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A global space-time estimate for dispersive operators through its local estimate [☆]

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A B S T R A C T

We will show that a local space-time estimate implies a global space-time estimate for dispersive operators. In order for this implication we consider a Littlewood-Paley type square function estimate for dispersive operators in a time variable and a generalization of Tao's epsilon removal lemma in mixed norms. By applying this implication to the fractional Schrödinger equation in $\mathbb{R}^{2+1}$ we obtain the sharp global space-time estimates with optimal regularity from the previous known local ones.

1. Introduction

Let us consider a Cauchy problem of a dispersive equation in $\mathbb{R}^{n+1}$

$$\begin{cases} i\partial_t u + \Phi(D)u = 0, \\ u(0) = f, \end{cases} \tag{1.1}$$

where $\Phi(D)$ is the corresponding Fourier multiplier to the function Φ. We assume that $\Phi \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is a real-valued function satisfying the following conditions:

Condition 1.1.

- $|\nabla\Phi(\xi)| \neq 0$ for all $\xi \neq 0$.

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- There is a constant $\mu \geq 1$ such that $\mu^{-1} \leq |\Phi(\xi)| \leq \mu$ for any ξ with $|\xi| = 1$.
- There is a constant $m \geq 1$ such that $\Phi(\lambda\xi) = \lambda^m \Phi(\xi)$ for all $\lambda > 0$ and all $\xi \neq 0$.
- The Hessian $H_\Phi(\xi)$ of Φ has rank at least 1 for all $\xi \neq 0$.

The solution u to (1.1) becomes the Schrödinger operator $e^{-it\Delta}f$ if $\Phi(\xi) = |\xi|^2$ and the wave operator $e^{it\sqrt{-\Delta}}f$ if $\Phi(\xi) = |\xi|$. When $\Phi(\xi) = |\xi|^m$ for $m > 1$, the solution is called the fractional Schrödinger operator $e^{it(\sqrt{-\Delta})^{m/2}}f$.

Let $e^{it\Phi(D)}f$ denote the solution to (1.1). Our interest is to find suitable pairs (q, r) which satisfy the global space-time estimate

$$\|e^{it\Phi(D)}f\|_{L^q_x(\mathbb{R}^n;L^r_t(\mathbb{R}))} \leq C\|f\|_{\dot{H}^s(\mathbb{R}^n)}, \tag{1.2}$$

where $\dot{H}^s(\mathbb{R}^n)$ denotes the homogeneous L^2 Sobolev space of order s. By scaling invariance the regularity $s = s(r, q)$ should be defined as

$$s = n(\frac{1}{2} - \frac{1}{q}) - \frac{m}{r}. \tag{1.3}$$

This problem for $\mu = 1$ has been studied by many researchers. For the Schrödinger operator, Planchon [15] conjectured that the estimate (1.2) is valid if and only if $r \geq 2$ and $\frac{n+1}{q} + \frac{1}{r} \leq \frac{n}{2}$. Kenig–Ponce–Vega [11] showed the conjecture is true for $n = 1$. In higher dimensions $n \geq 2$ it was proven by Vega [22] that (1.2) holds for $q \geq \frac{2(n+2)}{n}$ and $\frac{n+1}{q} + \frac{1}{r} \leq \frac{n}{2}$. When $n = 2$ Rogers [16] showed it for $2 \leq r < \infty$, $q > \frac{16}{5}$ and $\frac{3}{q} + \frac{1}{r} < 1$, and later the excluded endline $\frac{3}{q} + \frac{1}{r} = 1$ was obtained by Lee–Rogers–Vargas [12]. When $n \geq 3$, Lee–Rogers–Vargas [12] improved the previous known result to $r \geq 2$, $q > \frac{2(n+3)}{n+1}$ and $\frac{n+1}{q} + \frac{1}{r} = \frac{n}{2}$. Recently it is shown by Du–Kim–Wang–Zhang [6] that the estimate (1.2) with $r = \infty$, that is, the maximal estimate fails for $n \geq 3$. For a case of the wave operator it is known that (1.2) holds for (r, q) pairs such that $2 \leq r \leq q$, $q \neq \infty$ and $\frac{1}{r} + \frac{n-1}{2q} \leq \frac{n-1}{4}$ (see [9,10,14,19]) or such that $q = \infty$ and $2 \leq r < \infty$ (see [7, Proposition 4]). Particularly, when $r = \infty$, Rogers–Villarroya [17] showed that (1.2) with regularity $s > n(\frac{1}{2} - \frac{1}{q}) - \frac{1}{r}$ is valid for $q \geq \frac{2(n+1)}{n-1}$. For the fractional Schrödinger operator the known range of (r, q) for which the estimates hold is that $2 \leq r \leq q$, $q \neq \infty$ and $\frac{n}{2q} + \frac{1}{r} \leq \frac{n}{4}$ (see [1,2,4,13,21]).

The case of $\mu > 1$ has an interesting in its own right. The solution u is formally written as

$$u(t, x) = e^{it\Phi(D)}f(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t\Phi(\xi))}\hat{f}(\xi)d\xi.$$

From this form we see that the space-time Fourier transform of u is supported in the surface $S = \{(\xi, \Phi(\xi))\}$. It is known that the operator u is related to the curvature of S such as the sign of Gaussian curvature and the number of nonvanishing principle curvature. The Schrödinger operator corresponds to a paraboloid which has a positive Gaussian curvature, and the wave operator corresponds to a cone whose Gaussian curvature is zero. We are also interested in operators corresponding to a surface with negative Gaussian curvature. When $\mu > 1$ there is a surface with negative Gaussian curvature. For instance, the surface $\{(\xi_1, \xi_2, \xi_1^4 + 2\xi_1^3\xi_2 - 2\xi_1\xi_2^3 + \xi_2^4)\}$ has negative Gaussian curvature on a neighborhood of the point $(1, 0, 1)$.

In this paper we will establish a local-to-global approach as follows.

