Journal of Applied Analysis and Computation
Volume 11, Number 6, December 2021, 2921–2946
Website:http://www.jaac-online.com
DOI:10.11948/20210063

MULTIPLICITY OF SOLUTIONS FOR DOUBLE PHASE EQUATIONS WITH CONCAVE-CONVEX NONLINEARITIES

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Abstract This paper is devoted to the study of the L^∞-bound of solutions to a double-phase problem with concave-convex nonlinearities by applying the De Giorgi's iteration method and the localization method. Employing this and a variant of Ekeland's variational principle, we provide the existence of at least two distinct nontrivial solutions belonging to L^∞-space when the convex term does not satisfy the Ambrosetti-Rabinowitz condition in general. In addition, our problem has a sequence of multiple small energy solutions whose L^∞-norms converge to zero. To achieve this result, we utilize the modified functional method and the dual fountain theorem as the main tools.

Keywords Double phase equations, De Giorgi iteration, modified functional methods, dual fountain theorem.

MSC(2010) 35B45, 35J20, 35J62, 46E30.

1. Introduction

The study of differential equations and variational problems involving double phase operator has been paid to a great deal of attention in the recent decades; see [4, 7, 10–12, 23, 26, 30]. Such operator can be corroborated as a model for many physical phenomena which arise in the research of elasticity, strongly anisotropic materials and Lavrentiev's phenomenon; see [39–42] for more details. In particular, Zhikov examined the behavior of strongly anisotropic materials and found that their hardening properties varied sharply with the point. This phenomenon is described the following functional

$$\int_\Omega (|\nabla v|^p + a(x)|\nabla v|^q)\, dx, \tag{1.1}$$

where the function $a(\cdot)$ was used as an aid to regulating the mixture between two different materials, with power hardening of rates p and q, respectively. The functional (1.1) belongs to the class of the integral functionals with nonstandard growth

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condition. Recently, Colombo and Mingione [11] have established the regularity theory for minimizers of (1.1) and obtained sharp results for $q > p$ and $a(\cdot) > 0$. More, recently some noteworthy local regularity results for minimizers of two phase functionals have been produced by Mingione and coworkers in [4, 11, 12]. The study on unbalanced double phase Dirichlet problems with variable growth was considered by the recent works of Cencelj-Radulescu-Repovs [7]. Also, Colasuonno-Squassina [10] dealt with eigenvalue problems for Dirichlet double phase operators. A remarkable inquiry of some of the recent works on two phase equations, can be found in Radulescu [30].

This paper is concerned with the following double-phase problem by the case of a combined effect of concave-convex nonlinearities:

$$\begin{cases} -\operatorname{div}\left(|\nabla v|^{p-2}\nabla v + a(x)|\nabla v|^{q-2}\nabla v\right) = \lambda\varrho(x)|v|^{\gamma-2}v + \mu h(x,v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, $1 < \gamma < p < q < N$, $q/p < 1 + 1/N$, $\varrho \in L^\infty(\Omega)$ is nonnegative function, λ and μ are positive real parameters, $a : \Omega \to [0, +\infty)$ belongs to $L^1(\Omega)$ and $h : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition.

From a pure mathematical point of view, many researchers have extensively studied about nonlinear elliptic equations involving the concave-convex nonlinearities (see [5, 6, 8, 14, 19, 21, 31, 37, 38]) since the celebrated paper [1] of Ambrosetti, Brezis and Cerami for the Laplacian problem:

$$\begin{cases} -\triangle v = \lambda|v|^{q-2}v + |v|^{h-2}v & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < q < 2 < h < 2^* := \begin{cases} \frac{2N}{N-2} & \text{if } N > 2, \\ +\infty & \text{if } N = 1, 2. \end{cases}$

In particular, the multiplicity result of solutions to the concave-convex-type elliptic problems driven by a nonlocal integro-differential operator has been proposed in [8]; see also [5, 25, 38] and [19] for $p(x)$-Laplacian equations. For quasilinear elliptic equations involving nonhomogeneous operators which subject to Dirichlet boundary conditions, the authors in [6] obtained the existence and multiplicity of solutions by making use of the well-known Nehari manifold method as the main tool. Very recently, Kim *et al.* [20] have studied the following concave-convex problems of Schrödinger type

$$\begin{cases} -\operatorname{div}(\varphi'(|\nabla v|^2)\nabla v) + V(x)|v|^{\alpha-2}v = \lambda\rho(x)|v|^{r-2}v + h(x,v), \\ v(x) \to 0, \quad \text{as } |x| \to \infty, \end{cases} \quad \text{in } \mathbb{R}^N,$$

where $N \geq 2$, $1 < p < q < N$, $1 < \alpha \leq p^* q'/p'$, $\alpha < q$, $1 < r < \min\{p, \alpha\}$, $\varphi(t)$ behaves like $t^{q/2}$ for small t and $t^{p/2}$ for large t, and p' and q' are the conjugate exponents of p and q, respectively. They proved the existence of two distinct nontrivial solutions when the convex term h fulfils the condition of Ambrosetti-Rabinowitz type in [2], that is, there exists a constant $\theta > 0$ such that $\theta > q$ and

$$0 < \theta H(x,t) \leq h(x,t)t, \quad \text{for all } t \in \mathbb{R} \setminus \{0\} \text{ and } x \in \mathbb{R}^N, \tag{1.3}$$

where $H(x,t) = \int_0^t h(x,s)\,ds$. Also they established this existence result when (1.3) was superseded by the condition originally introduced by Oanh and Phuong [29], namely, there exist $\nu > \alpha$ and $M > 0$ such that

$$h(x,t)t - \nu H(x,t) \geq -\varrho\,|t|^{\alpha} - \beta(x) \quad \text{for all} \quad x \in \mathbb{R}^N \quad \text{and} \quad |t| \geq M,$$

where $\varrho \geq 0$ and $\beta \in L^1(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ with $\beta(x) \geq 0$.

In this regard, the first aim is to establish the existence result that (1.2) admits at least two distinct nontrivial solutions belonging to L^{∞}-space when the condition on h does not satisfy the Ambrosetti-Rabinowitz condition in general. The second aim is to investigate the existence of small energy solutions for problem (1.2) whose L^{∞}-norms converge to zero, depends only on the local behavior and conditions on $h(x,t)$, and only a sufficiently small t is required. In order to achieve these main results, we firstly show the uniform boundedness for weak solutions to problem (1.2). However, as far as we know, there are no results about L^{∞}-bound for weak solutions to the double-phase problem involving concave-convex nonlinearities. To overcome this difficulty, we employ the De Giorgi's iteration method and a truncated energy technique, originally given in [34], as the primary tools for obtaining this result. In particular this double phase operator has more complex nonlinearities than the p-Laplacian and the fractional p-Laplacian, so more elaborate analysis has to be carefully carried out. Next, with the help of this, taking into account the mountain pass theorem and a variant of Ekeland's variational principle, we obtain our first existence result. Finally we get the existence of a sequence of infinitely many small energy solutions whose converge to 0 in L^{∞}-norm. This is originally motivated by Wang [35] that nonlinear boundary value problems

$$\begin{cases} -\Delta v = \lambda\,|v|^{q-1}\,v + h(x,v) & \text{in } \Omega, \\ v = 0 & \text{on } \mathbb{R}^N \backslash \Omega \end{cases}$$

have a sequence of solutions, where $0 < q < 1$ and h is regarded as a perturbation term. He made use of the modified functional method and global variational formulation in [18] as the main tools, in order to establish this existence result that is a local phenomenon and is forced by the sublinear term. However, we design our consequence in a different approach from the previous works [9, 17, 24, 28, 32, 35]. More precisely, in contrast to aforementioned papers which establish the existence of such a sequence of solutions belonging to the L^{∞} space, we point out that we take the dual fountain theorem instead of global variational formulation into account. As we know, these results that apply the dual fountain theorem to derive the existence of small energy solutions to elliptic equations of variational type do not ensure the boundedness of solutions; see [13, 33] and the references therein. On the other hand, our arguments together with the modified functional method and the dual fountain theorem allow us to obtain the existence of multiple small-energy solutions converging to zero in L^{∞} space.

As seen above, the main purpose of the present paper is to ensure the existence and uniform boundedness of nontrivial solutions for double-phase problem (1.2) with the nonlinear term of concave-convex type, by virtue of variational tools such as the mountain pass theorem (see [2]), a variant of Ekeland's variational principle (see [3]) and the dual fountain theorem (see [36]). To the best of our knowledge, the present paper is the first attempt to study the existence and regularity type results for the concave-convex-type double phase problems.

This paper's outline is the following: we firstly present some necessary preliminary knowledge of function spaces. Next we give the variational framework associated with problem (1.2) and then we establish the results about L^∞-bound for weak solutions to the double-phase problem involving concave-convex nonlinearities by applying the De Giorgi's iteration method and a truncated energy technique. Finally, under suitable conditions on the convex term h, we carry out various existence results of nontrivial solutions by utilizing as the major tools the variational principle.

