ANALYTIC RANKS OF ELLIPTIC CURVES
OVER NUMBER FIELDS

PETER J. CHO

(Communicated by Amanda Folsom)

ABSTRACT. Let E be an elliptic curve over $\mathbb{Q}$. Then, we show that the average
analytic rank of E over cyclic extensions of degree l over $\mathbb{Q}$ with l a prime not
equal to 2, is at most $2 + r_{\mathbb{Q}}(E)$, where $r_{\mathbb{Q}}(E)$ is the analytic rank of the elliptic
curve E over $\mathbb{Q}$. This bound is independent of the degree l. Using a recent
result of Bhargava, Taniguchi and Thorne [Improved error estimates for the
Davenport–Heilbronn theorems, arxiv.org/abs/2107.12819, 2021], we obtain a
non-trivial upper bound on the average analytic rank of E over S_3-fields.

1. INTRODUCTION

Let E be an elliptic curve defined over $\mathbb{Q}$ with conductor Q_E. For a number field
F, let $E(F)$ be the group of F-rational points of the elliptic curve E. Let $L_F(s, E)$
be the normalized L-function of E over the field F so that its central point is $\frac{1}{2}$.
We omit the subscript F from $L_F(s, E)$ when F is the field of rational numbers.
We are interested in the behavior of the analytic ranks of the L-functions $L_F(s, E)$
when F varies over cyclic extensions of prime degree l over $\mathbb{Q}$. For a prime $l \geq 2$,
we denote the family of all cyclic extensions F of degree l over $\mathbb{Q}$ by C_l. Then, for
a number field F in C_l we have

$$(1.1) \qquad L_F(s, E) = L(s, E) \prod_{\chi} L(s, E \times \chi),$$

where χ runs over the $(l-1)$ primitive l-th order Dirichlet characters corresponding
to the field F.

For quadratic fields F, the average analytic rank of $L(s, E \times \chi)$ is expected to
be $\frac{1}{2}$ regardless of the analytic rank $r_{\mathbb{Q}}(E)$ of $L(s, E)$ by Goldfeld's conjecture. So
we have the following statement equivalent to Goldfeld's conjecture [7].

Conjecture 1.1 (Goldfeld's conjecture). *Let E be an elliptic curve over $\mathbb{Q}$. Then,
the average analytic rank of E over quadratic fields F is $\frac{1}{2} + r_{\mathbb{Q}}(E)$.*

Goldfeld's conjecture says that a half of the twisted L-functions $L(s, E \times \chi)$
do not vanish at the central point and the other half of them vanish to order 1
at the central point. However, the story seems different for cyclic extensions of

prime degree $l \geq 3$. David, Fearnley and Kisilevsky [6] conjectured that for a fixed elliptic curve E and a fixed prime $l \geq 7$, there are only finitely many primitive Dirichlet characters χ of order l for which $L(s, E \times \chi)$ vanishes at the central point $s = 1/2$. Also, they conjectured that only a small number of twisted L-functions $L(s, E \times \chi)$ vanish for $l = 3$ and 5. As a direct consequence of the conjecture, we have Conjecture 1.2.

Conjecture 1.2. *Let $l \geq 3$ be a prime and E an elliptic curve over $\mathbb{Q}$. Then, the average analytic rank of $E(F)$ over the family C_l is $r_{\mathbb{Q}}(E)$.*

We can relate this conjecture with Diophantine Stability introduced by Mazur and Rubin [11] recently. Let K be a number field. Suppose V is an irreducible algebraic variety over K. If L is a field containing K, we say that V is diophantine-stable for L/K if $V(L) = V(K)$. For a given elliptic curve E over $\mathbb{Q}$, if $r_F(E) = r_{\mathbb{Q}}(E)$ for a number field F, then under the Birch and Swinnerton-Dyer conjecture, the algebraic rank $r_F^{\mathrm{alg}}(E)$ of $E(F)$ is equal to the algebraic rank $r_{\mathbb{Q}}^{\mathrm{alg}}(E)$ of $E(\mathbb{Q})$. By Merel's uniform bound [14, Theorem 7.5.1] on the size of $E(F)_{\mathrm{tor}}$, we can see that there are only finitely many number fields F of degree l for which $E(F)_{\mathrm{tor}} > E(\mathbb{Q})_{\mathrm{tor}}$. Even if $r_F^{\mathrm{alg}}(E) = r_{\mathbb{Q}}^{\mathrm{alg}}(E)$ and $E(F)_{\mathrm{tor}} = E(\mathbb{Q})_{\mathrm{tor}}$, there could be a F-rational point not belonging to $E(\mathbb{Q})$ unfortunately.[1] However, these results still give strong conjectural evidence that an elliptic curve E over $\mathbb{Q}$ is diophantine-stable for $L/\mathbb{Q}$ for most cyclic fields L of prime degree $l \geq 3$.

We can understand these two seemingly different phenomena through Katz and Sarnak's n-level density conjecture for families of L-functions. Their philosophy is that the distribution of low-lying zeros of L-functions in a natural family is governed by one of the five classical matrix groups $O, SO(even), SO(odd), USp$, and U, which we call the symmetry type of the family. We refer to [13] for the introduction of the conjecture.