Theorem 1.2. *Let $\mathbb{I} = (0, 1)$ be a unit interval and $\mathbb{B} = B(0, 1)$ a unit ball in $\mathbb{R}^n$. Let $q_0, r_0 \in [2, \infty)$, $s(r, q)$ defined as (1.3) and Φ satisfy Condition 1.1. Suppose that the local estimate*

$$\|e^{it\Phi(D)}f\|_{L^{q_0}_x(\mathbb{B};L^{r_0}_t(\mathbb{I}))} \leq C_\epsilon\|f\|_{H^{s(r_0,q_0)+\epsilon}(\mathbb{R}^n)} \tag{1.4}$$

holds for all $\epsilon > 0$. Then for any $q_0 < q < \infty$ and $r_0 < r < \infty$, the global estimate

$$\|e^{it\Phi(D)}f\|_{L^q_x(\mathbb{R}^n;L^r_t(\mathbb{R}))} \le C\|f\|_{\dot{H}^{s(r,q)}(\mathbb{R}^n)} \tag{1.5}$$

holds, where $H^s(\mathbb{R}^n)$ denotes the inhomogeneous L^2-Sobolev space of order s and $\dot{H}^s(\mathbb{R}^n)$ denotes homogeneous one.

The maximal estimate, which is (1.4) with $r_0 = \infty$, is related to pointwise convergence problems. When $n = 2$ it was proven that the maximal estimates with $m > 1$ and $\mu = 1$ are valid for $q_0 = 3$ and $s > \frac{1}{3}$ (see [3,5]). By interpolating with a Strichartz estimate

$$\begin{aligned}\|e^{it\Phi(D)}f\|_{L^4_x(\mathbb{B};L^4_t(\mathbb{I}))} &\le \|e^{it\Phi(D)}f\|_{L^4_x(\mathbb{R}^2;L^4_t(\mathbb{R}))}\\ &\le \|e^{it\Phi(D)}f\|_{L^4_t(\mathbb{R};L^4_x(\mathbb{R}^2))}\\ &\le C\|f\|_{\dot{H}^{\frac{2-m}{4}}(\mathbb{R}^2)} \le C\|f\|_{H^{\frac{2-m}{4}}(\mathbb{R}^2)},\end{aligned}$$

we have (1.5) for the line $\frac{3}{q} + \frac{1}{r} = 1$ with $r \ge 4$. The case of $2 \le r < 4$ follows from [22] (see also [12]). By Theorem 1.2, we can obtain the following global space-time estimates which is the Planchon conjecture for $n = 2$ except the endline.

Corollary 1.3. *Let $m > 1$ and $\mu = 1$. For $2 \le r < \infty$ and $\frac{3}{q} + \frac{1}{r} < 1$, the global estimate*

$$\|e^{it\Phi(D)}f\|_{L^q_x(\mathbb{R}^2;L^r_t(\mathbb{R}))} \le C\|f\|_{\dot{H}^{1-\frac{2}{q}-\frac{m}{r}}(\mathbb{R}^2)}.$$

Notation. Throughout this paper let $C > 0$ denote various constants that vary from line to line, which possibly depend on n, q, r, m and μ. We use $A \lesssim B$ to denote $A \le CB$, and if $A \lesssim B$ and $B \lesssim A$ we denote by $A \sim B$.

2. Proof of Theorem 1.2

In this section we prove Theorem 1.2 by using two propositions. In subsection 2.1 we consider a Littlewood–Paley type inequality by which the initial data f can be assumed to be Fourier supported in $\{1/2 \le |\xi| \le 2\}$. In subsection 2.2 we prove a mixed norm version of Tao's ε-removable lemma by which the global estimates with a compact Fourier support are reduced to local ones. In subsection 2.3 we show the two propositions imply Theorem 1.2.

2.1. A Littlewood-Paley type inequality

We discuss a Littlewood-Paley type inequality for the operator $e^{it\Phi(D)}$ in a time variable.

Let a cut-off function $\phi \in C_0^\infty([\frac{1}{2}, 2])$ satisfy $\sum_{k\in\mathbb{Z}} \phi(2^{-k}x) = 1$ for $x \ne 0$. We define Littlewood-Paley projection operators P_k and $\widetilde{P_k}$ by

$$\widehat{P_k f}(\xi) = \phi(2^{-k}|\xi|)\hat{f}(\xi) \quad \text{and} \quad \widehat{\widetilde{P_k} g}(\tau) = \phi(2^{-mk}|\tau|)\hat{g}(\tau)$$

for $\xi \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$, respectively.

Lemma 2.1. *Suppose that Φ satisfies Condition 1.1. Then for $1 < r < \infty$,*

$$\big\|e^{it\Phi(D)}f(x)\big\|_{L^r_t(\mathbb{R})} \le C_{m,\mu}\Big\|\Big(\sum_{\substack{j,k\in\mathbb{Z}:\\ |k-j|\le\frac{\log_2\mu}{m}+2}} |\widetilde{P_j}e^{it\Phi(D)}P_kf(x)|^2\Big)^{1/2}\Big\|_{L^r_t(\mathbb{R})}$$

for all Schwartz functions $f \in \mathcal{S}(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n \setminus \{0\}$.

Proof. For simplicity,

$$F(t) := e^{it\Phi(D)}f(x) \quad \text{and} \quad F_k(t) := e^{it\Phi(D)}P_kf(x).$$

Since the projection operators are linear, we have an identity

$$F(t) = \sum_{j\in\mathbb{Z}} \widetilde{P_j}F(t) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \widetilde{P_j}F_k(t).$$

We claim that $\widetilde{P_j}F_k(t) = 0$ if

$$|k-j| > \frac{\log_2\mu}{m} + 2. \tag{2.1}$$

Indeed, the Fourier transform $\widehat{f}$ of a Schwartz function f may be written as

$$\widehat{f}(\tau) = \lim_{R\to\infty}\frac{1}{2\pi}\int_{\mathbb{R}} e^{it\tau}\psi\Big(\frac{t}{R}\Big)f(t)dt,$$

where $\psi \in C_0^\infty\big([-2,2]\big)$ with $\psi = 1$ in $[-1,1]$. Using this equation we have

$$\widehat{\widetilde{P_j}F_k}(\tau) = \frac{1}{(2\pi)^{n+1}}\phi\Big(\frac{|\tau|}{2^{mj}}\Big)\lim_{R\to\infty}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\Big(\int_{\mathbb{R}} e^{it(\tau+\Phi(\xi))}\psi\Big(\frac{t}{R}\Big)dt\Big)\phi\Big(\frac{|\xi|}{2^k}\Big)\hat{f}(\xi)d\xi.$$