2. Preliminaries

In order to consider problem (1.2), we need some elementary facts on the space $W^{1,\mathcal{H}}(\Omega)$ which is called Musielak-Orlicz-Sobolev space. For this reason, we will recall some properties involving the Musielak-Orlicz spaces, which can be found in [10, 16] and references therein.

Denote by $N(\Omega)$ the set of all generalized N-functions. For $1 < p < q$ and $0 \le a(\cdot) \in L^1(\Omega)$, we define

$$\mathcal{H}(x,t) := t^p + a(x)t^q, \quad \forall (x,t) \in \Omega \times [0,+\infty).$$

It is clear that $\mathcal{H} \in N(\Omega)$ is locally integrable and

$$\mathcal{H}(x,2t) \le 2^q \mathcal{H}(x,t), \quad \forall (x,t) \in \Omega \times [0,+\infty),$$

which is called condition $(\triangle_2)$.

The Musielak-Orlicz space $L^{\mathcal{H}}(\Omega)$ is defined by

$$L^{\mathcal{H}}(\Omega) := \left\{ v : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega \mathcal{H}(x,|v|)\, dx < +\infty \right\},$$

endowed with the Luxemburg norm

$$\|v\|_{\mathcal{H}} := \inf \left\{ \lambda > 0 : \int_\Omega \mathcal{H}\left(x, \left|\frac{v}{\lambda}\right|\right) dx \le 1 \right\}.$$

The Musielak-Orlicz-Sobolev space $W^{1,\mathcal{H}}(\Omega)$ is defined by

$$W^{1,\mathcal{H}}(\Omega) := \{ v \in L^{\mathcal{H}}(\Omega) : |\nabla v| \in L^{\mathcal{H}}(\Omega) \},$$

and it is equipped with the norm

$$\|v\|_{1,\mathcal{H}} := \|v\|_{\mathcal{H}} + \|\nabla v\|_{\mathcal{H}}.$$

Here we write $\|\nabla v\|_{\mathcal{H}} = \| \, |\nabla v| \, \|_{\mathcal{H}}$ for convenience in writing. We denote by $W_0^{1,\mathcal{H}}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$.

Lemma 2.1 (Proposition 2.1, [26]). *Set*

$$\rho_{\mathcal{H}}(v) = \int_\Omega (|v|^p + a(x)|v|^q)\, dx.$$

For $v \in L^{\mathcal{H}}(\Omega)$, we have *(i)* *for* $v \neq 0, \|v\|_{\mathcal{H}} = \lambda$ *iff* $\rho_{\mathcal{H}}(\frac{v}{\lambda}) = 1$;

(ii) $\|v\|_{\mathcal{H}} < 1(= 1; > 1)$ *iff* $\rho_{\mathcal{H}}(v) < 1(= 1; > 1)$;

(iii) *if* $\|v\|_{\mathcal{H}} > 1$, *then* $\|v\|^p_{\mathcal{H}} \leq \rho_{\mathcal{H}}(v) \leq \|v\|^q_{\mathcal{H}}$;

(iv) *if* $\|v\|_{\mathcal{H}} < 1$, *then* $\|v\|^q_{\mathcal{H}} \leq \rho_{\mathcal{H}}(v) \leq \|v\|^p_{\mathcal{H}}$;

In the following, the notation $X \hookrightarrow Y$ means that the space X is *continuously* imbedded into the space Y, while $X \hookrightarrow\hookrightarrow Y$ means that X is *compactly* imbedded into Y.

Lemma 2.2 ([10]). *Let us put* $p^* = Np/(N-p)$, *if* $p < N$, $p^* := +\infty$ *otherwise. Then the followings hold.*

(i) *The spaces* $W^{1,\mathcal{H}}(\Omega)$ *and* $W_0^{1,\mathcal{H}}(\Omega)$ *are separable reflexible Banach space.*

(ii) *If* $p \neq N$, *then*

$$W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega) \quad \text{for all } r \in [1, p^*].$$

If $p = N$, *then*

$$W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega) \quad \text{for all } r \in [1, +\infty].$$

(iii) *If* $p \leq N$, *then*

$$W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega) \quad \text{for all } r \in [1, p^*).$$

If $p > N$, *then*

$$W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega).$$

(iv) *[Poincaré inequality] There exists a constant* $C > 0$ *such that*

$$\|v\|_{\mathcal{H}} \leq C\|\nabla v\|_{\mathcal{H}}$$

for any $v \in W_0^{1,\mathcal{H}}(\Omega)$.

3. Variational setting and the main results

In view of Lemma 2.2 (iv), we know that $\|\nabla v\|_{\mathcal{H}}$ and $\|v\|_{1,\mathcal{H}}$ are equivalent norms on $W_0^{1,\mathcal{H}}(\Omega)$. Hence we equip the space $W_0^{1,\mathcal{H}}(\Omega)$ with the equivalent norm $\|\nabla v\|_{\mathcal{H}}$. Throughout this paper, let $X := W_0^{1,\mathcal{H}}(\Omega)$ with the norm

$$\|v\|_X = \inf\left\{\lambda > 0 : \int_\Omega \mathcal{H}\left(x, \left|\frac{\nabla v}{\lambda}\right|\right) dx \leq 1\right\}.$$

Let us define the functional $\Phi : X \to \mathbb{R}$ by

$$\Phi(v) = \int_\Omega \left(\frac{1}{p}|\nabla v|^p + \frac{a(x)}{q}|\nabla v|^q\right) dx.$$

Then it is standard to check that $\Phi \in C^1(X, \mathbb{R})$, and double-phase operator

$$-\mathrm{div}(|\nabla v|^{p-2}\nabla v + a(x)|\nabla v|^{q-2}\nabla v)$$

is the derivative operator of Φ in the weak sense. We denote $\Phi' : X \to X^*$ with

$$\langle \Phi'(v), u\rangle = \int_\Omega (|\nabla v|^{p-2}\nabla v \cdot \nabla u + a(x)|\nabla v|^{q-2}\nabla v \cdot \nabla u)\, dx.$$

for all $u, v \in X$. Here X^* denotes the dual space of X and $\langle \cdot, \cdot\rangle$ denotes the pairing between X and X^*.

Lemma 3.1 (Proposition 3.1, [26]). *The double phase operator $\Phi' : X \to X^*$ has the following properties:*

(i) Φ' is a continuous, bounded and strictly monotone operator;

(ii) Φ' is a mapping of type (S_+), i.e. if $v_n \rightharpoonup v$ in X and

$$\limsup_{n\to\infty} \langle \Phi'(v_n) - \Phi'(v), v_n - v\rangle \le 0,$$

then $v_n \to v$ in X;

(iii) Φ' is a homeomorphism.

Definition 3.1. We say that $v \in X$ is a weak solution of problem (1.2) if

$$\begin{aligned}&\int_\Omega \left(|\nabla v|^{p-2} \nabla v \cdot \nabla u + a(x) |\nabla v|^{q-2} \nabla v \cdot \nabla u\right) dx \\ =&\lambda \int_\Omega \varrho(x) |v|^{\gamma-2} vu\, dx + \mu \int_\Omega h(x, v)u\, dx,\end{aligned}$$

for any $u \in X$.

We assume that

$(h1)$ $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function;

$(h2)$ there exists a nonnegative function $\sigma \in L^\infty(\Omega)$ such that

$$|h(x, t)| \le \sigma(x)|t|^{r-1},$$

for all $(x, t) \in \Omega \times \mathbb{R}$ and $q < r < p^*$, where $p^* = \frac{Np}{N-p}$ is the critical exponent;

$(h3)$ $\lim_{|t|\to\infty} \frac{H(x,t)}{|t|^q} = \infty$ uniformly for almost all $x \in \Omega$, where $H(x, t) = \int_0^t h(x, s)\, ds$.

Let the functional Ψ_1, Ψ_2 and $\Psi : X \to \mathbb{R}$ be defined by

$$\Psi_1(v) = \frac{1}{\gamma} \int_\Omega \varrho(x) |v|^\gamma\, dx,\ \Psi_2(v) = \int_\Omega H(x, v)\, dx \ \text{ and } \ \Psi(v) = \lambda\Psi_1(v) + \mu\Psi_2(v).$$

Then, it is easy to check that $\Psi \in C^1(X, \mathbb{R})$ and its Fréchet derivative is

$$\langle \Psi'(v), u\rangle = \lambda \int_\Omega \varrho(x) |v|^{\gamma-2} vu\, dx + \mu \int_\Omega h(x, v)u\, dx$$

for any $v, u \in X$. Subsequently, the functional $\varphi : X \to \mathbb{R}$ is defined by

$$\varphi(v) = \Phi(v) - \Psi(v).$$

Then it follows that the functional $\varphi \in C^1(X, \mathbb{R})$ and its Fréchet derivative is

$$\begin{aligned}\langle \varphi'(v), u\rangle = &\int_\Omega \left(|\nabla v|^{p-2}\nabla v\cdot\nabla u + a(x)|\nabla v|^{q-2}\nabla v\cdot\nabla u\right)dx\\ &-\lambda\int_\Omega \varrho(x)|v|^{\gamma-2}vu\,dx - \mu\int_\Omega h(x,v)u\,dx\end{aligned}$$

for any $v, u \in X$.

