we can prove (ii) and (iii) and complete the proof of Proposition 2. $\qquad\square$

The following corollary is a direct application of Proposition 2.

Corollary 8. *Under the same assumptions as in Proposition 2, we assume that $\lambda > 0$ satisfies*

$$\int_{\Omega_\lambda} \left(\left| g_i(y) \right| + \left| g_o(y) \right| \right) dy \leq 1. \tag{36}$$

Then, $\det(\mathscr{I} - \mathscr{R}_\lambda) > 0$. *In particular, equation* (2)–(3) *has a unique solution u_λ satisfying* (11).

Proof. By (12), we have

$$\det(\mathscr{I} - \mathscr{R}_\lambda) = \left(1 - \int_{\Omega_\lambda} g_i(y)\phi_\lambda(y)\,dy \right)\left(1 - \int_{\Omega_\lambda} g_o(y)\psi_\lambda(y)\,dy \right)$$
$$- \int_{\Omega_\lambda} g_i(y)\psi_\lambda(y)\,dy \int_{\Omega_\lambda} g_o(y)\phi_\lambda(y)\,dy. \tag{37}$$

Let $\int_{\Omega_\lambda} |g_i(y)|\,dy = A_{\lambda,i}$ and $\int_{\Omega_\lambda} |g_o(y)|\,dy = A_{\lambda,o}$. Then, by (36), we have $A_{\lambda,i} + A_{\lambda,o} \leq 1$. Because $0 \leq \phi_\lambda, \psi_\lambda \leq 1$, $\phi_\lambda \not\equiv 1$ and $\psi_\lambda \not\equiv 1$ in $\overline{\Omega_\lambda}$, we have $\int_{\Omega_\lambda} |g_i(y)|\phi_\lambda(y)\,dy < A_{\lambda,i}$, $\int_{\Omega_\lambda} |g_i(y)|\psi_\lambda(y)\,dy < A_{\lambda,i}$, $\int_{\Omega_\lambda} |g_o(y)|\phi_\lambda(y)\,dy < A_{\lambda,o}$ and $\int_{\Omega_\lambda} |g_o(y)|\psi_\lambda(y)\,dy < A_{\lambda,o}$. Thus, $1 - \int_{\Omega_\lambda} g_i(y)\phi_\lambda(y)\,dy > 1 - A_{\lambda,i} \geq 0$ and $1 - \int_{\Omega_\lambda} g_o(y)\psi_\lambda(y)\,dy > 1 - A_{\lambda,o} \geq 0$. Along with (37), we arrive at

$$\det(\mathscr{I} - \mathscr{R}_\lambda) > \left(1 - A_{\lambda,i} \right)\left(1 - A_{\lambda,o} \right) - A_{\lambda,i} A_{\lambda,o} = 1 - A_{\lambda,i} - A_{\lambda,o} \geq 0,$$

implying that $\mathscr{I} - \mathscr{R}_\lambda$ is invertible. Along with Proposition 2 (i), we obtain the uniqueness of the solution u_λ of (2)–(3) when (36) holds. Therefore, the proof of Corollary 8 is complete. $\qquad \square$

Finally, let us consider the case that g_i and g_o are non-negative. Applying the technique of the maximum principle, we have the following result.

Corollary 9. *Under the same assumptions as in Proposition 2, we assume that both g_i and g_o are non-negative, and $\lambda > 0$ satisfies*

$$\int_{\Omega_\lambda} g_i(y)\,dy < 1 \qquad and \qquad \int_{\Omega_\lambda} g_o(y)\,dy < 1. \tag{38}$$

Then, one of the following statements holds:

(S1) $\det(\mathscr{I} - \mathscr{R}_\lambda) \neq 0$ *and* (2)–(3) *has a unique solution;*
(S2) $\det(\mathscr{I} - \mathscr{R}_\lambda) = 0$ *and* (2)–(3) *has no solution.*