From a work of Rubinstein [13], when χ is quadratic, we can see that the symmetry type for the family of L-functions $L(s, E \times \chi)$ is O and the average of analytic ranks $r_F(E)$ is at most $2.5 = 2 + 0.5$. Heath-Brown [8] lowered the bound to 1.5. If Katz and Sarnak's one-level conjecture is true for a test function with arbitrarily large compact support, the average analytic rank would be $1/2$, which is Goldfeld's conjecture.

In [5], the author and Park computed the one-level density for families of L-functions $L(s, \pi \times \chi)$ for a cuspidal representation π of $GL_M(A_{\mathbb{Q}})$. From it, when χ is a primitive character of prime degree $l > 2$, we can determine that the symmetry type for the family of $L(s, E \times \chi)$ is U. Under the one-level density conjecture for the symmetry type U, the average analytic rank becomes $r_{\mathbb{Q}}(E)$.

In this article, we make partial progress toward Conjecture 1.2. Let E be an elliptic curve defined over $\mathbb{Q}$, and F a field in C_l. By (1.1), we have

$$r_F(E) = r_{\mathbb{Q}}(E) + \sum_{\chi} \mathrm{ord}_{s=1/2} L(s, E \times \chi).$$

---

[1]For example, consider an elliptic curve E which is given by the Weierstrass equation $y^2 = x^3 + 9$. Then $E(\mathbb{Q}) \cong \mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Let K be the field by adjoining a root α of $x^4 + 8x^3 - 72x + 72$ to $\mathbb{Q}$. Then $E(K) \cong \mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and $E(K)$ contains a new point $P = (\alpha, -\frac{1}{2}\alpha^3 - 3\alpha^2 + 9)$ of infinite order such that $2P = (-2, 1)$.

Hence the average of $r_F(E)$ is given by

$$r_{\mathbb{Q}}(E) + \text{the average of } \mathrm{ord}_{s=1/2} L(s, E \times \chi),$$

where the average is taken over some subfamily of C_l which we describe now. We consider primitive characters χ with conductor q_χ coprime to Q_E. The condition $(q_\chi, Q_E) = 1$ determines the (not analytic but ordinary) conductor $q(E \times \chi)$ of $L(s, E \times \chi)$ completely, which is $q_\chi^2 Q_E$ by a work of Barthel and Ramakrishnan [1].

Define

$$C_{l,Q_E} = \{F \in C_l | (q_F, Q_E) = 1\},$$

where q_F is the conductor of the field F. Now let ω be a non-negative Schwartz class function. We define

$$\mathfrak{C}_{l,Q_E}(X) = \sum_{F \in C_{l,Q_E}} \omega\left(\frac{q_F}{X}\right).$$

We show that the average analytic rank has a nice uniform upper bound independent of the degree l.

Theorem 1.3. *Assume GRH.[2] Let E be an elliptic curve over $\mathbb{Q}$, l be a prime ≥ 3. Then,*

$$\lim_{X \to \infty} \frac{\sum_{F \in C_{l,Q_E}(X)} r_F(E) \omega\left(\frac{q_F}{X}\right)}{\mathfrak{C}_{l,Q_E}(X)} \leq 2 + r_{\mathbb{Q}}(E).$$

In Section 5, we also give an upper bound on the average analytic rank over some non-abelian fields. A number field F of degree d is an S_d-field if its normal closure $\widehat{F}$ over $\mathbb{Q}$ is an S_d Galois extension. For example, quadratic fields are S_2-fields.

For an S_d-field F we have

$$L_F(s, E) = L(s, E) L(s, E \times \rho),$$

where ρ is the $(d-1)$-dimensional standard representation of the symmetry group S_d.

Let $\mathcal{S}_{d,Q_E}$ be the family of S_d-fields F with discriminant D_F coprime to Q_E and $\mathcal{S}_d$ be the family of S_d-fields with no restriction on discriminant. For a positive number X, let

$$\mathcal{S}_{d,Q_E}(X) = \{F \in \mathcal{S}_{d,Q_E} | |D_F| \leq X\},$$

where D_F is the discriminant of the field F. In [10], Lemke Oliver and Thorne showed that there is a constant $c_d > 0$ such that in $\mathcal{S}_d(X)$, for any $\epsilon > 0$, there are $\gg_{E,\epsilon} X^{c_d - \epsilon}$ S_d-fields F with $r_F(E) > r_{\mathbb{Q}}(E)$.

For S_3-fields, using a recent result of Bhargava, Taniguchi and Thorne [2] we have our second main result.

Theorem 1.4. *Assume GRH.[3] Let E be an elliptic curve E over $\mathbb{Q}$. The average analytic rank $r_F(E)$ over S_{3,Q_E} is bounded by $7.5 + r_{\mathbb{Q}}(E)$.*

Remark 1.5. We also have analogues of Theorem 1.4 for S_4-fields and S_5-fields, which are mentioned at the end of Section 5. These poor bounds are due to the poor error terms of the counting functions for S_4-fields and S_5-fields. See [4].

---

[2]We need GRH for the following L-functions: $\zeta(s)$, Dirichlet L-functions with χ mod $2l$, Hecke L-functions over $K = \mathbb{Q}(\zeta_l)$ with characters of order l, $\zeta_K(s)$ and $L(s, E \times \chi)$ for primitive Dirichlet characters of order l.