First of all we present the L^∞-bound of solutions to the problem (1.2). To do this, we need the following important Lemma which is given in the paper [34, Lemma 2.2].

Lemma 3.2. *Let $\{\mathcal{Z}_n\}_{n=1}^\infty$ be a sequence of positive numbers, satisfying the recursion inequality*

$$\mathcal{Z}_{n+1} \le cb^n \mathcal{Z}_n^{1+\delta}, \qquad n = 0, 1, 2, \cdots$$

for some $b > 1$, $c > 0$ and $\delta > 0$. If $\mathcal{Z}_0 \le \min\{1, c^{(-1)/\delta}b^{(-1)/\delta^2}\}$ then $\mathcal{Z}_n \le 1$ for some $n \in \mathbb{N}\cup\{0\}$. Moreover,

$$\mathcal{Z}_n \le \min\left\{1, c^{(-1)/\delta}b^{(-1)/\delta^2}b^{(-n)/\delta}\right\}$$

for any $n \ge n_0$, where n_0 is the smallest $n \in \mathbb{N}\cup\{0\}$ satisfying $\mathcal{Z}_n \le 1$. In particular, $\mathcal{Z}_n \to 0$ as $n \to \infty$.

Invoking Lemma 3.2, we prove the following consequence, which is a regularity type result via De Giorgi technique and the localization method.

Proposition 3.1. *Assume that $(h1)$ and $(h2)$ hold. If v is a weak solution of the problem (1.2), then $v \in L^\infty(\Omega)$ and there exist positive constants C, η independent of v such that*

$$\|v\|_{L^\infty(\Omega)} \le C\|v\|_{L^r(\Omega)}^\eta.$$

Proof. Let $\mathcal{A}_k = \{x \in \Omega : v(x) > k\}$, $\widetilde{\mathcal{A}}_k = \{x \in \Omega : -v(x) > k\}$ for $k > 0$. Note that $|\mathcal{A}_k|$ and $|\widetilde{\mathcal{A}}_k|$ are finite for any $k \in \mathbb{N}$, where $|\cdot|$ denotes the Lebesgue measure on Ω. Taking a test function $u = (v-k)_+ \in X$, we obtain from Definition 3.1 that

$$\int_\Omega \left(|\nabla v|^{p-2} + a(x)|\nabla v|^{q-2}\right)\nabla v\cdot\nabla u\,dx = \lambda\int_\Omega \varrho(x)|v|^{\gamma-2}vu\,dx + \mu\int_\Omega h(x,v)u\,dx.$$

Equivalently,

$$\begin{aligned}&\int_{\mathcal{A}_k}\left(|\nabla v|^{p-2} + a(x)|\nabla v|^{q-2}\right)|\nabla v|^2\,dx\\ =&\lambda\int_{\mathcal{A}_k}\varrho(x)|v|^{\gamma-2}v(v-k)\,dx + \mu\int_{\mathcal{A}_k}h(x,v)(v-k)\,dx.\end{aligned}$$

Hence, since $v \ge v-k > 0$ on A_k, by assumption $(h2)$, we note that

$$\begin{aligned}&\int_{\mathcal{A}_k}\left(|\nabla v|^p + a(x)|\nabla v|^q\right)dx\\ =&\lambda\int_{\mathcal{A}_k}\varrho(x)|v|^{\gamma-2}v(v-k)\,dx + \mu\int_{\mathcal{A}_k}h(x,v)(v-k)\,dx\end{aligned}$$

$$
\begin{aligned}
&\leq \lambda\|\varrho\|_{L^\infty(\Omega)} \int_{\mathcal{A}_k} |v|^{\gamma-2} v(v-k)\,dx + \mu \int_{\mathcal{A}_k} \sigma(x)|v|^{r-1}(v-k)\,dx \\
&\leq \lambda\|\varrho\|_{L^\infty(\Omega)} \int_{\mathcal{A}_k} |v|^{\gamma}\ dx + \mu\|\sigma\|_{L^\infty(\Omega)} \int_{\mathcal{A}_k} |v|^{r}\,dx \\
&\leq \left(1+k^{\gamma-r}\right)\left(\lambda\|\varrho\|_{L^\infty(\Omega)} + \mu\|\sigma\|_{L^\infty(\Omega)}\right) \int_{\mathcal{A}_k} v^r dx. \qquad (3.1)
\end{aligned}
$$

Put $k_n := k_*(2-1/2^n), n = 0,1,2,\cdots$, with $k_*>0$ specified later and

$$\mathcal{Z}_n := \int_{\mathcal{A}_{k_n}} (v-k_n)^r\,dx.$$

Since $k_* \leq k_n \leq k_{n+1} \leq 2k_*$ for all $n \in \mathbb{N}$, we have

$$\int_{\mathcal{A}_{k_n}} (v-k_n)^r\,dx \geq \int_{\mathcal{A}_{k_{n+1}}} v^r \left(1-\frac{k_n}{k_{n+1}}\right)^r dx \geq \int_{\mathcal{A}_{k_{n+1}}} \frac{v^r}{2^{r(n+2)}}\,dx$$

and therefore

$$\mathcal{Z}_n \geq \int_{\mathcal{A}_{k_{n+1}}} \frac{v^r}{2^{r(n+2)}}\,dx.$$

Thus

$$\int_{\mathcal{A}_{k_{n+1}}} v^r\,dx \leq d_1^{n+2}\mathcal{Z}_n, \qquad (3.2)$$

where $d_1 = 2^r > 1$. For the Lebesgue measure of $\mathcal{A}_{k_{n+1}}$, we deduce that

$$\left|\mathcal{A}_{k_{n+1}}\right| \leq \int_{\mathcal{A}_{k_{n+1}}} \left(\frac{v-k_n}{k_{n+1}-k_n}\right)^r dx \leq \int_{\mathcal{A}_{k_n}} \left(\frac{2^{n+1}}{k_*}\right)^r (v-k_n)^r\,dx.$$

So one has

$$|\mathcal{A}_{k_{n+1}}| \leq \frac{d_1^{n+1}}{k_*^r}\mathcal{Z}_n. \qquad (3.3)$$

Note that $1+k_*^{\gamma-r} \leq 2(1+k_*^{-r})$. Then it follows from relations (3.1)–(3.3) that we obtain

$$
\begin{aligned}
&\int_{\mathcal{A}_{k_{n+1}}} \left(|\nabla v|^p + a(x)\,|\nabla v|^q\right)\,dx \\
&\leq \left(1+k_{n+1}^{\gamma-r}\right)\left(\lambda\|\varrho\|_{L^\infty(\Omega)} + \mu\|\sigma\|_{L^\infty(\Omega)}\right) \int_{\mathcal{A}_{k_{n+1}}} v^r dx \\
&\leq \left(1+k_*^{\gamma-r}\right)\left(\lambda\|\varrho\|_{L^\infty(\Omega)} + \mu\|\sigma\|_{L^\infty(\Omega)}\right) d_1^{n+2}\mathcal{Z}_n + |\mathcal{A}_{k_{n+1}}| \\
&\leq \left(1+k_*^{\gamma-r}\right)\left(\lambda\|\varrho\|_{L^\infty(\Omega)} + \mu\|\sigma\|_{L^\infty(\Omega)}\right) d_1^{n+2}\mathcal{Z}_n + \frac{d_1^{n+1}}{k_*^r}\mathcal{Z}_n \\
&\leq d_1^n\mathcal{Z}_n\left[2\left(1+k_*^{-r}\right)\left(\lambda\|\varrho\|_{L^\infty(\Omega)} + \mu\|\sigma\|_{L^\infty(\Omega)}\right) d_1^2 + d_1k_*^{-r}\right] \\
&\leq d_1^n\mathcal{Z}_n\left[2\left(1+k_*^{-r}\right)\left(\lambda\|\varrho\|_{L^\infty(\Omega)} + \mu\|\sigma\|_{L^\infty(\Omega)}\right) d_1^2 + d_1k_*^{-r} + 2d_1 + d_1k_*^{-r}\right] \\
&\leq 2\left(1+k_*^{-r}\right)\left[\left(\lambda\|\varrho\|_{L^\infty(\Omega)} + \mu\|\sigma\|_{L^\infty(\Omega)}\right) d_1^2 + d_1\right] d_1^n\mathcal{Z}_n \\
&= d_2d_1^n\mathcal{Z}_n, \qquad (3.4)
\end{aligned}
$$

where $\widetilde{C} := \lambda\|\varrho\|_{L^\infty(\Omega)} + \mu\|\sigma\|_{L^\infty(\Omega)}$ and $d_2 := 2(1+k_*^{-r})\left(\widetilde{C}d_1^2 + d_1\right)$. Define

$$\tilde{r} := \begin{cases} \frac{r+p^*}{2} & \text{if } p^* < \infty, \\ r+1 & \text{if } p^* = \infty. \end{cases}$$