Proof. We fix $\lambda > 0$ satisfying (38). First, we prove that the equation (2)–(3) has at most one solution. Suppose by contradiction that (2)–(3) has at least two distinct solutions U_1 and U_2. For $U := U_1 - U_2$, we have

$$-\Delta U(x) + \vec{\mathsf{a}}(x) \cdot \nabla U(x) + h(x)U(x) = 0 \quad \text{in} \quad \Omega_\lambda, \tag{39}$$

where U is subject to the integral-type boundary conditions

$$\begin{cases} U(x) = \displaystyle\int_{\Omega_\lambda} g_i(y)U(y)\,dy, & x \in \Gamma_{\lambda,i} := \left\{ \lambda z : z \in \partial\Omega_{\text{in}} \right\}, \\[2mm] U(x) = \displaystyle\int_{\Omega_\lambda} g_o(y)U(y)\,dy, & x \in \Gamma_{\lambda,o} := \left\{ \lambda z : z \in \partial\Omega_{\text{out}} \right\}. \end{cases} \tag{40}$$

To obtain a contradiction, we consider two situations without loss of generality.

(**Case 1**) U attains both its maximum and minimum values at the boundary points. Since g_i and g_o are both non-negative, by (38) and (40), one immediately obtains

$$\max_{\Omega_\lambda} U \leq 0 \leq \min_{\Omega_\lambda} U.$$

Hence, $U \equiv 0$, implying that $U_1 = U_2$, a contradiction.

(**Case 2**) U attains its maximum value at an interior point x_λ and its minimum value on the boundary. Then, by (4) and (39), we have $\max_{\Omega_\lambda} U = U(x_\lambda) \le 0$. On the other hand,

$$\min_{\Omega_\lambda} U \ge 0$$

holds trivially due to (38). This also implies $U_1 = U_2$ and leads to a contradiction.

Hence, by (**Case 1**)–(**Case 2**), equations (2)–(3) has at most one solution. Furthermore, if $\lambda > 0$ satisfies $\det(\mathscr{I} - \mathscr{R}_\lambda) \ne 0$, then by Proposition 2 (i), equation (2)–(3) has a unique solution, i.e. (S1) holds. If $\lambda > 0$ satisfies $\det(\mathscr{I} - \mathscr{R}_\lambda) = 0$, then by Proposition 2 (iii), there holds $\det \mathscr{C}_{\Psi_\lambda} \ne 0$, and the equation (2)–(3) has no solution, i.e., (S2) holds. Therefore, the proof of Corollary 9 is completed. $\qquad\square$

3.2. *Proof of Theorem 3*

Proof. First, we show that there exists $\lambda_* > 0$ such that $\det(\mathscr{I} - \mathscr{R}_\lambda) > 0$ for $\lambda \in (0, \lambda_*)$. Recall $\Omega := \Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}$. One can check that

$$\int_{\Omega_\lambda} \left(\left| g_{\text{i}}(y) \right| + \left| g_{\text{o}}(y) \right| \right) \mathrm{d}y = \lambda^N \int_{\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}} \left(\left| g_{\text{i}}(\lambda z) \right| + \left| g_{\text{o}}(\lambda z) \right| \right) \mathrm{d}z$$
$$\le \frac{1}{(\mathrm{dist}\,(0, \partial\Omega_{\text{in}}))^N} \int_{\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}} |\lambda z|^N \left(\left| g_{\text{i}}(\lambda z) \right| + \left| g_{\text{o}}(\lambda z) \right| \right) \mathrm{d}z. \tag{41}$$

Here we have verified $0 < \mathrm{dist}(0, \partial\Omega_{\text{in}}) \le \min_{z \in \overline{\Omega_{\text{out}}} \setminus \Omega_{\text{in}}} |z|$ since $0 \in \Omega_{\text{in}}$. Note also that

$$\lim_{\lambda \to 0+} \max_{z \in \overline{\Omega_{\text{out}}} \setminus \Omega_{\text{in}}} |\lambda z| = 0$$

since $\Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}$ is a bounded domain. Thus, by (7) and (41), we have

$$\lim_{\lambda \to 0+} \int_{\Omega_\lambda} \left(\left| g_{\text{i}}(y) \right| + \left| g_{\text{o}}(y) \right| \right) \mathrm{d}y = 0,$$

asserting that there exists $\lambda_* > 0$ such that (36) holds for $\lambda \in (0, \lambda_*)$. Then, by Corollary 8, we obtain $\det(\mathscr{I} - \mathscr{R}_\lambda) > 0$ for $\lambda \in (0, \lambda_*)$.