In the right side of the above equation, we see that the range of (τ,ξ) is contained in

$$2^{m(j-1)} \le |\tau| \le 2^{m(j+1)} \quad \text{and} \quad 2^{(k-1)} \le |\xi| \le 2^{(k+1)}.$$

From Condition 1.1 we have a bound

$$\mu^{-1}2^{m(k-1)} \le |\Phi(\xi)| \le \mu 2^{m(k+1)}.$$

Then it follows that for k and j satisfying (2.1),

$$|\tau + \Phi(\xi)| > 0.$$

By the integration by parts it implies that there exists a constant $C_0 > 0$ such that

$$\Big|\int_{\mathbb{R}} e^{it\tau}e^{it\Phi(\xi)}\psi\Big(\frac{t}{R}\Big)dt\Big| \le \frac{1}{C_0R}.$$

From this estimate and the Lebesgue dominated convergence theorem we obtain $\widehat{\widetilde{P_j}F_k} = 0$, which implies the claim.

By the claim, the Littlewood-Paley theory and the Cauchy-Schwarz inequality,

$$\begin{aligned}\big\|e^{it\Phi(D)}f(x)\big\|_{L_t^r(\mathbb{R})} &= \Big\|\sum_{j\in\mathbb{Z}}\widetilde{P_j}\Big(\sum_{k\in\mathbb{Z}}F_k(\cdot,x)\Big)\Big\|_{L_t^r(\mathbb{R})} \\ &\le C\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big|\sum_{k\in\mathbb{Z}:|k-j|\le\frac{\log_2\mu}{m}+2}\widetilde{P_j}F_k(\cdot,x)\Big|^2\Big)^{1/2}\Big\|_{L_t^r(\mathbb{R})} \\ &\le C_{m,\mu}\Big\|\Big(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}:|k-j|\le\frac{\log_2\mu}{m}+2}|\widetilde{P_j}F_k(\cdot,x)|^2\Big)^{1/2}\Big\|_{L_t^r(\mathbb{R})}.\end{aligned}$$

This is the desired inequality. □

Using the above lemma we can have the following proposition.

Proposition 2.2. *Let $2 \le q, r < \infty$. Suppose that Φ satisfies Condition 1.1. If the estimate*

$$\|e^{it\Phi(D)}f\|_{L_x^q(\mathbb{R}^n;L_t^r(\mathbb{R}))} \le C\|f\|_{L^2(\mathbb{R}^n)} \tag{2.2}$$

holds for all f with $\operatorname{supp}\hat{f} \subset \{1/2 \le |\xi| \le 2\}$, then the estimate

$$\|e^{it\Phi(D)}f\|_{L_x^q(\mathbb{R}^n;L_t^r(\mathbb{R}))} \le C_{m,\mu}\|f\|_{\dot{H}^{\frac{n}{2}-\frac{n}{q}-\frac{m}{r}}(\mathbb{R}^n)}$$

holds for all f.

Proof. The Minkowski inequality and Lemma 2.1 allow that

$$\big\|e^{it\Phi(D)}f\big\|_{L_x^q(\mathbb{R}^n;L_t^r(\mathbb{R}))} \le C_{m,\mu}\Big\|\Big(\sum_{|k-j|\le\frac{\log_2\mu}{m}+2}\Big\|\widetilde{P_j}\big(e^{it\Phi(D)}P_kf\big)\Big\|_{L_t^r(\mathbb{R})}^2\Big)^{1/2}\Big\|_{L_x^q(\mathbb{R}^n)}.$$

Since $\widetilde{P_j}$ is bounded in L^p, it is bounded by

$$C_{m,\mu}\Big\|\Big(\sum_{k\in\mathbb{Z}}\|e^{it\Phi(D)}P_kf\|_{L_t^r(\mathbb{R})}^2\Big)^{1/2}\Big\|_{L_x^q(\mathbb{R}^n)}.$$

By the Minkowski inequality we thus have

$$\big\|e^{it\Phi(D)}f\big\|_{L_x^q(\mathbb{R}^n;L_t^r(\mathbb{R}))} \le C_{m,\mu}\Big(\sum_{k\in\mathbb{Z}}\|e^{it\Phi(D)}P_kf\|_{L_x^q(\mathbb{R}^n;L_t^r(\mathbb{R}))}^2\Big)^{1/2}.$$

Apply (2.2) to the right side of the above estimate after rescaling. Then we obtain

$$\begin{aligned}\|e^{it\Phi(D)}f\|_{L_x^q(\mathbb{R}^n;L_t^r(\mathbb{R}))} &\le C_{m,\mu}\Big(\sum_{k\in\mathbb{Z}}2^{2k(\frac{n}{2}-\frac{n}{q}-\frac{m}{r})}\|P_kf\|_2^2\Big)^{1/2} \\ &= C_{m,\mu}\|f\|_{\dot{H}^{\frac{n}{2}-\frac{n}{q}-\frac{m}{r}}(\mathbb{R}^n)}. \quad \square\end{aligned}$$

2.2. Local-to-global arguments

We will show that the global estimate (2.2) is obtained from its local estimate. Adopting the arguments in [20], we consider the dual estimate of (2.2).

Let $S = \{(\xi, \Phi(\xi)) \in \mathbb{R}^n \times \mathbb{R} : 1/2 \le |\xi| \le 2\}$ be a compact hypersurface with the induced (singular) Lebesgue measure $d\sigma$. We define the Fourier restriction operator $\mathfrak{R}$ for a compact surface S by the restriction of $\hat{f}$ to S, i.e.,

$$\mathfrak{R}f = \hat{f}\big|_S.$$

Its adjoint operator $\mathfrak{R}^* f = \widehat{f d\sigma}$ can be written as $e^{it\Phi(D)}g$ with

$$\hat{g}(\xi) := f(\xi, \Phi(\xi)) J_\Phi(\xi),$$

where J_Φ is the Jacobian determinant of Φ.