Using the Hölder inequality and Lemma 2.2, we get

$$\begin{aligned} \int_{\mathcal{A}_{k_{n+1}}} (v-k_{n+1})_+^r \, dx &\le \left(\int_\Omega \left\{(v-k_{n+1})_+^r\right\}^{\frac{\tilde{r}}{r}} \, dx \right)^{\frac{r}{\tilde{r}}} \left|\mathcal{A}_{k_{n+1}}\right|^{\frac{\tilde{r}-r}{\tilde{r}}} \\ &= \|(v-k_{n+1})_+\|_{L^{\tilde{r}}(\Omega)}^r \left|\mathcal{A}_{k_{n+1}}\right|^{1-\frac{r}{\tilde{r}}} \\ &\le C_{imb}^r \|(v-k_{n+1})_+\|_X^r \left|\mathcal{A}_{k_{n+1}}\right|^{1-\frac{r}{\tilde{r}}}, \end{aligned} \tag{3.5}$$

where C_{imb} is a imbedding constant of $X \hookrightarrow L^{\tilde{r}}(\Omega)$. By Lemma 2.1 and (3.4), we get

$$\begin{aligned} \|(v-k_{n+1})_+\|_X^\tau &\le \int_\Omega \left(\left|\nabla (v-k_{n+1})_+\right|^p + a(x)\left|\nabla (v-k_{n+1})_+\right|^q \right) dx \\ &= \int_{\mathcal{A}_{k_{n+1}}} \left(|\nabla v|^p + a(x)|\nabla v|^q\right) dx \\ &\le d_2 d_1^n \mathcal{Z}_n \end{aligned} \tag{3.6}$$

where τ is either p or q. We deduce from (3.3), (3.5), (3.6) and Lemma 2.2 that

$$\begin{aligned} \mathcal{Z}_{n+1} &= \int_{A_{k_{n+1}}} (v-k_{n+1})^r \, dx \\ &\le C_{imb}^r \|(v-k_{n+1})_+\|_X^r \left|\mathcal{A}_{k_{n+1}}\right|^{1-\frac{r}{\tilde{r}}} \\ &\le C_{imb}^r \left(d_2 d_1^n \mathcal{Z}_n\right)^{\frac{r}{\tau}} \left|\mathcal{A}_{k_{n+1}}\right|^{1-\frac{r}{\tilde{r}}} \\ &\le C_{imb}^r \left[2(1+k_*^{-r})\left(\widetilde{C}d_1^2+d_1\right) d_1^n \mathcal{Z}_n\right]^{\frac{r}{\tau}} \left(\frac{d_1^{n+1}}{k_*^r}\mathcal{Z}_n\right)^{1-\frac{r}{\tilde{r}}} \\ &= C_{imb}^r \left(2\left(\widetilde{C}d_1^2+d_1\right)\right)^{\frac{r}{\tau}} \left(1+k_*^{-r}\right)^{\frac{r}{\tau}} d_1^{n\frac{r}{\tau}+(n+1)\left(1-\frac{r}{\tilde{r}}\right)} k_*^{-r\left(1-\frac{r}{\tilde{r}}\right)} \mathcal{Z}_n^{\frac{r}{\tau}+1-\frac{r}{\tilde{r}}} \\ &\le C_0 C_{imb}^r \left(2\left(\widetilde{C}d_1^2+d_1\right)\right)^{\frac{r}{\tau}} d_1^{1-\frac{r}{\tilde{r}}} \left(1+k_*^{-r\frac{r}{\tau}}\right) k_*^{-r\left(1-\frac{r}{\tilde{r}}\right)} d_1^{n\left(1-\frac{r}{\tilde{r}}+\frac{r}{\tau}\right)} \mathcal{Z}_n^{1-\frac{r}{\tilde{r}}+\frac{r}{\tau}} \end{aligned}$$

for a positive constant C_0. In other words,

$$\mathcal{Z}_{n+1} \le d_3 \left(k_*^{-r\left(1-\frac{r}{\tilde{r}}\right)} + k_*^{-r\left(1-\frac{r}{\tilde{r}}+\frac{r}{\tau}\right)} \right) d_1^{n(1+\delta)} \mathcal{Z}_n^{1+\delta}, \quad n \in \mathbb{N}\cup\{0\},$$

where $d_3 = C_0 C_{imb}^r (2(\widetilde{C}d_1^2+d_1))^{\frac{r}{\tau}} d_1^{1-\frac{r}{\tilde{r}}}$ and $\delta = \frac{r}{\tau} - \frac{r}{\tilde{r}}$. This implies

$$\mathcal{Z}_{n+1} \le d_3 \left(k_*^{-\alpha_1} + k_*^{-\alpha_2}\right) b^n \mathcal{Z}_n^{1+\delta} \tag{3.7}$$

where

$$0 < \alpha_1 := r\left(1-\frac{r}{\tilde{r}}\right) < \alpha_2 := r\left(1-\frac{r}{\tilde{r}}+\frac{r}{\tau}\right) \text{ and } b := d_1^{1+\delta}.$$

Applying Lemma 3.2 with (3.7), we obtain that

$$\mathcal{Z}_n = \int_\Omega (v - k_n)_+^r \, dx \to 0 \quad \text{as} \quad n \to \infty, \tag{3.8}$$

provided that

$$\mathcal{Z}_0 \le \min \left\{ 1, d_3^{-\frac{1}{\delta}} \left(k_*^{-\alpha_1} + k_*^{-\alpha_2} \right)^{-\frac{1}{\delta}} b^{-\frac{1}{\delta^2}} \right\}.$$

We note that for k_* large enough, it is $\mathcal{Z}_0 \le 1$ since $|\mathcal{A}_{k_*}| \to 0$ as $k_* \to \infty$. Moreover, observe that

$$\mathcal{Z}_0 = \int_{A_{k_*}} (v - k_*)^r \, dx \le \int_\Omega v_+^r \, dx. \tag{3.9}$$

Meanwhile,

$$\int_\Omega v_+^r \, dx \le d_3^{-\frac{1}{\delta}} \left(k_*^{-\alpha_1} + k_*^{-\alpha_2} \right)^{-\frac{1}{\delta}} b^{-\frac{1}{\delta^2}}$$

is equivalent to

$$k_*^{-\alpha_1} + k_*^{-\alpha_2} \le d_3^{-1} b^{-\frac{1}{\delta}} \left(\int_\Omega v_+^r \, dx \right)^{-\delta}. \tag{3.10}$$

Moreover,

$$\begin{cases} 2k_*^{-\alpha_1} \le d_3^{-1} b^{-\frac{1}{\delta}} \left(\int_\Omega v_+^r \, dx \right)^{-\delta}, \\ 2k_*^{-\alpha_2} \le d_3^{-1} b^{-\frac{1}{\delta}} \left(\int_\Omega v_+^r \, dx \right)^{-\delta} \end{cases}$$

is equivalent to

$$\begin{cases} k_* \ge (2d_3)^{\frac{1}{\alpha_1}} b^{\frac{1}{\delta \alpha_1}} \left(\int_\Omega v_+^r \, dx \right)^{\frac{\delta}{\alpha_1}}, \\ k_* \ge (2d_3)^{\frac{1}{\alpha_2}} b^{\frac{1}{\delta \alpha_2}} \left(\int_\Omega v_+^r \, dx \right)^{\frac{\delta}{\alpha_2}}. \end{cases}$$

Hence, by choosing

$$k_* = \max \left\{ (2d_3)^{\frac{1}{\alpha_1}} b^{\frac{1}{\delta \alpha_1}} \left(\int_\Omega v_+^r \, dx \right)^{\frac{\delta}{\alpha_1}}, (2d_3)^{\frac{1}{\alpha_2}} b^{\frac{1}{\delta \alpha_2}} \left(\int_\Omega v_+^r \, dx \right)^{\frac{\delta}{\alpha_2}} \right\},$$

we obtain the inequality (3.10). Combining this and (3.9), we deduce the relation (3.8). Since $k_n \uparrow 2k_*$, the relation (3.8) and the Lebesgue dominated convergence theorem infer that

$$\int_\Omega (v - 2k_*)_+^r \, dx = 0.$$

Therefore, $(v - 2k_*)_+ = 0$ almost everywhere in Ω and hence $\operatorname{ess\,sup}_\Omega v \le 2k_*$. By replacing v with $-v$ and $\mathcal{A}_k$ with $\widetilde{\mathcal{A}}_k$, we have analogously that v is bounded from below. Therefore

$$\|v\|_{L^\infty(\Omega)} \le C \max \left\{ \left(\int_\Omega |v|^r \, dx \right)^{\frac{\delta}{\alpha_1}}, \left(\int_\Omega |v|^r \, dx \right)^{\frac{\delta}{\alpha_2}} \right\},$$

where C is a positive constant independent of v. This completes the proof. $\square$

Next we give the following useful lemmas which play a crucial role in establishing the existence of at least two distinct nontrivial solutions to the problem (1.2).