On the other hand, by (31)–(32), one may check that, as $\lambda \gg 1$,

$$\left| \int_{\Omega_\lambda} g_{\text{i}}(y) \phi_\lambda(y) \, \mathrm{d}y \right| \le \max_{\overline{\Omega_\lambda}} \left| g_{\text{i}} \right| \left(\int_{\Omega_\lambda} \exp\left(-M^* \lambda^{-\frac{\kappa}{2}} \mathrm{dist}(y, \partial\Omega_\lambda) \right) \mathrm{d}y \right)$$
$$\le \lambda^N \max_{z \in \overline{\Omega}} \left| g_{\text{i}}(\lambda z) \right| \left(\int_\Omega \exp\left(-M^* \lambda^{1-\frac{\kappa}{2}} \mathrm{dist}(z, \partial\Omega) \right) \mathrm{d}z \right). \tag{42}$$

Furthermore, we carefully deal with the last term of (42). In order to achieve a more refined estimate, we set a constant $\tau \in (0, 1 - \frac{\kappa}{2})$ independent of λ such that as $\lambda > 0$ is sufficiently large, the subdomain

$$\Omega_{\lambda^{-\tau}}^* := \left\{ z \in \Omega : \mathrm{dist}(z, \partial\Omega) > \lambda^{-\tau} \right\}$$

is nonempty, and each interior point of $\Omega \setminus \overline{\Omega_{\lambda^{-\tau}}^*}$ has a unique nearest point on $\partial\Omega$. Note that $\partial\Omega_{\lambda^{-\tau}}^*$ is smooth and

$$\max_{z \in \overline{\Omega_{\lambda^{-\tau}}^*}} \exp\left(-M^* \lambda^{1-\frac{\kappa}{2}} \mathrm{dist}(z, \partial\Omega) \right) \le \exp\left(-M^* \lambda^{1-\frac{\kappa}{2}-\tau} \right).$$

Utilizing the coarea formula (cf. [7, Theorems 3.10 and 3.14]), one may check that, as $\lambda \gg 1$,

$$\int_\Omega \exp\left(-M^* \lambda^{1-\frac{\kappa}{2}} \operatorname{dist}(z, \partial\Omega)\right) \mathrm{d}z$$

$$= \int_0^{\lambda^{-\tau}} \exp\left(-M^* \lambda^{1-\frac{\kappa}{2}} t\right) \mathscr{H}^{N-1}\left(\{z \in \Omega : \operatorname{dist}(z, \partial\Omega_{\mathrm{in}}) = t\}\right) \mathrm{d}t$$

$$\quad + \int_0^{\lambda^{-\tau}} \exp\left(-M^* \lambda^{1-\frac{\kappa}{2}} t\right) \mathscr{H}^{N-1}\left(\{z \in \Omega : \operatorname{dist}(z, \partial\Omega_{\mathrm{out}}) = t\}\right) \mathrm{d}t$$

$$\quad + \int_{\Omega^*_{\lambda^{-\tau}}} \exp\left(-M^* \lambda^{1-\frac{\kappa}{2}} \operatorname{dist}(z, \partial\Omega)\right) \mathrm{d}z \tag{43}$$

$$\leq 2\left(\mathscr{H}^{N-1}(\partial\Omega_{\mathrm{in}}) + \mathscr{H}^{N-1}(\partial\Omega_{\mathrm{out}})\right) \int_0^{\lambda^{-\tau}} \exp\left(-M^* \lambda^{1-\frac{\kappa}{2}} t\right) \mathrm{d}t$$

$$\quad + \mathscr{H}^N(\Omega) \exp\left(-M^* \lambda^{1-\frac{\kappa}{2}-\tau}\right)$$

$$\leq \frac{2\lambda^{\frac{\kappa}{2}-1}}{M^*}\left(\mathscr{H}^{N-1}(\partial\Omega_{\mathrm{in}}) + \mathscr{H}^{N-1}(\partial\Omega_{\mathrm{out}})\right) + \mathscr{H}^N(\Omega) \exp\left(-M^* \lambda^{1-\frac{\kappa}{2}-\tau}\right) \leq C_3 \lambda^{\frac{\kappa}{2}-1},$$

where $\mathscr{H}^d$ represents the d-dimensional Hausdorff measure on $\mathbb{R}^N$, and C_3 is a positive constant independent of λ. Since $0 < \lambda^{-\tau} \ll 1$, in (43) we have used

$$\exp\left(-M^* \lambda^{1-\frac{\kappa}{2}-\tau}\right) \ll \lambda^{\frac{\kappa}{2}-1}$$