[3]We need GRH for $L(s, E \times \rho)$, $L(s, \mathrm{Sym}^2(E))$ and $L(s, \Lambda^2(E))$.

In Section 2, we introduce an explicit formula we use which is one of the main tools for one-level density. In Section 3, we recall some preliminaries on primitive Dirichlet characters and lemmas for proof of Theorem 1.3. Sections 4 and 5 are devoted to the proof of Theorems 1.3 and 1.4.

2. EXPLICIT FORMULA

Let $L(s, f)$ be an entire L-function with conductor $q(f)$ and gamma factor $\gamma(f, s)$ which satisfies the standard functional equation:

$$\Lambda(s, f) = q(f)^{\frac{s}{2}} \gamma(f, s) L(s, f) = \omega_f \Lambda(1 - s, \overline{f}),$$

where ω_f is the root number of modulus 1. Let $\Lambda_f(n) = a_f(n)\Lambda(n)$ be the n-th coefficient of the Dirichlet series $-\frac{L'}{L}(s, f) = \sum_{n=1}^{\infty} \frac{\Lambda_f(n)}{n^s}$. If the Euler factor of $L(s, f)$ at the place p is $\prod_{i=1}^{d} \left(1 - \frac{\alpha_i(p)}{p^s}\right)^{-1}$, then $a_f(p^k) = \sum_{i=1}^{d} \alpha_i(p)^k$ and $\Lambda_f(p^k) = a_f(p^k) \log p$. By [9, Theorem 5.12], we have the following explicit formula.

Lemma 2.1. *Let ϕ be an even Schwartz class function such that its Fourier transform $\widehat{\phi}$ is compactly supported. Let $L(s, f)$ be an L-function as above. For a parameter $L > 0$, we have*

$$\sum_{\rho = \frac{1}{2} + i\gamma} \phi\left(\gamma \frac{\log L}{2\pi}\right) = \widehat{\phi}(0) \frac{\log q(f)}{\log L} - \frac{1}{\log L} \sum_n \left(\frac{\Lambda_f(n)}{\sqrt{n}} + \frac{\Lambda_{\overline{f}}(n)}{\sqrt{n}}\right) \widehat{\phi}\left(\frac{\log n}{\log L}\right)$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\gamma'}{\gamma}(f, \frac{1}{2} + it) + \frac{\gamma'}{\gamma}(\overline{f}, \frac{1}{2} - it)\right) \phi\left(\frac{t \log L}{2\pi}\right) dt,$$

where the sum is over non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $L(s, f)$ with multiplicity.

We can show using [12, Lemma 12.14] that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\gamma'}{\gamma}(f, \frac{1}{2} + it) + \frac{\gamma'}{\gamma}(f, \frac{1}{2} - it)\right) \phi\left(\frac{t \log L}{2\pi}\right) dt \ll \frac{1}{\log L}.$$

We assume that f satisfies the Ramanujan-Petersson conjecture. This assumption is true for the L-functions we consider. Then we have

$$\frac{1}{\log L} \sum_{n = p^k, k \geq 3} \left(\frac{\Lambda_f(n)}{\sqrt{n}} + \frac{\Lambda_{\overline{f}}(n)}{\sqrt{n}}\right) \widehat{\phi}\left(\frac{\log n}{\log L}\right) \ll \frac{1}{\log L},$$

by absolute convergence of the Dirichlet series. Therefore,

$$(2.1) \quad \sum_{\gamma} \phi\left(\gamma \frac{\log L}{2\pi}\right) = \widehat{\phi}(0) \frac{\log q(f)}{\log L} - \frac{1}{\log L} \sum_p \left(\frac{\Lambda_f(p)}{\sqrt{p}} + \frac{\Lambda_{\overline{f}}(p)}{\sqrt{p}}\right) \widehat{\phi}\left(\frac{\log p}{\log L}\right)$$

$$- \frac{1}{\log L} \sum_p \left(\frac{\Lambda_f(p^2)}{p} + \frac{\Lambda_{\overline{f}}(p^2)}{2}\right) \widehat{\phi}\left(\frac{2 \log p}{\log L}\right) + O\left(\frac{1}{\log L}\right).$$

We will use (2.1) for our one-level density computation.

3. Cyclic extensions of degree l

For a prime $l \geq 3$, let F be a cyclic extension of degree l over $\mathbb{Q}$. There is an $(l-1)$-to-1 correspondence between primitive Dirichlet characters χ of order l and cyclic extensioms F of degree l over $\mathbb{Q}$. Thus, counting cyclic extensions of degree l over $\mathbb{Q}$ can be reduced to counting primitive Dirichlet characters.

In [5], the author and Park summarize the following well-known results for primitive Dirichlet characters of prime order l.

Proposition 3.1. *Assume that l is a prime.*

(1) *When $l = 2$, q_χ is the conductor of a primitive quadratic character χ if and only if $q_\chi = 2^b m$ where m is an odd square-free integer and $b = 0$, 2 or 3.*

(2) *When $l > 2$, q_χ is the conductor of a primitive character of order l if and only if*

$$q_\chi = l^b \prod_{q \equiv 1 \mod l}^{finite} q, \quad b{=}0 \ or \ 2.$$

(3) *Let q be the conductor of a primitive character of order l with $\gcd(q, l) = 1$. Then, the number of primitive characters of order l with conductor q is $(l-1)^{\omega(q)}$, where $\omega(n)$ is the number of distinct prime divisors of n.*

Remark 3.2. Since there are $(l-1)$ primitive Dirichlet character of order l with conductor l^2, we can see that the number of primitive characters of order l with conductor q is also $(l-1)^{\omega(q)}$.