Lemma 3.3. *Let $(h1)$–$(h3)$ hold and let μ be fixed. Furthermore, we assume*

$(h4)$ *$H(x,t) \geq 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$.*

Then the functional φ satisfies the followings:

(1) There exists a positive constant λ^* such that for any $\lambda \in (0, \lambda^*)$ we can choose $R > 0$ and $0 < \beta < 1$ such that $\varphi(v) \geq R > 0$ for all $v \in X$ with $\|v\|_X = \beta$;

(2) There exists $w \in C_c^\infty(\Omega)$ with $w > 0$ such that $\varphi(tw) \to -\infty$ as $t \to +\infty$;

(3) There exists $\omega \in C_c^\infty(\Omega)$ with $\omega > 0$ such that $\varphi(t\omega) < 0$ as $t \to 0+$.

Proof. Let us prove the condition (1). By Lemma 2.2, there exists a positive constant C_1 such that $\|v\|_{L^\tau(\Omega)} \leq C_1 \|v\|_X$ for any τ with $1 < \tau < p^*$. Assume that $\|v\|_X < 1$. Then it follows from $(h2)$ and Lemma 2.1 that

$$\begin{aligned}
\varphi(v) &= \int_\Omega \left(\frac{1}{p} |\nabla v|^p + \frac{a(x)}{q} |\nabla v|^q \right) dx - \frac{\lambda}{\gamma} \int_\Omega \varrho(x) |v|^\gamma \, dx - \mu \int_\Omega H(x,v) \, dx \\
&\geq \frac{1}{q} \int_\Omega \left(|\nabla v|^p + a(x) |\nabla v|^q \right) dx - \frac{\lambda}{\gamma} \|\varrho\|_{L^\infty(\Omega)} \|v\|_{L^\gamma(\Omega)}^\gamma - \frac{\mu \|\sigma\|_{L^\infty(\Omega)}}{r} \|v\|_{L^r(\Omega)}^r \\
&\geq \frac{1}{q} \|v\|_X^q - \frac{\lambda}{\gamma} \|\varrho\|_{L^\infty(\Omega)} C_1^\gamma \|v\|_X^\gamma - \frac{\mu \|\sigma\|_{L^\infty(\Omega)}}{r} C_1^r \|v\|_X^r \\
&= \left(\frac{1}{q} - \frac{\lambda}{\gamma} C_2 \|v\|_X^{\gamma - q} - \frac{\mu}{r} C_3 \|v\|_X^{r-q} \right) \|v\|_X^q
\end{aligned} \tag{3.11}$$

for positive constants C_2, C_3. Let us define the function $g_\lambda : (0, \infty) \to \mathbb{R}$ by

$$g_\lambda(s) = \frac{\lambda}{\gamma} C_2 s^{\gamma - q} + \frac{\mu}{r} C_3 s^{r-q}.$$

Then it is clear that g_λ has a local minimum at the point $s_0 = \left(\frac{\lambda r C_2 (q - \gamma)}{\mu \gamma C_3 (r - q)} \right)^{\frac{1}{r - \gamma}}$ and so

$$\lim_{\lambda \to 0^+} g_\lambda(s_0) = 0.$$

Thus there is $\lambda^* > 0$ such that for each $\lambda \in (0, \lambda^*)$, there exist $R > 0$ and $\beta > 0$ small enough such that $\varphi(v) \geq R > 0$ for any $v \in X$ with $\|v\|_X = \beta$.

Next we show the condition (2). For any $M_0 > 0$, it follows from $(h2)$ and $(h3)$ that there exists a constant $C_{M_0} > 0$ such that

$$H(x,t) \geq M_0 |t|^q - C_{M_0} \tag{3.12}$$

for all $(x,t) \in \Omega \times \mathbb{R}$. Take $w \in C_c^\infty(\Omega)$ with $w > 0$. It follows from (3.12) that

$$\begin{aligned}
\varphi(sw) &= \int_\Omega \left(\frac{1}{p} |\nabla sw|^p + \frac{a(x)}{q} |\nabla sw|^q \right) dx \\
&\quad - \frac{\lambda}{\gamma} \int_\Omega \varrho(x) |sw|^\gamma \, dx - \mu \int_\Omega H(x, sw) \, dx
\end{aligned}$$

$$\begin{aligned}
&\leq s^q \int_\Omega \left(\frac{1}{p} |\nabla w|^p + \frac{a(x)}{q} |\nabla w|^q \right) dx - \frac{\lambda s^\gamma}{\gamma} \int_\Omega \varrho(x) |w|^\gamma \, dx \\
&\quad - \mu M_0 s^q \int_\Omega |w|^q \, dx + \mu C_{M_0} |\Omega| \\
&\leq s^q \left(\int_\Omega \left(\frac{1}{p} |\nabla w|^p + \frac{a(x)}{q} |\nabla w|^q \right) dx - \mu M_0 \int_\Omega |w|^q \, dx \right) \\
&\quad - \frac{\lambda s^\gamma}{\gamma} \int_\Omega \varrho(x) |w|^\gamma \, dx + \mu C_{M_0} |\Omega|
\end{aligned}$$

for sufficiently large $s \geq 1$. We see that $\varphi(sw) \to -\infty$ as $s \to \infty$.

Finally we remain to prove the condition (3). Choose $\omega \in C_c^\infty(\Omega)$ such that $\omega > 0$. Let λ be fixed. For $s > 0$ small enough, we obtain from $(h4)$ that

$$\begin{aligned}
\varphi(s\omega) &= \int_\Omega \left(\frac{1}{p} |\nabla s\omega|^p + \frac{a(x)}{q} |\nabla s\omega|^q \right) dx - \frac{\lambda}{\gamma} \int_\Omega \varrho(x) |s\omega|^\gamma \, dx - \mu \int_\Omega H(x, s\omega) \, dx \\
&\leq s^p \int_\Omega \left(\frac{1}{p} |\nabla \omega|^p + \frac{a(x)}{q} |\nabla \omega|^q \right) dx - \frac{\lambda s^\gamma}{\gamma} \int_\Omega \varrho(x) |\omega|^\gamma \, dx.
\end{aligned}$$

Since $p > \gamma$, we see that $\varphi(s\omega) < 0$ as $s \to 0+$. □

Lemma 3.4. *Assume that $(h1)$–$(h2)$ hold. Then Ψ and Ψ' are weakly strongly continuous on X for any $\lambda, \mu > 0$.*

Proof. Let $\{y_n\}$ be a sequence in X such that $y_n \rightharpoonup y$ in X as $n \to \infty$. Since $\{y_n\}$ is bounded in X, Lemma 2.2 guarantees that there exists a subsequence $\{y_{n_k}\}$ such that

$$y_{n_k} \to y \quad \text{a.e. in } \Omega \quad \text{and} \quad y_{n_k} \to y \quad \text{in } L^r(\Omega) \tag{3.13}$$

as $k \to \infty$. First we prove that Ψ is weakly strongly continuous in X. By the convergence principle, there exists a function $f \in L^r(\Omega)$ such that $|y_{n_k}| \leq f$ for all $k \in \mathbb{N}$. Therefore, it follows from $(h2)$ and the Young inequality that

$$\begin{aligned}
&\frac{\lambda}{\gamma} \int_\Omega \varrho(x) |y_{n_k}|^\gamma \, dx + \mu \int_\Omega |H(x, y_{n_k})| \, dx \\
&\leq \frac{\lambda}{\gamma} \int_\Omega |\varrho(x)| \, |y_{n_k}|^\gamma \, dx + \frac{\mu}{r} \int_\Omega \sigma(x) |y_{n_k}|^r \, dx \\
&\leq \frac{\lambda}{\gamma} \int_\Omega \left(\frac{r-\gamma}{r} |\varrho(x)|^{\frac{r}{r-\gamma}} + \frac{\gamma}{r} |y_{n_k}|^r \right) dx + \frac{\mu \|\sigma\|_{L^\infty(\Omega)}}{r} \int_\Omega |y_{n_k}|^r \, dx \\
&\leq C_4 \left[\int_\Omega \left(|\varrho(x)|^{\frac{r}{r-\gamma}} + |f|^r \right) dx + \int_\Omega |f|^r \, dx \right]
\end{aligned}$$

for some positive constant C_4, and so the integral at the left-hand side is dominated by an integrable function. Since h is the Carathéodory function by $(h1)$, it follows from (3.13) that

$$\frac{\varrho(x)}{\gamma} |y_{n_k}|^\gamma \to \frac{\varrho(x)}{\gamma} |y|^\gamma \quad \text{and} \quad H(x, y_{n_k}) \to H(x, y)$$

as $k \to \infty$ for almost all $x \in \Omega$. Therefore, Lebesgue's dominated convergence theorem tells us that

$$\frac{\lambda}{\gamma} \int_\Omega \varrho(x) |y_{n_k}|^\gamma \, dx + \mu \int_\Omega H(x, y_{n_k}) \, dx \to \frac{\lambda}{\gamma} \int_\Omega \varrho(x) |y|^\gamma \, dx + \mu \int_\Omega H(x, y) \, dx$$

as $k \to \infty$, that is, $\Psi(y_{n_k}) \to \Psi(y)$ as $k \to \infty$. Thus Ψ is weakly strongly continuous in X.