(as $\lambda \gg 1$), and the fact that

$$\mathscr{H}^{N-1}\left(\{z \in \Omega : \operatorname{dist}(z, \partial\Omega_{\mathrm{in}}) = t\}\right) \leq 2\mathscr{H}^{N-1}(\partial\Omega_{\mathrm{in}})$$

and

$$\mathscr{H}^{N-1}\left(\{z \in \Omega : \operatorname{dist}(z, \partial\Omega_{\mathrm{out}}) = t\}\right) \leq 2\mathscr{H}^{N-1}(\partial\Omega_{\mathrm{out}})$$

as $0 < t \leq \lambda^{-\tau} \ll 1$. Hence, by (7) and (42)–(43), one arrives at

$$\left|\int_{\Omega_\lambda} g_{\mathrm{i}}(y)\phi_\lambda(y)\,\mathrm{d}y\right| \leq C_3 \lambda^{N+\frac{\kappa-2}{2}} \max_{z \in \overline{\Omega}}|g_{\mathrm{i}}(\lambda z)| \xrightarrow{\lambda \to \infty} 0. \tag{44}$$

Similarly, we can obtain

$$\int_{\Omega_\lambda} g_{\mathrm{o}}(y)\psi_\lambda(y)\,\mathrm{d}y, \int_{\Omega_\lambda} g_{\mathrm{i}}(y)\psi_\lambda(y)\,\mathrm{d}y, \int_{\Omega_\lambda} g_{\mathrm{o}}(y)\phi_\lambda(y)\,\mathrm{d}y \xrightarrow{\lambda \to \infty} 0. \tag{45}$$

As a consequence, by (12)–(13), (37) and (44)–(45), we have

$$\det(\mathscr{I} - \mathscr{R}_\lambda) \xrightarrow{\lambda \to \infty} 1, \quad \det\mathscr{C}_{\psi_\lambda} \xrightarrow{\lambda \to \infty} b_{\mathrm{i}}, \quad \det\mathscr{C}_{\phi_\lambda} \xrightarrow{\lambda \to \infty} b_{\mathrm{o}}. \tag{46}$$

Therefore, by (33) and (46), there exists a sufficiently large number λ^* such that $\det(\mathscr{I} - \mathscr{R}_\lambda) > 0$ for $\lambda \in (\lambda^*, \infty)$. Consequently,

$$\det(\mathscr{I} - \mathscr{R}_\lambda) > 0 \qquad \text{as} \quad \lambda \in (0, \lambda_*) \cup (\lambda^*, \infty). \tag{47}$$

By (47) and Proposition 2 (i), we obtain the uniqueness of solutions to (2)–(3) with $\lambda \in (0, \lambda_*) \cup (\lambda^*, \infty)$.

It remains to prove (14). By (35) and (46), we have $|B_{\lambda,\mathrm{i}}| + |B_{\lambda,\mathrm{o}}| \leq 2(b_{\mathrm{i}} + b_{\mathrm{o}})$ for sufficiently large $\lambda > 0$. On the other hand, as $\lambda \gg 1$, by (11) and (31)–(32), we arrive at an interior estimate of u_λ:

$$|u_\lambda(x)| \leq |B_{\lambda,\mathrm{i}}|\phi_\lambda(x) + |B_{\lambda,\mathrm{o}}|\psi_\lambda(x) \leq (|B_{\lambda,\mathrm{i}}| + |B_{\lambda,\mathrm{o}}|)\exp\left(-M^* \lambda^{-\frac{\kappa}{2}} \operatorname{dist}(x, \partial\Omega_\lambda)\right), \quad \forall\, x \in \overline{\Omega_\lambda}.$$