Let $\rho > 0$ be the decay of $\widehat{d\sigma}$, i.e.,

$$|\widehat{d\sigma}(x)| \lesssim (1 + |x|)^{-\rho}, \qquad x \in \mathbb{R}^{n+1}. \tag{2.3}$$

It is known that ρ is determined by the number of nonzero principal curvatures of the surface S, which is equal to the rank of the Hessian H_Φ. Specifically, if H_Φ has rank at least k then

$$\rho = k/2,$$

see [18, subsection 5.8, VIII]. From Condition 1.1 we have $k \ge 1$.

When a function f has a compact Fourier support, the $\widehat{f d\sigma}$ decays away from the support of $\hat{f}$ because of the decay of $\widehat{d\sigma}$. Thus if f and g are compactly Fourier supported and their supports are far away from each other then the interaction between $\widehat{f d\sigma}$ and $\widehat{g d\sigma}$ is negligible.

Definition 2.3. A finite collection $\{Q(z_i, R)\}_{i=1}^N$ of balls in $\mathbb{R}^{n+1}$ with radius $R > 0$ is called (N, R)-*sparse* if the centers $\{z_i\}$ are $(NR)^\gamma$-separated where $\gamma := n/\rho$ (≥ 2).

From the definition of (N, R)-sparse we have a kind of orthogonality as follows. Let ϕ be a radial Schwartz function such that $\phi > 0$ on the ball $B(0, \delta^{-1})$, $\phi \ge 1/2$ on the unit ball $B(0, 1)$, and the Fourier transform $\hat{\phi}$ is supported in the ball $B(0, \delta)$ where $0 < \delta < 1$ is a constant.

Lemma 2.4 *([20, in the proof of Lemma 3.2]). Let $\{Q(z_i, R)\}_{i=1}^N$ be a (N, R)-sparse collection for $R > 1$ and $\phi_i(z) = \phi(R^{-1}(z - z_i))$ for $i = 1, \cdots, N$. Then there is a constant C independent of N such that*

$$\Big\| \sum_{i=1}^N f_i * \hat{\phi}_i\big|_S \Big\|_2 \le C R^{1/2} \Big(\sum_{i=1}^N \|f_i\|_2^2 \Big)^{1/2} \tag{2.4}$$

for all $f_i \in L^2(\mathbb{R}^{n+1})$.

A proof of the above lemma is given in Appendix A.

Let $\mathbb{I}_R = (0, R)$ denote an R-interval and $\mathbb{B}_R$ the ball of radius R centered at the origin in $\mathbb{R}^n$. Using Lemma 2.4 we have an intermediate result.

Proposition 2.5. *Let $R > 1$ and $1 < q, r \le 2$. Suppose that there is a constant $A(R)$ such that*

$$\|\mathfrak{R}(\chi_{\mathbb{I}_R \times \mathbb{B}_R} f)\|_{L^2(d\sigma)} \le A(R)\|f\|_{L^q_x(\mathbb{R}^n;L^r_t(\mathbb{R}))} \tag{2.5}$$

for all $f \in L^q_x(\mathbb{R}^n; L^r_t(\mathbb{R}))$. Then for any (N, R)-sparse collection $\{Q(z_i, R)\}_{i=1}^N$ there is a constant C independent of N such that

$$\|\mathfrak{R}f\|_{L^2(d\sigma)} \le CA(R)\|f\|_{L^q_x(\mathbb{R}^n;L^r_t(\mathbb{R}))} \tag{2.6}$$

for all f supported in $\cup_{i=1}^N Q(z_i, R)$.

Proof. Let $f_i = f\chi_{Q(z_i,R)}$. Then,

$$\mathfrak{R}f_i = \hat{f}_i\big|_S = \widehat{f_i\phi_i}\big|_S = (\hat{f}_i * \hat{\phi}_i)\big|_S,$$

where $\phi_i(z)$ is defined as in Lemma 2.4. Since $\hat{\phi}_i$ is supported on the ball $B(0, \frac{2}{3R})$, we may restrict the support of $\hat{f}_i$ to a $O(1/R)$-neighborhood of the surface S and write

$$\mathfrak{R}f_i = (\hat{f}_i\big|_{\mathcal{N}_{1/R}(S)} * \hat{\phi}_i)\big|_S$$

where $\mathcal{N}_{1/R}(S)$ is a $O(1/R)$-neighborhood of the surface S. Let $\tilde{\mathfrak{R}}$ be another restriction operator defined by $\tilde{\mathfrak{R}}f = \hat{f}\big|_{\mathcal{N}_{1/R}(S)}$. If f is supported in $\cup_{i=1}^N Q(z_i, R)$, we write

$$\mathfrak{R}f = \sum_{i=1}^N (\tilde{\mathfrak{R}}f_i * \hat{\phi}_i)\big|_S.$$

By Lemma 2.4,

$$\|\mathfrak{R}f\|_{L^2(d\sigma)} \le CR^{1/2}\Big(\sum_{i=1}^N \|\tilde{\mathfrak{R}}f_i\|^2_{L^2(\mathcal{N}_{1/R}(S))}\Big)^{1/2}.$$

Since the estimate (2.5) is translation invariant, by a slice argument we have

$$\|\tilde{\mathfrak{R}}f_i\|_{L^2(\mathcal{N}_{1/R}(S))} \le CR^{-1/2}A(R)\|f_i\|_{L^q_x(\mathbb{R}^n;L^r_t(\mathbb{R}))}.$$

By combining the previous two estimates,

$$\|\mathfrak{R}f\|_{L^2(d\sigma)} \le CA(R)\Big(\sum_{i=1}^N \|f_i\|^2_{L^q_x(\mathbb{R}^n;L^r_t(\mathbb{R}))}\Big)^{1/2}.$$

If $1 \le r \le q \le 2$ then by $\ell^r \subset \ell^q \subset \ell^2$,

$$\begin{aligned}\Big(\sum_{i=1}^N \|f_i\|^2_{L^q_x(\mathbb{R}^n;L^r_t(\mathbb{R}))}\Big)^{1/2} &\le \Big(\sum_{i=1}^N \|f_i\|^q_{L^q_x(\mathbb{R}^n;L^r_t(\mathbb{R}))}\Big)^{1/q} \\ &= \Big(\int_{\mathbb{R}^n} \sum_{i=1}^N \|f_i\|^q_{L^r_t(\mathbb{R})} dx\Big)^{1/q}\end{aligned}$$

$$\le \Big(\int_{\mathbb{R}^n} \Big(\sum_{i=1}^{N} \|f_i\|_{L_t^r(\mathbb{R})}^r \Big)^{q/r} dx \Big)^{1/q}$$
$$= \|f\|_{L_x^q(\mathbb{R}^n;L_t^r(\mathbb{R}))}.$$