Recall that E is an elliptic curve over $\mathbb{Q}$ with conductor Q_E. We want to count the fields in C_{l,Q_E} by considering the primitive Dirichlet characters χ of order l with conductor q_χ coprime to Q_E. This can be achived by the following generating series:

$$\left(1 + \frac{(l-1)}{l^{2s}}\right)^{1-\delta_{l|Q_E}} \prod_{p \equiv 1 \pmod l, p \nmid Q_E} \left(1 + \frac{(l-1)}{p^s}\right) = \sum_{q=1}^{\infty} \frac{a(q)}{q^s},$$

where $a(q)$ is the number of primitive Dirichlet characters χ of order l with conductor q coprime to Q_E, which is $(l-1)^{\omega(q)}$. In [5], we showed that the Dirichlet series $\prod_{p \equiv 1 \pmod l} \left(1 + \frac{(l-1)}{p^s}\right)$ has meromorphic continuation for $\Re(s) > 1/4$ with a simple pole at $s = 1$ and a pole of a finite order at $s = 1/3$. Since the term $\frac{p^s}{p^s+(l-1)}$ for a prime divisor p of Q_E congruent to 1 modulo l has poles on the line $\Re(s) = \log_p(l-1) < 1$, the Dirichlet series $\sum_{q=1}^{\infty} a(q)q^{-s}$ is meromorphic with a simple pole at $s = 1$ for $\Re(s) > H_{Q_E}$ for some constant H_{Q_E} with $\frac{1}{3} < H_{Q_E} < 1$.

We count the primitive characters with a weight. Let ω be a non-negative Schwartz class function. Then, we define

$$W_{Q_E}(X) = \sum_{\chi}^{*} \omega\left(\frac{q_\chi}{X}\right) = \sum_{q} \omega\left(\frac{q}{X}\right) a(q),$$

where the first sum is over all primitive characters of order l with conductors q_χ coprime to Q_E.

Lemma 3.3. *Under GRH, for any $\varepsilon > 0$*

$$W_{Q_E}(X) = R_{\omega,l,Q_E} X + O_{\omega,l,Q_E,\varepsilon}(X^{H_{Q_E}+\varepsilon}) \ for \ some \ constant \ R_{\omega,l,Q_E}.$$

Proof. The proof is essentially the same as that of [5, Lemma 3.8]. □

Lemma 3.4. *Under GRH,*

$$\sum_{\chi}^{*} \omega\left(\frac{q_\chi}{X}\right) \log q_\chi = W_{Q_E}(X) \log X + O_{\omega, l, Q_E}(X).$$

Proof. The proof is essentially the same as that of [5, Lemma 3.5]. □

Lemma 3.5. *Under GRH when n is not a l-th power,*

$$\sum_{\chi}^{*} \omega\left(\frac{q_\chi}{X}\right) \chi(n) \ll_{\omega, \epsilon} n^\epsilon X^{1/2+\epsilon}.$$

Proof. The proof is essentially the same with that of [5, Lemma 3.9]. □

4. Proof of Theorem 1.3

Let f be the modular form of weight 2 with level Q_E which corresponds to the elliptic curve E and χ a primitive Dirichlet character of order l with conductor q_χ coprime to Q_E. Then, the conductor $q(f \times \chi)$ of $L(s, f \times \chi)(= L(s, E \times \chi))$ is exactly $Q_E q_\chi^2$ by a work of Barthel and Ramakrishnan [1].

The one-level density for an L-function $L(s, f \times \chi)$ is defined to be

$$D_X(f \times \chi, \phi) = \sum_{\gamma_{f \times \chi}} \phi\left(\gamma_{f \times \chi} \frac{\log L}{2\pi}\right),$$

where $\gamma_{f \times \chi}$ denote the imaginary part of a generic non-trivial zero and $L = X^2$ for a parameter X. Let ϕ be an even Schwartz class function such that its Fourier transform $\widehat{\phi}$ is compactly supported. Then, by Weil's explicit formula (2.1), we have

$D_X(f \times \chi, \phi)$

$$= \widehat{\phi}(0) \frac{\log c_{f \times \chi}}{\log L} - \frac{1}{\log L} \sum_p \frac{\log p}{p^{1/2}} \left(a_{f \times \chi}(p)\widehat{\phi}\left(\frac{\log p}{\log L}\right) + a_{f \times \overline{\chi}}(p)\widehat{\phi}\left(\frac{\log p}{\log L}\right)\right)$$

$$- \frac{1}{\log L} \sum_p \frac{\log p}{p} \left(a_{f \times \chi}(p^2)\widehat{\phi}\left(\frac{2\log p}{\log L}\right) + a_{f \times \overline{\chi}}(p^2)\widehat{\phi}\left(\frac{2\log p}{\log L}\right)\right) + O\left(\frac{1}{\log L}\right)$$

$$= \widehat{\phi}(0) \frac{2\log q_\chi}{\log L} - \frac{1}{\log L} \sum_p \frac{\log p}{p^{1/2}} \left(a_{f \times \chi}(p)\widehat{\phi}\left(\frac{\log p}{\log L}\right) + a_{f \times \overline{\chi}}(p)\widehat{\phi}\left(\frac{\log p}{\log L}\right)\right)$$

$$- \frac{1}{\log L} \sum_p \frac{\log p}{p} \left(a_{f \times \chi}(p^2)\widehat{\phi}\left(\frac{2\log p}{\log L}\right) + a_{f \times \overline{\chi}}(p^2)\widehat{\phi}\left(\frac{2\log p}{\log L}\right)\right) + O\left(\frac{1}{\log L}\right).$$

Note that $a_{f \times \chi}(n) = a_f(n) \times \chi(n)$.