Therefore, we obtain (14) with $C^* = 2(b_{\mathrm{i}} + b_{\mathrm{o}})$ and complete the proof of Theorem 3. $\qquad\qquad \square$
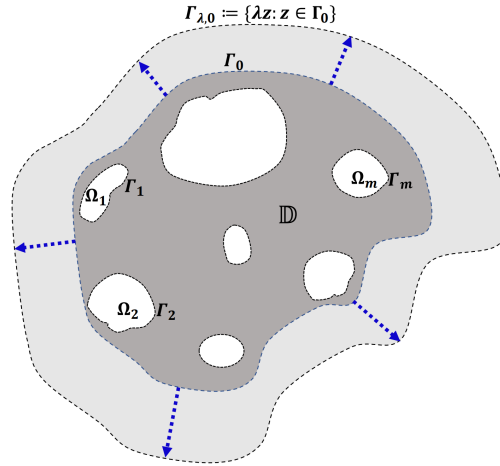
Figure 2. The dark grey region shows a domain $\mathbb{D} := \Omega_0 \setminus \bigcup_{1 \le k \le m} \overline{\Omega_k}$ with many boundary components Γ_j's, where $\Gamma_j := \partial \Omega_j$, $j = 0, 1, \ldots, m$. Here, Ω_0 and $\Omega_k \Subset \Omega_0$ are bounded smooth domains in $\mathbb{R}^N$, $N \ge 2$, and all $\overline{\Omega_k}$'s are disjoint to each other. Furthermore, we assume $0 \notin \overline{\mathbb{D}}$. The corresponding expanding domain is defined by $\mathbb{D}_\lambda := \{\lambda z : z \in \mathbb{D}\}$ with $m + 1$ boundary components $\Gamma_{\lambda,j}$'s.

4. A uniqueness result in the expanding domain with many boundary components

Let $\mathbb{D}_\lambda$ be a bounded domain with $m + 1$ smooth boundary components Γ_j's, $j = 0, 1, \ldots, m$. For the definitions of $\mathbb{D}_\lambda$ and $\Gamma_{\lambda,j}$'s, we refer the readers to the caption of Figure 2. Note that $\vec{a}$ and h are well-defined in $\overline{\mathbb{D}_\lambda}$ since $0 \notin \overline{\mathbb{D}_\lambda}$. It suffices to consider the equation

$$-\Delta v_\lambda(x) + \vec{a}(x) \cdot \nabla v_\lambda(x) + h(x) v_\lambda(x) = 0 \quad \text{in} \quad \mathbb{D}_\lambda, \tag{48}$$

with the integral-type boundary conditions

$$v_\lambda(x) = b_j + \int_{\mathbb{D}_\lambda} g_j(y) v_\lambda(y) \, dy, \quad x \in \Gamma_{\lambda,j} := \{\lambda z : z \in \Gamma_j\}, \quad j = 0, 1, \ldots, m, \tag{49}$$

where $b_j \in \mathbb{R}$ and $g_j \in C(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ are independent of λ. Here, we will focus primarily on the uniqueness problem of (48)–(49).

Because there are $m + 1$ boundary components $\Gamma_{\lambda,j}$'s, for each $\eta \in \{0, 1, \ldots, m\}$, let $\phi_{\lambda,\eta}$ be the unique solution of the equation

$$\begin{cases} -\Delta \phi_{\lambda,\eta}(x) + \vec{a}(x) \cdot \nabla \phi_{\lambda,\eta}(x) + h(x) \phi_{\lambda,\eta}(x) = 0 \quad \text{in} \quad \mathbb{D}_\lambda, \\ \phi_{\lambda,\eta}(x) = 1, \quad x \in \Gamma_{\lambda,\eta}, \qquad \phi_{\lambda,\eta}(x) = 0, \ x \in \Gamma_{\lambda,j} \quad (j \ne \eta). \end{cases} \tag{50}$$

Owing to (4), ϕ_{λ,η_1} and ϕ_{λ,η_2} are linearly independent ($\eta_1 \ne \eta_2$). Hence, each solution v_λ of (48)–(49) (if it exists) is a linear combination of all $\phi_{\lambda,\eta}$'s with the coefficients determined by (49). Therefore, we can apply a similar argument as in the proofs of Proposition 2 and Theorem 3 to arrive at the following result.