If $1 \le q \le r \le 2$ one can use the embedding $\ell^r \subset \ell^2$ and the Minkowski inequality to get

$$\Big(\sum_{i=1}^{N} \|f_i\|_{L_x^q(\mathbb{R}^n;L_t^r(\mathbb{R}))}^2 \Big)^{1/2} \le \Big(\sum_{i=1}^{N} \|f_i\|_{L_x^q(\mathbb{R}^n;L_t^r(\mathbb{R}))}^r \Big)^{1/r}$$
$$\le \Big(\int_{\mathbb{R}^n} \Big(\sum_{i=1}^{N} \|f_i\|_{L_t^r(\mathbb{R})}^r \Big)^{q/r} dx \Big)^{1/q}$$
$$= \|f\|_{L_x^q(\mathbb{R}^n;L_t^r(\mathbb{R}))}.$$

Therefore we have (2.6). □

We now extend the (N, R)-sparse sets to the whole space. For this we need the following decomposition lemma.

Lemma 2.6 *([20]). Let E be a subset in $\mathbb{R}^n$ with $|E| > 1$. Suppose that E is a finite union of finitely overlapping cubes of side-length $c \sim 1$. Then for each $K \in \mathbb{N}$, there are subsets $E_1, E_2, \cdots, E_K$ of E with*

$$E = \bigcup_{k=1}^{K} E_k$$

such that each E_k has $O(|E|^{1/K})$ number of $(O(|E|), |E|^{O(\gamma^{k-1})})$-sparse collections

$$\mathbf{S}_1, \mathbf{S}_2, \cdots, \mathbf{S}_{O(|E|^{1/K})}$$

of which the union $\mathbf{S}_1 \cup \mathbf{S}_2 \cup \cdots \cup \mathbf{S}_{O(|E|^{1/K})}$ is a covering of E_k.

This lemma is a precise version of Lemma 3.3 in [20]. A detailed proof can be found in Appendix A.
Using the above lemma we have the following proposition.

Proposition 2.7. *Let $1 < q_0, r_0 < \infty$. Suppose that for any $\epsilon > 0$ and any (N, R)-sparse collection $\{Q(z_i, R)\}_{i=1}^N$ in $\mathbb{R}^{n+1}$, the estimate*

$$\|\mathfrak{R} f\|_{L^2(d\sigma)} \le C_\epsilon R^\epsilon \|f\|_{L_x^{q_0}(\mathbb{R}^n;L_t^{r_0}(\mathbb{R}))} \tag{2.7}$$

holds for all f supported in $\cup_{i=1}^N Q(z_i, R)$. Then for any $1 \le q < q_0$ and $1 \le r < r_0$, the estimate

$$\|\mathfrak{R} f\|_{L^2(d\sigma)} \le C \|f\|_{L_x^q(\mathbb{R}^n;L_t^r(\mathbb{R}))}$$

holds for all $f \in L_x^q(\mathbb{R}^n; L_t^r(\mathbb{R}))$.

Proof. By interpolation (see [8]), it suffices to show that for $1 \le q < q_0$ and $1 \le r < r_0$, the restricted type estimate

$$\|\mathfrak{R} \chi_E\|_{L^2(d\sigma)} \le C \|\chi_E\|_{L^q(\mathbb{R}^n;L^r(\mathbb{R}))} \tag{2.8}$$

for all subset E in $\mathbb{R}^{n+1}$. We may assume $|E| > 1$, otherwise the estimate is trivial. Indeed,

$$\|\mathfrak{R}\chi_E\|_{L^2(d\sigma)} \leq C\|\mathfrak{R}\chi_E\|_{L^\infty(S)} \leq C|E| \leq C.$$

Since the set S is compact, there is a bump function $\varphi \in C_0^\infty$ supported in a cube of sidelength $2c \sim 1$ and centered at the origin such that $\hat{\varphi}$ is positive on a cube of sidelength $(2c)^{-1}$ that contains S. By the Poisson summation formula we may assume that $\sum_{k\in c\mathbb{Z}^{n+1}} \varphi(\cdot - k) = 1$.

Let c-lattice cubes $\{\Delta_k\}$ cover the set E. We claim that if the estimate

$$\|\mathfrak{R}(\sum_k \chi_{\Delta_k})\|_{L^2(d\sigma)} \leq C\Big\| \sum_k \chi_{\Delta_k}\Big\|_{L^q(\mathbb{R}^n;L^r(\mathbb{R}))}$$

holds for any $1 \leq q < q_0$ and $1 \leq r < r_0$, then we have (2.8). By interpolation the above estimate implies that for any $1 \leq q < q_0$ and $1 \leq r < r_0$,

$$\|\mathfrak{R}(\sum_k a_k\chi_{\Delta_k})\|_{L^2(d\sigma)} \leq C\Big\| \sum_k a_k\chi_{\Delta_k}\Big\|_{L^q(\mathbb{R}^n;L^r(\mathbb{R}))},$$

for all real sequences $\{a_k\}$. Let φ_k be a translation of φ which is supported in $2\Delta_k$. Since φ_k decays rapidly away from Δ_k, the above inequality implies

$$\|\mathfrak{R}(\sum_k a_k\varphi_k)\|_{L^2(d\sigma)} \leq C\Big\| \sum_k a_k\varphi_k\Big\|_{L^q(\mathbb{R}^n;L^r(\mathbb{R}))} + C^{-N} \tag{2.9}$$

where $N \geq 1$ is a large number.