Next, we show that Ψ' is weakly strongly continuous in X^*. First of all, we note that

$$\begin{aligned}
&\int_\Omega \left|\varrho(x)|y_{n_k}|^{\gamma-2}y_{n_k} - \varrho(x)|y|^{\gamma-2}y\right|^{\gamma'} dx \\
&= C_5 \int_\Omega |\varrho(x)|^{\frac{1}{\gamma-1}} |\varrho(x)| \left(|y_{n_k}|^\gamma + |y|^\gamma\right) dx \\
&\le C_6 \int_\Omega |\varrho(x)| \left(|y_{n_k}|^\gamma + |y|^\gamma\right) dx \\
&\le C_6 \int_\Omega \frac{2(r-\gamma)}{r} |\varrho(x)|^{\frac{r}{r-\gamma}} + \frac{\gamma}{r} |y_{n_k}|^r + \frac{\gamma}{r} |y|^r \, dx
\end{aligned} \tag{3.14}$$

for some positive constants C_5, C_6. Due to $(h2)$, we obtain

$$\begin{aligned}
\int_\Omega |h(x,y_{n_k}) - h(x,y)|^{r'} dx &\le C_7 \int_\Omega |h(x,y_{n_k})|^{r'} + |h(x,y)|^{r'} dx \\
&\le C_8 \int_\Omega |y_{n_k}|^r + |y|^r \, dx
\end{aligned} \tag{3.15}$$

for some positive constants C_7, C_8. Invoking (3.13)–(3.15) and the convergence principle, one has

$$\left|\varrho(x)\,|y_{n_k}|^{\gamma-2} - \varrho(x)\,|y|^{\gamma-2}\right|^{\gamma'} \le k_1(x)$$

and

$$|h(x,y_{n_k}) - h(x,y)|^{r'} \le k_2(x)$$

for almost all $x \in \Omega$ and for some $k_1, k_2 \in L^1(\Omega)$, and so $\varrho(x)|y_{n_k}|^{\gamma-2}y_{n_k} \to \varrho(x)|y|^{\gamma-2}y$ and $h(x,y_{n_k}) \to h(x,y)$ as $k \to \infty$ for almost all $x \in \Omega$. This together with Lebesgue convergence theorem yields that

$$\begin{aligned}
&\|\Psi'(y_{n_k}) - \Psi'(y)\|_{X^*} \\
&= \sup_{\|u\|_X \le 1} |\langle \Psi'(y_{n_k}) - \Psi'(y), u\rangle| \\
&= \sup_{\|u\|_X \le 1} \left|\lambda \int_\Omega (\varrho(x)|y_{n_k}|^{\gamma-2}y_{n_k} - \varrho(x)|y|^{\gamma-2}y)u\,dx + \mu \int_\Omega (h(x,y_{n_k}) - h(x,y))u\,dx\right| \\
&\le C_9 \left(\lambda \|\varrho(x)|y_{n_k}|^{\gamma-2}y_{n_k} - \varrho(x)|y|^{\gamma-2}y\|_{L^{\gamma'}(\Omega)} + \mu \|h(x,y_{n_k}) - h(x,y)\|_{L^{r'}(\Omega)}\right) \to 0
\end{aligned}$$

as $k \to \infty$ for some positive constant C_9. Consequently, we derive that $\Psi'(y_{n_k}) \to \Psi'(y)$ in X as $k \to \infty$. This completes the proof. $\square$

Definition 3.2. We say that φ satisfies the Cerami condition ((C)-condition for short) in X, if any (C)-sequence $\{z_n\} \subset X$, i.e. $\{\varphi(z_n)\}$ is bounded and $\|\varphi'(z_n)\|_{X^*}(1+\|z_n\|_X) \to 0$ as $n \to \infty$, has a convergent subsequence in X.

Lemma 3.5. *It is assumed that $(h1)$–$(h3)$ hold. In addition,*

$(h5)$ *There exist $\nu > q$, $K > 0$ and $\zeta \ge 0$ such that*

$$h(x,t)t - \nu H(x,t) \ge -\zeta\,|t|^p - \eta(x)$$

for all $(x,t) \in \Omega \times \mathbb{R}$ with $|t| \ge K$ and for some $\eta \in L^1(\Omega)$ with $\eta(x) \ge 0$.

$(h6)$ $H(x,t)=o(|t|^p)$ as $t\to 0$ for $x\in\Omega$ uniformly.

Then, the functional φ satisfies the (C)-condition for any $\lambda,\mu>0$.

Proof. Let $\{z_n\}$ be a (C)-sequence in X, that is,

$$\sup_{n\in\mathbb{N}}|\varphi(z_n)|\le\mathcal{K}_1 \text{ and } \langle\varphi'(z_n),z_n\rangle=o(1)\to 0, \tag{3.16}$$

as $n\to\infty$, where $\mathcal{K}_1$ is a positive constant. Suppose that $\{z_n\}$ is a bounded sequence satisfying (3.16). Then $\{z_n\}$ has a weakly convergent subsequence in X. Without loss of generality, we suppose that

$$z_n\rightharpoonup z \text{ in } X \text{ as } n\to\infty.$$

By Lemma 3.4, Ψ' is weakly strongly continuous, and so $\Psi'(z_n)\to\Psi'(z)$ in X^* as $n\to\infty$. In addition, using (3.16), we know that

$$\langle\varphi'(z_n),z_n-z\rangle\to 0 \text{ and } \langle\varphi'(z),z_n-z\rangle\to 0$$

as $n\to\infty$, and thus

$$\langle\varphi'(z_n)-\varphi'(z),z_n-z\rangle=o(1).$$

From this, we have

$$\langle\Phi'(z_n)-\Phi'(z),z_n-z\rangle=\langle\Psi'(z_n)-\Psi'(z),z_n-z\rangle+\langle\varphi'(z_n)-\varphi'(z),z_n-z\rangle\to 0.$$

Since X is reflexive and Φ' is a mappaing of type (S_+) by Lemma 3.1, we infer that

$$z_n\to z \text{ in } X \text{ as } n\to\infty.$$

Hence it needs only be proved that $\{z_n\}$ is bounded in X. To this end, arguing by contradiction, it is assumed that $\|z_n\|_X>1$ and $\|z_n\|_X\to\infty$ as $n\to\infty$, and a sequence $\{y_n\}$ is defined by $y_n=z_n/\|z_n\|_X$. Then, up to a subsequence, still denoted by $\{y_n\}$, we obtain $y_n\rightharpoonup y_0$ in X as $n\to\infty$, and by Lemma 2.2,

$$y_n\to y_0 \text{ a.e. in } \Omega, \quad\text{and}\quad y_n\to y_0 \text{ in } L^p(\Omega) \text{ as } n\to\infty. \tag{3.17}$$

According to $(h2)$ and $(h6)$, one has

$$\int_{|z_n|\le K} H(x,z_n)-\frac{1}{\nu}h(x,z_n)\,z_n\,dx\le(1+\nu^{-1})\left(\|\sigma\|_{L^\infty(\Omega)}K^r+K^p\right)|\Omega|=:\mathcal{K}_0.$$

Combining this with $(h5)$ and Lemmas 2.1 and 2.2 one has

$$\begin{aligned}
&\mathcal{K}_1+o(1)\\
&\ge\varphi(z_n)-\frac{1}{\nu}\langle\varphi'(z_n),z_n\rangle\\
&=\int_\Omega\left(\frac{1}{p}-\frac{1}{\nu}\right)|\nabla z_n|^p+\left(\frac{1}{q}-\frac{1}{\nu}\right)a(x)\,|\nabla z_n|^q\,dx\\
&\quad-\lambda\left(\frac{1}{\gamma}-\frac{1}{\nu}\right)\int_\Omega\varrho(x)\,|z_n|^\gamma\,dx+\mu\int_\Omega\left(\frac{1}{\nu}h(x,z_n)\,z_n-H(x,z_n)\right)dx\\
&\ge\int_\Omega\left(\frac{1}{p}-\frac{1}{\nu}\right)|\nabla z_n|^p+\left(\frac{1}{q}-\frac{1}{\nu}\right)a(x)\,|\nabla z_n|^q\,dx
\end{aligned}$$

$$
\begin{aligned}
&-\lambda\left(\frac{1}{\gamma}-\frac{1}{\nu}\right)\int_\Omega \varrho(x)\,|z_n|^\gamma\;dx+\mu\int_{|z_n|>K}\left(\frac{1}{\nu}h\left(x,z_n\right)z_n-H\left(x,z_n\right)\right)\,dx-\mu\mathcal{K}_0\\
\geq&\left(\frac{1}{q}-\frac{1}{\nu}\right)\int_\Omega |\nabla z_n|^p+a(x)\,|\nabla z_n|^q\;dx\\
&-\lambda\left(\frac{1}{\gamma}-\frac{1}{\nu}\right)\|\varrho\|_{L^{\frac{p}{p-\gamma}}(\Omega)}\|z_n\|^\gamma_{L^p(\Omega)}-\frac{\mu}{\nu}\int_\Omega\left(\zeta\,|z_n|^p+\eta(x)\right)\,dx-\mu\mathcal{K}_0\\
\geq&\left(\frac{1}{q}-\frac{1}{\nu}\right)\|z_n\|_X^p-\lambda C_{10}\left(\frac{1}{\gamma}-\frac{1}{\nu}\right)\|\varrho\|_{L^{\frac{p}{p-\gamma}}(\Omega)}\|z_n\|_X^\gamma\\
&-\frac{\mu\zeta}{\nu}\|z_n\|^p_{L^p(\Omega)}-\frac{\mu}{\nu}\|\eta\|_{L^1(\Omega)}-\mu\mathcal{K}_0
\end{aligned}
$$

for some constant C_{10}, so that

$$
1\leq\frac{\mu\zeta}{\nu\left(\frac{1}{q}-\frac{1}{\nu}\right)}\limsup_{n\to\infty}\|y_n\|^p_{L^p(\Omega)}=\frac{\mu\zeta}{\nu\left(\frac{1}{q}-\frac{1}{\nu}\right)}\|y_0\|^p_{L^p(\Omega)}. \tag{3.18}
$$