Theorem 10. *Assume that b_j's, $j = 0, 1, \ldots, m$, are constants independent of λ, $h \in C^\alpha(\mathbb{R}^N; \mathbb{R})$ and $\vec{a} \in C^\alpha(\mathbb{R}^N; \mathbb{R}^N)$ satisfying (4). Let $g_j \in C(\mathbb{R}^N; \mathbb{R})$ satisfy (7) with $g = g_j$. Then, there hold the followings:*

(i) *Equation* (48)*–*(49) *has a unique solution if and only if $\lambda > 0$ satisfies*

$$\det(\mathscr{I}_{m+1} - \mathscr{G}_\lambda) \ne 0,$$

where $\mathscr{I}_{m+1}$ is the $(m+1) \times (m+1)$ identity matrix, and

$$\mathscr{G}_\lambda = \big[\mathsf{G}_{\lambda,ij}\big]_{(m+1)\times(m+1)} \quad with \quad \mathsf{G}_{\lambda,ij} = \int_{\mathbb{D}_\lambda} g_{i-1}(y)\phi_{\lambda,j-1}(y)\,\mathrm{d}y, \ i,j \in \{1,\ldots,m+1\}.$$

(ii) *There exist* $\widetilde{\lambda}_*$ *and* $\widetilde{\lambda}^*$ *with* $0 < \widetilde{\lambda}_* < \widetilde{\lambda}^*$ *such that for* $\lambda \in (0,\widetilde{\lambda}_*) \cup (\widetilde{\lambda}^*,\infty)$, *(48)–(49) has a unique solution* v_λ *satisfying*

$$v_\lambda(x) = \sum_{\eta=0}^{m} c_\eta \phi_{\lambda,\eta}(x),$$

where $\phi_{\lambda,\eta}$ *is the unique solution of* (50), *and*

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_m \end{bmatrix} = (\mathscr{I}_{m+1} - \mathscr{G}_\lambda)^{-1} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

Proof. The proof of (i) follows the same argument as in the proof of Proposition 2 (i). We omit the detail here because of its simplicity.

Now we state the proof of (ii). Applying a similar argument as (31) to (50), we can establish the following estimate for each $\phi_{\lambda,\eta}$:

$$0 \le \phi_{\lambda,\eta}(x) \le \exp\left(-M^* \lambda^{-\frac{\kappa}{2}} \mathrm{dist}\,(x,\partial\mathbb{D}_\lambda)\right), \quad \text{for all } x \in \overline{\mathbb{D}_\lambda}. \tag{51}$$

Moreover, by (7) with $g = g_{i-1}$ and (51) with $\eta = j-1$, we can use the same arguments as (42)–(43) to obtain $\lim_{\lambda\to\infty} \mathsf{G}_{\lambda,ij} = 0$. As a consequence, we arrive at $\lim_{\lambda\to\infty} \det(\mathscr{I}_{m+1} - \mathscr{G}_\lambda) = 1$. On the other hand, since $0 \le \phi_{\lambda,\eta} \le 1$ in $\overline{\mathbb{D}_\lambda}$ and g_j satisfies (7) with $g = g_j$ near $z = 0$, one can follow the same argument as (41) to obtain $\lim_{\lambda\to 0+} \mathsf{G}_{\lambda,ij} = 0$ which implies $\lim_{\lambda\to 0+} \det(\mathscr{I}_{m+1} - \mathscr{R}_\lambda) = 1$.

Consequently, there exist $\widetilde{\lambda}_*$ and $\widetilde{\lambda}^*$ with $0 < \widetilde{\lambda}_* < \widetilde{\lambda}^*$ such that for $\lambda \in (0,\widetilde{\lambda}_*) \cup (\widetilde{\lambda}^*,\infty)$, $\det(\mathscr{I}_{m+1} - \mathscr{R}_\lambda) \neq 0$ holds. Along with (i), we obtain the uniqueness result of (48)–(49) and complete the proof of Theorem 10. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5. Concluding remarks and future plans

For equation (2)–(3) in the domain Ω_λ defined by (1), let us consider the set

$$\mathfrak{S} := \{\lambda > 0 : \det(\mathscr{I} - \mathscr{R}_\lambda) = 0\},$$

where $\mathscr{I}$ and $\mathscr{R}_\lambda$ are defined by (12). Generally, $\mathfrak{S}$ is non-empty because Example 1 is the case. Proposition 2 provide a basic understanding that equation (2)–(3) has a unique solution (satisfying (11)) if and only if $\lambda \in \mathbb{R}^{>0} \setminus \mathfrak{S}$. Furthermore, Theorem 3 shows that under assumptions (4) and (7), $\mathfrak{S}$ is contained in a bounded interval $[\lambda_*, \lambda^*]$. For the general case when the domain Ω_λ is replaced with a bounded smooth domain $\mathbb{D}_\lambda$ with many boundary components, the corresponding uniqueness result is stated by Theorem 10. To the best of our knowledge, these results may contribute to the first understanding of the roles of domain size on the existence and uniqueness of the equation (2)–(3).