By replacing χ_E with $\sum_k \chi_E\varphi_k$,

$$\|\mathfrak{R}\chi_E\|_{L^2(d\sigma)} \leq C\Big\| \sum_k \hat{\chi}_E * \hat{\varphi}_k\Big|_S\Big\|_{L^2(d\sigma)}.$$

Using $\|\hat{f}\|_\infty \leq C\|f\|_1$ we have

$$|\hat{\chi}_E * \hat{\varphi}_k(z)| \leq C\int \chi_E\varphi_k.$$

Since $\frac{1}{|\Delta_k|}\hat{\varphi}_k$ is a positive Schwartz function and $\frac{1}{|\Delta_k|}\hat{\varphi}_k \lesssim 1$ on S, we have that for any $z \in S$,

$$|\hat{\chi}_E * \hat{\varphi}_k(z)| \leq Ca_k\hat{\varphi}_k(z),$$

where

$$a_k := \frac{1}{|\Delta_k|}\int \chi_E\varphi_k.$$

Thus,

$$\|\mathfrak{R}\chi_E\|_{L^2(d\sigma)} \leq C\Big\| \sum_k a_k\hat{\varphi}_k\Big|_S\Big\|_{L^2(d\sigma)}.$$

Apply (2.9). Then,

$$\|\mathfrak{R}\chi_E\|_{L^2(d\sigma)} \leq C\Big\| \sum_k a_k\varphi_k\Big\|_{L^q(\mathbb{R}^n;L^r(\mathbb{R}))} + C^{-N}. \tag{2.10}$$

Since the supports of φ_k are finitely overlapped, we have

$$\Big\|\sum_k a_k\varphi_k\Big\|_{L^q(\mathbb{R}^n;L^r(\mathbb{R}))} \le C\Big(\int_{\mathbb{R}^n}\Big(\sum_k \|a_k\varphi_k(x)\|^r_{L^r(\mathbb{R})}\Big)^{q/r}dx\Big)^{1/q}$$
$$\le C\Big(\sum_k a_k^q\|\varphi_k\|^q_{L^q(\mathbb{R}^n;L^r(\mathbb{R}))}\Big)^{1/q}.$$

By Hölder's inequality,

$$a_k = \frac{1}{|\Delta_k|}\int \chi_E\varphi_k \le C\frac{\|\chi_E\varphi_k\|_{L^q(\mathbb{R}^n;L^r(\mathbb{R}))}\|\varphi_k\|_{L^{q'}(\mathbb{R}^n;L^{r'}(\mathbb{R}))}}{\|\varphi_k\|_1}.$$

By calculation we can see $\|\varphi_k\|_{L^{q'}(\mathbb{R}^n;L^{r'}(\mathbb{R}))}\|\varphi_k\|_{L^q(\mathbb{R}^n;L^r(\mathbb{R}))} \sim \|\varphi_k\|_1$, and

$$a_k\|\varphi_k\|_{L^q(\mathbb{R}^n;L^r(\mathbb{R}))} \le C\|\chi_E\varphi_k\|_{L^q(\mathbb{R}^n;L^r(\mathbb{R}))}.$$

By inserting this estimate,

$$\Big\|\sum_k a_k\varphi_k\Big\|_{L^q(\mathbb{R}^n;L^r(\mathbb{R}))} \le C\Big(\sum_k \|\chi_E\varphi_k\|^q_{L^q(\mathbb{R}^n;L^r(\mathbb{R}))}\Big)^{1/q}.$$

Since the supports of φ_k are finitely overlapped, the above estimate is

$$\le C\Big\|\sum_k \chi_E\varphi_k\Big\|_{L^q(\mathbb{R}^n;L^r(\mathbb{R}))} = C\|\chi_E\|_{L^q(\mathbb{R}^n;L^r(\mathbb{R}))}.$$

By combining this estimate with (2.10) we obtain (2.8). The claim is proved.

By the claim, the set E in (2.8) can be considered as the union of c-cubes Δ_k. We denote by $\mathrm{proj}(E)$ the projection of E onto the x-plane. For each grid point $x \in c\,\mathbb{Z}^n \cap \mathrm{proj}(E)$, we define E_x to be the union of c-cubes in E that intersect $\mathbb{R}\times\{x\}$. Let E^j be the union of E_x which satisfies

$$2^{j-1} < \text{ the number of } c\text{- cubes contained in } E_x \le 2^{j+1}$$

for $j \in \mathbb{N}$, (see Fig. 1). Then,

$$E = \bigcup_{j\ge 1} E^j.$$

By using Lemma 2.6 with

$$K := \frac{\log(1/\epsilon)}{2\log\gamma} + 1,$$

the E^j is decomposed into E^j_k's which are covered by $O(|E^j|^{1/K})$ number of $(O(|E^j|), |E^j|^{C\gamma^{k-1}}))$-sparse collections. We apply (2.7) to these sparse collections and obtain

$$\|\mathfrak{R}\chi_{E^j_k}\|_{L^2(d\sigma)} \le C_\epsilon|E^j|^{1/K}(|E^j|^{C\gamma^{k-1}})^\epsilon\|\chi_{E^j_k}\|_{L^{q_0}_x(\mathbb{R}^n;L^{r_0}_t(\mathbb{R}))}.$$

Summing over k, we have

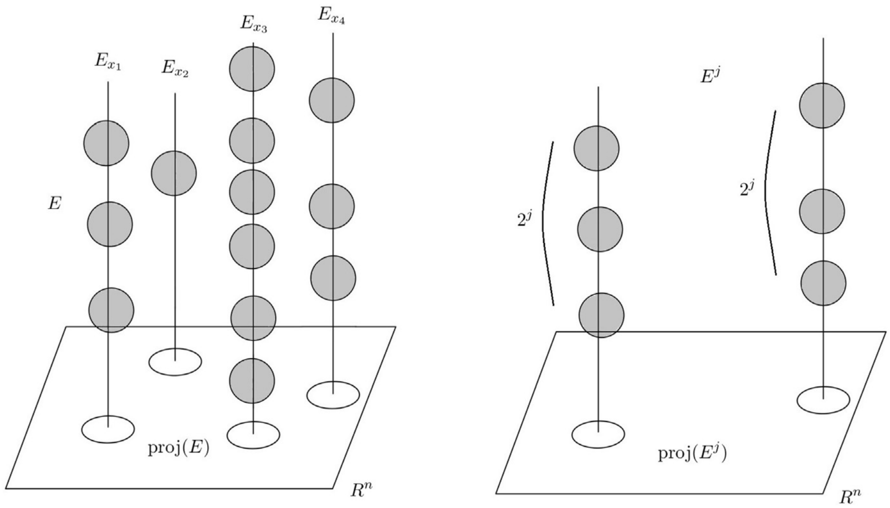

Fig. 1. The sets E, proj$E$, E_x and E^j in the proof of Proposition 2.7.