Hence, it follows from (3.18) that $y_0\neq 0$. However we will show that this is absurd. By Lemmas 2.1 and 2.2, we have

$$
\begin{aligned}
\varphi(z_n)&=\int_\Omega\left(\frac{1}{p}\,|\nabla z_n|^p+\frac{a(x)}{q}\,|\nabla z_n|^q\right)\,dx-\frac{\lambda}{\gamma}\int_\Omega\varrho(x)\,|z_n|^\gamma\;dx-\mu\int_\Omega H\left(x,z_n\right)\,dx\\
&\geq\frac{1}{q}\|z_n\|_X^p-\frac{\lambda C_{10}}{\gamma}\|\varrho\|_{L^{\frac{p}{p-\gamma}}(\Omega)}\|z_n\|_X^\gamma-\mu\int_\Omega H\left(x,z_n\right)\,dx.
\end{aligned}
$$

Since $|\varphi(z_n)|\leq\mathcal{K}_1$ for all $n\in\mathbb{N}$ and $\|z_n\|_X\to\infty$ as $n\to\infty$, we assert that

$$
\mu\int_\Omega H(x,z_n)\,dx\geq\frac{1}{q}\|z_n\|_X^p-C_{10}\frac{\lambda}{\gamma}\|\varrho\|_{L^{\frac{p}{p-\gamma}}(\Omega)}\|z_n\|_X^\gamma-\varphi(z_n)\to\infty \tag{3.19}
$$

as $n\to\infty$. Note that

$$
\begin{aligned}
\varphi(z_n)&=\int_\Omega\left(\frac{1}{p}\,|\nabla z_n|^p+\frac{a(x)}{q}\,|\nabla z_n|^q\right)\,dx-\frac{\lambda}{\gamma}\int_\Omega\varrho(x)\,|z_n|^\gamma\;dx-\mu\int_\Omega H(x,z_n)\,dx\\
&\leq\int_\Omega\left(\frac{1}{p}\,|\nabla z_n|^p+\frac{a(x)}{q}\,|\nabla z_n|^q\right)\,dx-\mu\int_\Omega H(x,z_n)\,dx.
\end{aligned}
$$

And so,

$$
\int_\Omega\left(\frac{1}{p}\,|\nabla z_n|^p+\frac{a(x)}{q}\,|\nabla z_n|^q\right)\,dx\geq\varphi(z_n)+\mu\int_\Omega H(x,z_n)\,dx. \tag{3.20}
$$

Taking assumption $(h3)$ into account, there is $t_0>1$ such that $H(x,t)>|t|^q$ for all $x\in\Omega$ and $|t|>t_0$. Owing to $(h2)$, we can choose $\hat{\mathcal{K}}>0$ such that $|H(x,t)|\leq\hat{\mathcal{K}}$ for all $(x,t)\in\Omega\times[-t_0,t_0]$. Therefore we can choose $\mathcal{K}_2\in\mathbb{R}$ such that $H(x,t)\geq\mathcal{K}_2$ for all $(x,t)\in\Omega\times\mathbb{R}$, and thus

$$
\frac{H(x,z_n)-\mathcal{K}_2}{\int_\Omega\left(\frac{1}{p}\,|\nabla z_n|^p+\frac{a(x)}{q}\,|\nabla z_n|^q\right)\,dx}\geq 0 \tag{3.21}
$$

for all $x \in \Omega$ and $n \in \mathbb{N}$. Set $\Omega_1 = \{x \in \Omega : y_0(x) \neq 0\}$. By convergence (3.17), we know that $|z_n(x)| = |y_n(x)| \, \|z_n\|_X \to \infty$ as $n \to \infty$ for all $x \in \Omega_1$. Thus, it follows from assumption $(h3)$ that, for all $x \in \Omega_1$,

$$\begin{aligned}
\lim_{n\to\infty} \frac{H(x,z_n) - \mathcal{K}_2}{\int_\Omega \left(\frac{1}{p}|\nabla z_n|^p + \frac{a(x)}{q}|\nabla z_n|^q\right) dx} &\geq \lim_{n\to\infty} \frac{H(x,z_n)}{\max\{\|z_n\|_X^q, \|z_n\|_X^p\}} \\
&= \lim_{n\to\infty} \frac{H(x,z_n)}{\|z_n\|_X^q} \\
&= \lim_{n\to\infty} \frac{H(x,z_n)}{|z_n|^q} |y_n(x)|^q = \infty.
\end{aligned} \tag{3.22}$$

Hence, we have that $|\Omega_1| = 0$. In fact, if $|\Omega_1| \neq 0$, using relations (3.19)–(3.22) and the Fatou lemma, we deduce that

$$\begin{aligned}
\frac{1}{\mu} &= \liminf_{n\to\infty} \frac{\int_\Omega H(x,z_n)\,dx}{\mu \int_\Omega H(x,z_n)\,dx + \varphi(z_n)} \\
&\geq \liminf_{n\to\infty} \int_\Omega \left(\frac{H(x,z_n)}{\int_\Omega \left(\frac{1}{p}|\nabla z_n|^p + \frac{a(x)}{q}|\nabla z_n|^q\right) dx} \right) dx \\
&\geq \liminf_{n\to\infty} \int_{\Omega_1} \left(\frac{H(x,z_n)}{\int_\Omega \left(\frac{1}{p}|\nabla z_n|^p + \frac{a(x)}{q}|\nabla z_n|^q\right) dx} \right) dx \\
&\quad - \limsup_{n\to\infty} \int_{\Omega_1} \left(\frac{\mathcal{K}_2}{\int_\Omega \left(\frac{1}{p}|\nabla z_n|^p + \frac{a(x)}{q}|\nabla z_n|^q\right) dx} \right) dx \\
&= \liminf_{n\to\infty} \int_{\Omega_1} \left(\frac{H(x,z_n) - \mathcal{K}_2}{\int_\Omega \left(\frac{1}{p}|\nabla z_n|^p + \frac{a(x)}{q}|\nabla z_n|^q\right) dx} \right) dx \\
&\geq \int_{\Omega_1} \left(\liminf_{n\to\infty} \frac{H(x,z_n) - \mathcal{K}_2}{\int_\Omega \left(\frac{1}{p}|\nabla z_n|^p + \frac{a(x)}{q}|\nabla z_n|^q\right) dx} \right) dx \\
&= \int_{\Omega_1} \left(\liminf_{n\to\infty} \frac{H(x,z_n)}{\int_\Omega \left(\frac{1}{p}|\nabla z_n|^p + \frac{a(x)}{q}|\nabla z_n|^q\right) dx} \right) dx \\
&\quad - \int_{\Omega_1} \left(\limsup_{n\to\infty} \frac{\mathcal{K}_2}{\int_\Omega \left(\frac{1}{p}|\nabla z_n|^p + \frac{a(x)}{q}|\nabla z_n|^q\right) dx} \right) dx = \infty,
\end{aligned} \tag{3.23}$$

which is impossible. Thus, $y_0(x) = 0$ for almost all $x \in \Omega$. Therefore, we conclude a contradiction. Thus, $\{z_n\}$ is bounded in X. This completes the proof. □

The following lemma is the variational principle of Ekeland's type in [3, 22], initially developed by C.-K. Zhong [43].

Lemma 3.6 (Corollary 2.2, [3], Corollary 2.10, [22])**.** *Let E be a Banach space and x_0 be a fixed point of E. Suppose that $h : E \to \mathbb{R} \cup \{+\infty\}$ is a lower semi-continuous* *function, not identically* $+\infty$*, bounded from below. Then, for every* $\varepsilon > 0$ *and* $y \in E$ *such that*

$$h(y) < \inf_E h + \varepsilon,$$

and every $\lambda > 0$*, there exists some point* $z \in E$ *such that*

$$h(z) \le h(y), \quad \|z - x_0\|_E \le (1 + \|y\|_E)(e^{\lambda} - 1),$$

and

$$h(x) \ge h(z) - \frac{\varepsilon}{\lambda(1 + \|z\|_E)} \|x - z\|_E \quad \textit{for all} \quad x \in E.$$

With the help of Lemmas 3.3, 3.5 and 3.6, we are in a position to derive our first major result.