Hence, we have

$$\frac{1}{W_{Q_E}(X)} \sum_{\chi}^{*} D_X(f \times \chi, \phi)\omega\left(\frac{q_\chi}{X}\right)$$

$$= \frac{\widehat{\phi}(0)}{W_{Q_E}(X)} \sum_{\chi}^{*} \frac{2\omega(q_\chi/X)\log q_\chi}{\log L} + S_1 + S_2 + O\left(\frac{1}{\log L}\right),$$

where

$$S_1 = -\frac{2}{W_{Q_E}(X)\log L}\sum_p \frac{\log p}{p^{1/2}}a_f(p)\widehat{\phi}\left(\frac{\log p}{L}\right)\left(\Re\sum_\chi^* \chi(p)\omega\left(\frac{q_\chi}{X}\right)\right),$$

$$S_2 = -\frac{2}{W_{Q_E}(X)\log L}\sum_p \frac{\log p}{p}a_f(p^2)\widehat{\phi}\left(\frac{2\log p}{L}\right)\left(\Re\sum_\chi^* \chi(p^2)\omega\left(\frac{q_\chi}{X}\right)\right).$$

By Lemma 3.4, the first sum is $\widehat{\phi}(0) + O\left(\frac{1}{\log L}\right)$. Now we assume that $\widehat{\phi}$ is supported in $(-1/2, 1/2)$. Then, by Lemma 3.5,

$$S_1 \ll \frac{1}{X\log X}\sum_{p<X^{1-2\epsilon}}\frac{p^\epsilon \log p}{p^{1/2}}X^{1/2+\epsilon} \ll \frac{1}{\log X},$$

$$S_2 \ll \frac{1}{X\log X}\sum_{p<X^{1/2-\epsilon}}\frac{p^\epsilon \log p}{p}X^{1/2+\epsilon} \ll \frac{1}{\log X}.$$

Theorem 4.1. *Let ϕ be an even Schwartz class function such that its Fourier $\widehat{\phi}$ is supported in $(-1/2, 1/2)$. Then,*

$$\lim_{X\to\infty}\frac{1}{W_{Q_E}(X)}\sum_\chi^* \omega\left(\frac{q_\chi}{X}\right)D_X(f\times\chi,\phi) = \widehat{\phi}(0).$$

Let $r_{E,\chi}$ denote the analytic rank of $L(s, f\times\chi)$. If ϕ is a non-negative valued function with $\phi(0) > 0$, by a trivial bound

$$r_{E,\chi}\phi(0) \le \sum_{\gamma_{f\times\chi}}\phi\left(\gamma_{f\times\chi}\frac{L}{2\pi}\right),$$

we have

$$\frac{\phi(0)}{W_{Q_E}(X)}\sum_\chi^* r_{E,\chi}\omega\left(\frac{q_\chi}{X}\right) \le \frac{1}{W_{Q_E}(X)}\sum_\chi^* D_X(f\times\chi,\phi)\omega\left(\frac{q_\chi}{X}\right),$$

and it implies

$$\lim_{X\to\infty}\frac{1}{W_{Q_E}(X)}\sum_\chi^* r_{E,\chi}\omega\left(\frac{q_\chi}{X}\right) = \lim_{X\to\infty}\frac{\sum_{F\in C_{l,Q_E}(X)}(r_F(E) - r_\mathbb{Q}(E))\omega\left(\frac{q_F}{X}\right)}{\mathfrak{C}_{l,Q_E}(X)} \le \frac{\widehat{\phi}(0)}{\phi(0)}.$$

In particular, we take $\phi(x) = \frac{\sin^2\left(2\pi\frac{1}{2}\sigma x\right)}{(2\pi x)^2}$. Then,

$$\widehat{\phi}(u) = \frac{1}{2}\left(\frac{1}{2}\sigma - \frac{1}{2}|u|\right) \text{ for } |u| \le \sigma,\ \phi(0) = \frac{\sigma^2}{4}\ , \text{ and } \widehat{\phi}(0) = \frac{\sigma}{4}.$$

By choosing $\sigma = 1/2$, Theorem 1.3 follows.

5. Proof of Theorem 1.4

The main tool for the one-level density of the family of elliptic curve L-functions over S_3-fields is counting cubic number fields with a finite number of local conditions. First, we introduce some notation and known results.

Let C denote a conjugacy class of the group S_3. There are three conjugacy classes $[(1)]$, $[(12)]$, and $[(123)]$ in S_3. We say that an S_3-field F satisfies the local condition $S_{p,C}$ if p is unramified in F and the conjugacy class of Frobenius automorphism

at p is C. There are two splitting types for a ramified prime in a S_3-field, which are partial ramification and total ramification and we denote them by r_1 and r_2 respectively. An S_3-field F is said to satisfy the local condition S_{p,r_i} if p is ramified in F and its splitting type is r_i.