$$
\begin{aligned}
\|\mathfrak{R}\chi_{E^j}\|_{L^2(d\sigma)} &\le \sum_{k=1}^{K} \|\mathfrak{R}\chi_{E^j_k}\|_{L^2(d\sigma)} \\
&\le C_\epsilon |E^j|^{1/K} (|E^j|^{C\gamma^{K-1}})^{\epsilon} \|\chi_{E^j}\|_{L^{q_0}_x(\mathbb{R}^n; L^{r_0}_t(\mathbb{R}))}
\end{aligned}
$$

where K is absorbed into C_ϵ. Since $|E^j| \le 2^{j+1}|\text{proj}\,(E^j)|$, we have

$$
\|\mathfrak{R}\chi_{E^j}\|_{L^2(d\sigma)} \le C_\epsilon 2^{j(\frac{1}{r_0}+\delta(\epsilon))} |\text{proj}(E^j)|^{\frac{1}{q_0}+\delta(\epsilon)},
$$

where

$$
\delta(\epsilon) := \frac{1}{K} + C\gamma^{K-1}\epsilon.
$$

Since $\lim_{\epsilon \to 0} \delta(\epsilon) = 0$, we can take $\epsilon > 0$ such that

$$
0 < \delta(\epsilon) + \epsilon \le \min\left(\frac{1}{q} - \frac{1}{q_0}, \frac{1}{r} - \frac{1}{r_0}\right).
$$

Thus,

$$
\begin{aligned}
\|\mathfrak{R}\chi_E\|_{L^2(d\sigma)} &\le \sum_{j\ge 1} \|\mathfrak{R}\chi_{E^j}\|_{L^2(d\sigma)} \\
&\le C_\epsilon \sum_{j\ge 1} 2^{j(\frac{1}{r_0}+\delta(\epsilon))} |\text{proj}(E^j)|^{\frac{1}{q_0}+\delta(\epsilon)} \\
&\le C \sum_{j\ge 1} 2^{-\epsilon j} 2^{\frac{1}{r}j} |\text{proj}(E^j)|^{\frac{1}{q}} \\
&\le C \sum_{j\ge 1} 2^{-\epsilon j} \|\chi_E\|_{L^q_x(\mathbb{R}^n; L^r_t(\mathbb{R}))} \\
&\le C \|\chi_E\|_{L^q_x(\mathbb{R}^n; L^r_t(\mathbb{R}))}. \quad \square
\end{aligned}
$$

Combining Proposition 2.5 and Proposition 2.7 we obtain an extension of Tao's epsilon removal lemma as follows.

Proposition 2.8. *Let $1 < q_0, r_0 \le 2$. Suppose that*

$$\|\mathfrak{R}(\chi_{\mathbb{I}_R \times \mathbb{B}_R} f)\|_{L^2(d\sigma)} \le C_\epsilon R^\epsilon \|f\|_{L_x^{q_0}(\mathbb{R}^n; L_t^{r_0}(\mathbb{R}))}$$

for all $\epsilon > 0$, $R > 1$ and all $f \in L_x^{q_0}(\mathbb{R}^n; L_t^{r_0}(\mathbb{R}))$. Then for any $1 \le q < q_0$ and $1 \le r < r_0$,

$$\|\mathfrak{R} f\|_{L^2(d\sigma)} \le C\|f\|_{L_x^q(\mathbb{R}^n; L_t^r(\mathbb{R}))}$$

for all $f \in L_x^q(\mathbb{R}^n; L_t^r(\mathbb{R}))$.

Now we are ready to prove Theorem 1.2. The theorem follows from Proposition 2.2 and Proposition 2.8 as follows.

2.3. Proof of Theorem 1.2

Let P_0 be the Littlewood-Paley projection operator as in subsection 2.1. By rescaling $x \mapsto 2^{-k}x$ and $t \mapsto 2^{-mk}t$, the estimate (1.4) implies

$$\|e^{it\Phi(D)} P_0 f\|_{L_x^{q_0}(\mathbb{B}_{2^k}; L_t^{r_0}(\mathbb{I}_{2^{mk}}))} \le C_\epsilon 2^{k\epsilon} \|P_0 f\|_{L^2(\mathbb{R}^n)}$$

for all $k \ge 1$ and $\epsilon > 0$. Since $m \ge 1$, we have

$$\|e^{it\Phi(D)} P_0 f\|_{L_x^{q_0}(\mathbb{B}_{2^k}; L_t^{r_0}(\mathbb{I}_{2^k}))} \le C_\epsilon 2^{k\epsilon} \|P_0 f\|_{L^2(\mathbb{R}^n)}.$$

By Proposition 2.8 and duality,

$$\|e^{it\Phi(D)} P_0 f\|_{L_x^q(\mathbb{R}^n; L_t^r(\mathbb{R}))} \le C\|P_0 f\|_{L^2(\mathbb{R}^n)}.$$

By Proposition 2.2, we obtain the desired estimate. □

Appendix A

A.1. Proof of Lemma 2.4

We divide the left side of (2.4) into two parts

$$\|\sum_{i=1}^N f_i * \hat{\phi}_i|_S\|_2^2 = \sum_i \|f_i * \hat{\phi}_i|_S\|_2^2 + \sum_{i \ne j} \int f_i * \hat{\phi}_i \overline{f_j * \hat{\phi}_j} d\sigma.$$

We may assume that $N \ge 2$ because if $N = 1$ then the estimate is trivial. By a basic restriction estimate we have $\|f_i * \hat{\phi}_i|_S\|_2 \lesssim R^{1/2}\|f_i\|_2$ (for details see [20,23]). Thus,

$$\sum_{i=1}^N \|f_i * \hat{\phi}_i|_S\|_2^2 \lesssim R \sum_{i=1}^N \|f_i\|_2^2. \tag{A.1}$$