We shall emphasise that for $\lambda_s \in \mathfrak{S}$, (2)–(3) has infinitely many solutions if $\det \mathscr{C}_{\psi_{\lambda_s}} = 0$, and has no solution if $\det \mathscr{C}_{\psi_{\lambda_s}} \neq 0$ (cf. Proposition 2 (ii)–(iii)). Therefore, examining the number of elements in $\mathfrak{S}$ plays a crucial role in the uniqueness of equation (2)–(3). Under assumptions (4) and (7), we conjecture that $\mathfrak{S}$ only has finitely many elements since Example 1 provides relevant evidence on this conjecture. The rigorous proof is a great challenge.

On the other hand, even if we consider only the annular domain in Example 1, a closer observation of equation (6) shows that λ^* is influenced by the domain geometry. Thus, it is

expected that for the domain $\Omega := \Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}$ with non-constant boundary mean curvature, the roots of equation $\det(\mathscr{I} - \mathscr{R}_\lambda) = 0$ may depend on the geometries of Ω_{in} and Ω_{out}. A question naturally arises at this point:

> *"How to identify the effect of the geometries of Ω_{in} and Ω_{out} on the roots of $\det(\mathscr{I} - \mathscr{R}_\lambda) = 0$ more precisely?"*

Furthermore, as shown in Examples 1 and 6, it is expected that, for an equation in a bounded domain with the integral-type boundary conditions, the relationship between the domain capacity and the boundary condition may affect the uniqueness result of the solutions, which will be our future research directions.

Appendix A. Proof of (22)

In this section, we state the proof of (22) for the sake of clarity. By (19) with $C_0 = \frac{5e}{2\pi(2\sin 1 + \cos 1)}$ and (21), one may check via simple calculations that

$$
\begin{aligned}
\det(\mathscr{I} - \mathscr{R}_\lambda) &= -\frac{e^{-2\lambda+1}}{e^{-1} - e^{-2\lambda+1}}\left[1 - \frac{5}{2\sin 1 + \cos 1}\left(\cos 1 - \cos\lambda + \frac{1}{5}(2\sin\lambda + \cos\lambda)\right)\right]\\
&= -\frac{2e^{-2\lambda+2}(\sin 1 - 2\cos 1 - \sin\lambda + 2\cos\lambda)}{(1 - e^{-2\lambda+2})(2\sin 1 + \cos 1)}\\
&= -\frac{4\sqrt{5}e^{-2\lambda+2}}{(1 - e^{-2\lambda+2})(2\sin 1 + \cos 1)}\cos\frac{1+\lambda-2\theta_0}{2}\sin\frac{1-\lambda}{2}, \quad \text{where } \theta_0 = \arcsin\frac{2}{\sqrt{5}}.
\end{aligned}
$$

Here we employed some trigonometric identities to obtain $(\sin 1 - 2\cos 1) - (\sin\lambda - 2\cos\lambda) = \sqrt{5}(\sin(1-\theta_0) - \sin(\lambda-\theta_0)) = 2\sqrt{5}\cos\frac{1+\lambda-2\theta_0}{2}\sin\frac{1-\lambda}{2}$. Note that $\lambda > 1$ and $\frac{4\sqrt{5}e^{-2\lambda+2}}{(1-e^{-2\lambda+2})(2\sin 1+\cos 1)} \neq 0$. Thus, we have

$$
\det(\mathscr{I} - \mathscr{R}_\lambda) = 0 \iff \cos\frac{1+\lambda-2\theta_0}{2}\sin\frac{1-\lambda}{2} = 0,
$$

which immediately implies (22), as desired.

Acknowledgement

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