Let $\mathcal{S} = (LC_{p_i})_{i=1}^k$ be a finite set of local conditions. Define the density of the set $\mathcal{S}$ by

$$|S_{p,C}| = \frac{|C|}{|S_3|(1+f(p))}, \quad |S_{p,r_i}| = \frac{c_i(p)}{1+f(p)}, \quad |S| = \prod_{i=1}^k |LC_{p_i}|,$$

where $c_1(p) = \frac{1}{p}$, $c_2(p) = \frac{1}{p^2}$, and $f(p) = \frac{1}{p} + \frac{1}{p^2}$.

Let

$$S_3(X, \mathcal{S}) = \{F \in S_3 \,|\, |D_F| \le X, F \text{ satisfies } \mathcal{S}\}.$$

By a recent work of Bhargava, Taniguchi and Thorne [2], we have

$$(5.1) \qquad |S_3(X, \mathcal{S})| = c_3|S|X + O\left(X^{\frac{5}{6}} + (\prod_{i=1}^k p_i)^{\frac{2}{3}} X^{\frac{2}{3}+\epsilon}\right)$$

for some explicit constant c_3.

We can compute the cardinality of $S_{3,Q_E}(X)$ by forcing all the prime divisors p of Q_E not to ramify. By (5.1), we have

$$(5.2) \qquad |S_{3,Q_E}| = c_{3,Q_E}X + O_{Q_E}\left(X^{\frac{5}{6}} + X^{\frac{2}{3}+\epsilon}\right),$$

where $c_{3,Q_E} = \left(\prod_{q|Q_E} \frac{1}{1+f(p)}\right) c_3$.

We define the one-level density for $L(s, f \times \rho)$ by

$$D_X(f \times \rho, \phi) = \sum_{\gamma_{f\times\rho}} \phi\left(\gamma_{f\times\rho} \frac{\log L}{2\pi}\right),$$

where $\gamma_{f\times\chi}$ denote the imaginary part of a generic non-trivial zero and $L = X^2$.

Once we show that for $supp(\widehat{\phi}) \subset [-\sigma, \sigma]$ with $\sigma < \frac{1}{7}$

$$(5.3) \qquad \frac{1}{|S_{3,Q_E}(X)|} \sum_{F \in S_{3,Q_E}(X)} D_X(f \times \rho, \phi) = \widehat{\phi}(0) + \frac{\phi(0)}{2} + O\left(\frac{1}{\log X}\right),$$

we have

$$\frac{\phi(0)}{|S_{3,Q_E}(X)|} \sum_{F \in S_{3,Q_E}(X)} r_{E\times\rho} = \frac{\phi(0)}{|S_{3,Q_E}(X)|} \sum_{F \in S_{3,Q_E}(X)} (r_F(E) - r_{\mathbb{Q}}(E)) \le \widehat{\phi}(0)$$

$$+ \frac{\phi(0)}{2} + O\left(\frac{1}{\log X}\right).$$

By taking

$$\phi(x) = \frac{\sin^2\left(2\pi \frac{1}{2}\sigma x\right)}{(2\pi x)^2}, \quad \widehat{\phi}(u) = \frac{1}{2}\left(\frac{1}{2}\sigma - \frac{1}{2}|u|\right) \text{ for } |u| \le \sigma$$

with $\sigma = \frac{1}{7}$, Theorem 1.4 follows.

Remark 5.1. The one-level density in (5.3) is different from that of Theorem 4.1, which means that the symmetry types for the two families are different. The symmetry type of the former one is U and the symmetry type of the latter one is O.

Now, it is left to show (5.3). Since the conductor $q(f \times \rho)$ of $L(s, f \times \rho)$ is $|D_F|^2 Q_E$ [1], by the explicit formula (2.1), we have

$$D_X(f \times \rho, \phi) = \widehat{\phi}(0) \frac{2 \log |D_F|}{\log L} - \frac{2}{\log L} \sum_p \frac{\log p}{p^{1/2}} a_{f \times \rho}(p) \widehat{\phi} \left(\frac{\log p}{\log L} \right)$$

$$- \frac{2}{\log L} \sum_p \frac{\log p}{p} a_{f \times \rho}(p^2) \widehat{\phi} \left(\frac{2 \log p}{\log L} \right) + O \left(\frac{1}{\log L} \right).$$

Since

$$a_{f \times \rho}(p) = a_f(p) a_\rho(p), \quad a_{f \times \rho}(p^2) = a_f(p^2) a_\rho(p^2),$$

we have

(5.4)

$$\frac{1}{|S_{3, Q_E}(X)|} \sum_{F \in S_{3, Q_E}(X)} D_X(f \times \rho, \phi) = \frac{2 \widehat{\phi}(0)}{|S_{3, Q_E}(X)| \log L} \sum_{F \in S_{3, Q_E}(X)} \log |D_F|$$

(5.5)
$$+ S_1 + S_2 + O \left(\frac{1}{\log L} \right),$$

where

$$S_1 = -\frac{2}{|S_{3, Q_E}(X)| \log L} \sum_p \frac{\log p}{p^{1/2}} a_f(p) \widehat{\phi} \left(\frac{\log p}{\log L} \right) \left(\sum_{F \in S_{3, Q_E}(X)} a_\rho(p) \right),$$

$$S_2 = -\frac{2}{|S_{3, Q_E}(X)| \log L} \sum_p \frac{\log p}{p} a_f(p^2) \widehat{\phi} \left(\frac{2 \log p}{\log L} \right) \left(\sum_{F \in S_{3, Q_E}(X)} a_\rho(p^2) \right).$$

We can determine $a_\rho(p)$ and $a_\rho(p^2)$ with the corresponding conjugacy class C and it is summarized in the table below.