By Parseval's identity,

$$\int f_i * \hat{\phi}_i \overline{f_j * \hat{\phi}_j} d\sigma = \int \overline{\check{f}_j \phi_j}((\check{f}_i \phi_i) * \widehat{d\sigma}),$$

where the ˇ denotes the inverse Fourier transform. It is bounded by

$$\big(\sup_{z,w}|\phi_j^{1/2}(z)\phi_i^{1/2}(w)\widehat{d\sigma}(z-w)|\big)\|\check{f}_i\phi_i^{1/2}\|_1\|\check{f}_j\phi_j^{1/2}\|_1.$$

By the Cauchy-Schwarz inequality and Plancherel's theorem,

$$\|\check{f}_i\phi_i^{1/2}\|_1 \lesssim R^{(n+1)/2}\|f_i\|_2.$$

By (2.3),

$$\sup_{z,w}|\phi_j^{1/2}(z)\phi_i^{1/2}(w)\widehat{d\sigma}(z-w)| \lesssim |z_i-z_j-2R|^{-\rho}.$$

Since $|z_i-z_j| \geq (NR)^\gamma$, $\gamma \geq 2$, $N \geq 2$ and $R > 1$,

$$|z_i-z_j-2R| \geq |z_i-z_j|-2R \gtrsim |z_i-z_j|/2.$$

Thus,

$$\sup_{z,w}|\phi_j^{1/2}(z)\phi_i^{1/2}(w)\widehat{d\sigma}(z-w)| \lesssim |z_i-z_j|^{-\rho}.$$

Combining these estimates we have

$$\begin{aligned}\sum_{i\neq j}\int f_i*\hat{\phi}_i\overline{f_j*\hat{\phi}_j}d\sigma &\lesssim R^{n+1}\sum_{i=1}^{N}\sum_{j\in\{1,2,\dots,N\},i\neq j}|z_i-z_j|^{-\rho}\|f_i\|_2\|f_j\|_2\\ &\lesssim R^{n+1}N\max_{i,j}|z_i-z_j|^{-\rho}\sum_{i=1}^{N}\|f_i\|_2^2.\end{aligned}$$

Since $|z_i-z_j| \geq (NR)^\gamma \geq N^{\frac{1}{\rho}}R^{\frac{n}{\rho}}$, it follows that

$$\sum_{i\neq j}\int f_i*\hat{\phi}_i\overline{f_j*\hat{\phi}_j}d\sigma \lesssim R\sum_{i=1}^{N}\|f_i\|_2^2.$$

From the above estimate and (A.1) we obtain (2.4). □

A.2. Proof of Lemma 2.6

Fix $K \in \mathbb{N}$. We define $R_0 = 1$ and R_k for $k = 1, 2, \cdots, K$ recursively by

$$R_k = |E|^\gamma R_{k-1}^\gamma. \tag{A.2}$$

From this definition we have $R_k = |E|^{\frac{\gamma^{k+1}-\gamma}{\gamma-1}}$. Let $E_0 = \emptyset$. We define E_k for $k = 1, 2, \cdots, K$ to be the set of all $x \in E \setminus \cup_{j=0,1,2,\cdots,k-1}E_j$ such that

$$|E \cap B(x, R_k)| \leq |E|^{k/K}. \tag{A.3}$$

Then, $E = \bigcup_{k=1}^{K} E_k$. From this construction it follows that for $x \in E_k$, $k = 2, 3, \cdots, K$,

$$|E \cap B(x, R_{k-1})| > |E|^{(k-1)/K}. \tag{A.4}$$

We cover E_k with finitely overlapping R_k-balls $\mathbf{C}_{E_k} := \{B_i = B(x_i, R_k) : x_i \in E_k\}$. Since E is a finite union of cubes of side-length $c \sim 1$, it is obvious that $\#\mathbf{C}_{E_k} \lesssim |E|$. For each $B_i \in \mathbf{C}_{E_k}$ we cover $E_k \cap B_i$ with finitely overlapping R_{k-1}-balls $\mathbf{C}_{E_k \cap B_i} := \{B'_{ij} = B'(y_j, R_{k-1}) : y_j \in E_k \cap B_i\}$, that is,

$$E_k \cap B_i = \bigcup_{B'_{ij} \in \mathbf{C}_{E_k \cap B_i}} E_k \cap B'_{ij}.$$

Since $((E \setminus E_k) \cap B'_{ij}) \subset ((E \setminus E_k) \cap B_i)$ for all j, we have

$$(E_k \cap B_i) \cup ((E \setminus E_k) \cap B_i) \supset \bigcup_{B'_{ij} \in \mathbf{C}_{E_k \cap B_i}} (E_k \cap B'_{ij}) \cup ((E \setminus E_k) \cap B'_{ij}),$$

thus

$$E \cap B_i \supset \bigcup_{B'_{ij} \in \mathbf{C}_{E_k \cap B_i}} E \cap B'_{ij}.$$

By finitely overlapping,

$$\#\mathbf{C}_{E_k \cap B_i} \lesssim \max_{B'_{ij} \in \mathbf{C}_{E_k \cap B_i}} \frac{|E \cap B_i|}{|E \cap B'_{ij}|}.$$

By (A.3) and (A.4) the above is bounded by $C|E|^{1/K}$, and we have $\#\mathbf{C}_{E_k \cap B_i} \le C|E|^{1/K}$ for all i. Thus,

$$E_k \subset \bigcup_{i=1}^{O(|E|)} \bigcup_{j=1}^{O(|E|^{1/K})} B'_{ij}.$$

We choose $O(R_k)$-separated balls $\{B'_{ij(i)}\}_{i=1}^{O(|E|)}$. Then it is a $(O(|E|), R_{k-1})$-sparse collection because of (A.2). Since $R_{k-1} = |E|^{O(\gamma^{k-1})}$ and every $B_i \in \mathbf{C}_{E_k}$ has the covering $\mathbf{C}_{E_k \cap B_i}$ of cardinality $O(|E|^{1/K})$, there are $O(|E|^{1/K})$ number of $(O(|E|), |E|^{O(\gamma^{k-1})})$-sparse collections $\mathbf{S}_1, \mathbf{S}_2, \cdots, \mathbf{S}_{O(|E|^{1/K})}$ such that

$$E_k \subset \bigcup_{j=1}^{O(|E|^{1/K})} \bigcup_{B' \in \mathbf{S}_j} B'. \quad \square$$

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