Conjugacy class	$a_\rho(p)$	$a_\rho(p^2)$
$[(1)]$	2	2
$[(12)]$	0	2
$[(123)]$	-1	-1

Note that for $a_f(p) = \alpha(p) + \overline{\alpha(p)} = \alpha + \overline{\alpha}$, we have the following relations:

$$a_f(p^2) = \alpha^2 + \overline{\alpha}^2 = \alpha^2 + 1 + \overline{\alpha}^2 - 1 = a_{Sym^2 f}(p) - a_{\Lambda^2 f}(p)$$

$$a_{f \times f}(p) = a_f(p)^2 = \alpha^2 + 2 + \overline{\alpha}^2 = \alpha^2 + 1 + \overline{\alpha}^2 + 1 = a_{Sym^2 f}(p) + a_{\Lambda^2 f}(p).$$

Since f is self-dual, $L(s, f \times f)$ has a simple pole at $s = 1$ and $L(s, f, \Lambda^2) = \prod_{q | Q_E} \left(1 - \frac{1}{p^s} \right) \zeta(s)$ also has a simple pole at $s = 1$, from the relations above, $L(s, Sym^2 f)$ is entire. Hence, under GRH we have

(5.6)
$$\theta_f(x) = \sum_{n \le x} a_f(p^2) \log p = -x + O \left(x^{\frac{1}{2}} (\log x)(\log(x^3 c_f)) \right)$$

for some constant $c_f > 0$ [9, Theorem 5.15].

By partial summation, we have

Lemma 5.2.

$$\sum_{F \in S_{3,Q_E}(X)} \log|D_F| = |S_{3,Q_E}(X)| \log X + O_{Q_E}(X).$$

By Lemma 5.2, we can estimate the first sum in (5.4):

$$\frac{2\widehat{\phi}(0)}{|S_{3,Q_E}(X)| \log L} \sum_{F \in S_{3,Q_E}(X)} \log|D_F| = \widehat{\phi}(0) + O\left(\frac{1}{\log X}\right).$$

To control the sum S_1, we need Lemma 5.3.

Lemma 5.3.

(5.7)
$$\sum_{F \in S_{3,Q_E}(X)} a_\rho(p) = O_{Q_E}\left(X^{\frac{5}{6}} + p^{\frac{2}{3}} X^{\frac{2}{3}+\epsilon}\right).$$

Proof. By (5.1) and the table ,

$$\sum_{F \in S_{3,Q_E}(X)} a_\rho(p) = c_3 X \prod_{q|Q_E} \frac{1}{1+f(q)} \cdot \left[\frac{1 \times 2 + 0 \times 3 + (-1) \times 2}{|S_3|(1+f(p))}\right]$$

$$+ O_{Q_E}\left(X^{\frac{5}{6}} + p^{\frac{2}{3}} X^{\frac{2}{3}+\epsilon}\right),$$

$$= O_{Q_E}\left(X^{\frac{5}{6}} + p^{\frac{2}{3}} X^{\frac{2}{3}+\epsilon}\right).$$

$\square$

Again by (5.1) and the table we can show that

Lemma 5.4.

(5.8)
$$\sum_{F \in S_{3,Q_E}(X)} a_\rho(p^2) = c_{3,Q_E} X + O_{Q_E}\left(\frac{1}{p}X + X^{\frac{5}{6}} + p^{\frac{2}{3}} X^{\frac{2}{3}+\epsilon}\right).$$

Assume that support of $\widehat{\phi} \subset [-\sigma, \sigma]$ for some $\sigma < \frac{1}{7}$. By Lemma 5.3 and (5.2),

(5.9)
$$S_1 \ll \frac{X^{\frac{5}{6}}}{X \log X} \sum_{p \le X^{2\sigma}} \frac{\log p}{p^{\frac{1}{2}}} + \frac{X^{\frac{2}{3}+\epsilon}}{X \log X} \sum_{p \le X^{2\sigma}} p^{\frac{2}{3}-\frac{1}{2}} \log p$$

$$\ll \frac{X^{\frac{5}{6}+\sigma}}{X \log X} + \frac{X^{\frac{2}{3}+\frac{7\sigma}{3}+\epsilon}}{X \log X} \ll \frac{1}{\log X}.$$

For S_2, we have

$$S_2 = -\frac{2}{|S_{3,Q_E}(X)| \log L} \sum_p \frac{\log p}{p} a_f(p^2) \widehat{\phi}\left(\frac{2\log p}{\log L}\right)$$

$$\left(c_{3,Q_E} X + O_{Q_E}\left(\frac{1}{p}X + X^{\frac{5}{6}} + p^{\frac{2}{3}} X^{\frac{2}{3}+\epsilon}\right)\right)$$

$$= -\frac{c_{3,Q_E} X}{|S_{3,Q_E}(X)|} \sum_p \frac{2a_f(p^2) \log p}{p \log L} \widehat{\phi}\left(\frac{2\log p}{\log L}\right)$$

$$+ O_{Q_E}\left(\frac{1}{\log X} + \frac{X^{\frac{5}{6}} \log\log X}{X} + \frac{X^{\frac{2}{3}+\frac{2\sigma}{3}+\epsilon}}{X \log X}\right).$$

By summation by parts we have

$$\sum_p \widehat{\phi}\left(\frac{2\log p}{\log L}\right)\frac{2a_f(p^2)\log p}{p\log L} = \int_1^\infty \widehat{\phi}\left(\frac{2\log t}{\log L}\right)\frac{2d\theta_f(t)}{t\log L}.$$

Using (5.6), we can show that

$$(5.10) \qquad S_2 = -\frac{c_{3,Q_E}X}{|S_{3,Q_E}(X)|}\sum_p \widehat{\phi}\left(\frac{2\log p}{\log L}\right)\frac{2a_f(p^2)\log p}{p\log L} + O_{Q_E}\left(\frac{1}{\log X}\right)$$

$$= \frac{1}{2}\phi(0) + O_{Q_E}\left(\frac{1}{\log X}\right).$$

By Lemma 5.2, (5.9) and (5.10), we establish the one-level density (5.3) for any $\sigma < \frac{1}{7}$.

For S_4-fields and S_5-fields, using the same arguments, we can take that $\sigma = 1/864$ and $1/2400$ respectively by a work of the author and Kim [3]. However, this gives a poor bound.

Acknowledgments

References

[1] L. Barthel and D. Ramakrishnan, *A nonvanishing result for twists of L-functions of* GL(*n*), Duke Math. J. **74** (1994), no. 3, 681–700, DOI 10.1215/S0012-7094-94-07425-5. MR1277950

[2] M. Bhargava, T. Taniguchi, and F. Thorne, *Improved error estimates for the Davenport–Heilbronn theorems*, `arXiv:2107.12819`, 2021.

[3] P. J. Cho and H. H. Kim, *Low lying zeros of Artin L-functions*, Math. Z. **279** (2015), no. 3-4, 669–688, DOI 10.1007/s00209-014-1387-2. MR3318244

[4] P. J. Cho and H. H. Kim, *Central limit theorem for Artin L-functions*, Int. J. Number Theory **13** (2017), no. 1, 1–14, DOI 10.1142/S1793042117500014. MR3573409

[5] P. J. Cho and J. Park, *Dirichlet characters and low-lying zeros of L-functions*, J. Number Theory **212** (2020), 203–232, DOI 10.1016/j.jnt.2019.12.001. MR4080062

[6] C. David, J. Fearnley, and H. Kisilevsky, *Vanishing of L-functions of elliptic curves over number fields*, Ranks of elliptic curves and random matrix theory, London Math. Soc. Lecture Note Ser., vol. 341, Cambridge Univ. Press, Cambridge, 2007, pp. 247–259, DOI 10.1017/CBO9780511735158.016. MR2322350

[7] D. Goldfeld, *Conjectures on elliptic curves over quadratic fields*, Number theory, Carbondale 1979 (Proc. Southern Illinois Conf., Southern Illinois Univ., Carbondale, Ill., 1979), Lecture Notes in Math., vol. 751, Springer, Berlin, 1979, pp. 108–118. MR564926

[8] D. R. Heath-Brown, *The average analytic rank of elliptic curves*, Duke Math. J. **122** (2004), no. 3, 591–623, DOI 10.1215/S0012-7094-04-12235-3. MR2057019

[9] H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications, vol. 53, American Mathematical Society, Providence, RI, 2004, DOI 10.1090/coll/053. MR2061214

[10] R. J. Lemke Oliver and F. Thorne, *Rank growth of elliptic curves in non-abelian extensions*, Int. Math. Res. Not. IMRN **24** (2021), 18411–18441, DOI 10.1093/imrn/rnz307. MR4365991

[11] B. Mazur and K. Rubin, *Diophantine stability*, Amer. J. Math. **140** (2018), no. 3, 571–616, DOI 10.1353/ajm.2018.0014. With an appendix by Michael Larsen. MR3805014

[12] H. L. Montgomery and R. C. Vaughan, *Multiplicative number theory. I. Classical theory*, Cambridge Studies in Advanced Mathematics, vol. 97, Cambridge University Press, Cambridge, 2007. MR2378655

[13] M. Rubinstein, *Low-lying zeros of L-functions and random matrix theory*, Duke Math. J. **109** (2001), no. 1, 147–181, DOI 10.1215/S0012-7094-01-10916-2. MR1844208

[14] J. H. Silverman, *The arithmetic of elliptic curves*, 2nd ed., Graduate Texts in Mathematics, vol. 106, Springer, Dordrecht, 2009, DOI 10.1007/978-0-387-09494-6. MR2514094

DEPARTMENT OF MATHEMATICAL SCIENCES, ULSAN NATIONAL INSTITUTE OF SCIENCE AND TECHNOLOGY, ULSAN, KOREA

Email address: petercho@unist.ac.kr