Theorem 3.1. *Let* $(h1)$*–*$(h6)$ *hold and let* $\mu > 0$ *be fixed. Then there exists a positive constant* λ^* *such that for any* $\lambda \in (0, \lambda^*)$*, problem* (1.2) *admits at least two nontrivial different solutions in* X *which belong to* $L^\infty(\Omega)$*.*

Proof. Thanks to Lemmas 3.3 and 3.5, there exists a positive number λ^* such that for all $\lambda \in (0, \lambda^*)$, the functional φ satisfies the mountain pass geometry and (C)-condition. By employing the mountain pass theorem, we infer that there exists a critical point $v_0 \in X$ of φ with $\varphi(v_0) = \overline{\ell} > 0 = \varphi(0)$. Hence v_0 is a nontrivial weak solution of the problem (1.2). Taking into account Lemma 3.3, there are $R > 0$ and $0 < \beta < 1$ such that $\varphi(v) \ge R > 0$ for all $v \in X$ with $\|v\|_X = \beta$. Let us denote $\ell := \inf_{v \in \overline{B}_\beta} \varphi(v)$ where $B_\beta := \{v \in X : \|v\|_X < \beta\}$ with a boundary ∂B_β. Then by (3.11) and Lemma 3.3 (3), we have $-\infty < \ell < 0$. Putting $0 < \epsilon < \inf_{v \in \partial B_\beta} \varphi(v) - \ell$, by Lemma 3.6, we can choose $v_\epsilon \in \overline{B}_\beta$ such that

$$\begin{cases} \varphi(v_\epsilon) \le \ell + \epsilon \\ \varphi(v_\epsilon) < \varphi(v) + \frac{\epsilon}{1 + \|v_\epsilon\|_X} \|v - v_\epsilon\|_X, \quad \text{for all} \quad v \in \overline{B}_\beta \quad v \ne v_\epsilon. \end{cases} \tag{3.24}$$

This implies that $v_\epsilon \in B_\beta$ since $\varphi(v_\epsilon) \le \ell + \epsilon < \inf_{v \in \partial B_\beta} \varphi(v)$. From these facts we have that v_ϵ is a local minimum of $\widehat{\varphi}(v) = \varphi(v) + \frac{\epsilon}{1 + \|v_\epsilon\|_X} \|v - v_\epsilon\|_X$. Now by taking $v = v_\epsilon + tw$ for $w \in B_1$ and sufficiently small $t > 0$, from (3.24), we deduce

$$0 \le \frac{\widehat{\varphi}(v_\epsilon + tw) - \widehat{\varphi}(v_\epsilon)}{t} = \frac{\varphi(v_\epsilon + tw) - \varphi(v_\epsilon)}{t} + \frac{\epsilon}{1 + \|v_\epsilon\|_X} \|w\|_X.$$

Therefore, letting $t \to 0+$, we get

$$\langle \varphi'(v_\epsilon), w \rangle + \frac{\epsilon}{1 + \|v_\epsilon\|_X} \|w\|_X \ge 0.$$

Replacing w by $-w$ in the argument above, we have

$$-\langle \varphi'(v_\epsilon), w \rangle + \frac{\epsilon}{1 + \|v_\epsilon\|_X} \|w\|_X \ge 0.$$

Thus, one has

$$(1 + \|v_\epsilon\|_X) \left| \langle \varphi'_\lambda(v_\epsilon), w \rangle \right| \le \epsilon \|w\|_X$$

for any $w \in \overline{B}_1$. Hence we know

$$(1 + \|v_\epsilon\|_X) \|\varphi'(v_\epsilon)\|_{X^*} \le \epsilon. \tag{3.25}$$

Using (3.24) and (3.25), we can choose a sequence $\{v_n\} \subset B_\beta$ such that

$$\begin{cases} \varphi(v_n) \to \ell \quad \text{as} \quad n \to \infty \\ (1 + \|v_n\|_X)\|\varphi'(v_n)\|_{X^*} \to 0 \quad \text{as} \quad n \to \infty. \end{cases} \tag{3.26}$$

Thus, $\{v_n\}$ is a bounded Cerami sequence in the reflexive Banach space X. According to Lemma 3.5, $\{v_n\}$ has a subsequence $\{v_{n_k}\}$ such that $v_{n_k} \to v_1$ in X as $k \to \infty$. This together with (3.26) yields that $\varphi(v_1) = \ell$ and $\varphi'(v_1) = 0$. Hence v_1 is a nontrivial solution of the given problem with $\varphi(v_1) < 0$ which is different from v_0. This completes the proof. □

Finally, according to the similar argument in [32, 35] with the cut-off method (Lemma 3.8), we establish the existence of a sequence of infinitely many weak solutions for problem (1.2) whose converges to 0 in L^∞-space. However, in contrast to [32, 35], we take the dual fountain theorem instead of global variational formulation into consideration. To do this, we need the following additional assumptions of h:

- $(h7)$ There exists a constant $s_0 > 0$ such that $h(x,t)$ is odd in $\Omega \times (-s_0, s_0)$ for t and $pH(x,t) - h(x,t)t > 0$ for all $x \in \Omega$ and for $0 < |t| < s_0$;
- $(h8)$ $\lim_{|t|\to 0} \frac{h(x,t)}{|t|^{p-2}t} = +\infty$ uniformly for all $x \in \Omega$.

Let us introduce the following lemmas which are useful in proving our second result.

Lemma 3.7. *Assume that* $(h1)$ *and* $(h2)$ *hold. If furthermore*

$$pH(x,t) - h(x,t)t > 0 \quad \textit{for all } x \in \Omega \textit{ and for } t \neq 0, \tag{3.27}$$

then

$$\varphi(v) = \langle \varphi'(v), v \rangle = 0 \quad \textit{if and only if} \quad v = 0.$$

Proof. Let $\varphi(v) = \langle \varphi'(v), v \rangle = 0$. Then we see that

$$\begin{aligned} 0 &= -p\, \varphi(v) \\ &= -p \int_\Omega \left(\frac{1}{p} |\nabla v|^p + \frac{a(x)}{q} |\nabla v|^q \right) dx + \frac{p\lambda}{\gamma} \int_\Omega \varrho(x)\, |v|^\gamma \, dx + p\mu \int_\Omega H(x,v)\, dx \\ &\geq - \int_\Omega (|\nabla v|^p + a(x)|\nabla v|^q) \, dx + \lambda \int_\Omega \varrho(x)\, |v|^\gamma \, dx + p\mu \int_\Omega H(x,v)\, dx, \end{aligned} \tag{3.28}$$

and

$$\langle \varphi'(v), v \rangle = \int_\Omega (|\nabla v|^p + a(x)|\nabla v|^q) \, dx - \lambda \int_\Omega \varrho(x)|v|^\gamma \, dx - \mu \int_\Omega h(x,v)v \, dx = 0. \tag{3.29}$$

It follows from relations (3.28) and (3.29) that

$$\int_\Omega (pH(x,v) - h(x,v)v) \, dx \leq 0.$$

Consequently, assumption (3.27) implies that $v = 0$. The converse is clear from definition of φ. □

Remark 3.1. Fix $s_1 \in (0, \frac{s_0}{2})$ and let us define a cut-off function $\chi \in C^1(\mathbb{R}, \mathbb{R})$ satisfying $\chi(t) = 1$ for $|t| \le s_1$, $\chi(t) = 0$ for $|t| \ge 2s_1$, $|\chi'(t)| \le 2/s_1$, and $\chi'(t)t \le 0$. So, we set

$$\widetilde{H}(x,t) = \chi(t)H(x,t) + (1-\chi(t))\xi|t|^p \quad \text{and} \quad \tilde{h}(x,t) = \frac{\partial}{\partial t}\widetilde{H}(x,t), \tag{3.30}$$

where ξ is a positive constant.

Lemma 3.8. *Assume that* $(h1)$, $(h2)$, $(h7)$ *and* $(h8)$ *hold. Then there exist* $s_2 \in (0, \frac{s_1}{2})$ *and* $\tilde{h} \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$ *such that* $\tilde{h}(x,t)$ *is odd for* t, $\widetilde{\mathfrak{H}}(x,t) \ge 0$ *and*

$$\widetilde{\mathfrak{H}}(x,t) = 0 \quad \textit{iff} \quad t \equiv 0 \quad \textit{or} \quad |t| \ge 2s_2,$$

where $\widetilde{\mathfrak{H}}(x,t) := p\widetilde{H}(x,t) - \tilde{h}(x,t)t$.

Proof. It is immediate to see that

$$p\widetilde{H}(x,t) - \tilde{h}(x,t)t = \chi(t)\mathfrak{H}(x,t) - \chi'(t)tH(x,t) + \chi'(t)t\xi|t|^p,$$

where $\mathfrak{H}(x,t) := pH(x,t) - h(x,t)t$. For $0 \le |t| \le s_2$ and $|t| \ge 2s_2$ the conclusion follows. By $(h7)$ and $(h8)$, we choose a sufficiently small $s_2 > 0$ such that $H(x,t) \ge \xi|t|^p$ for $s_2 \le |t| \le 2s_2$. Due to the assumption that $\chi'(t)t \le 0$ we get the conclusion. □

Let $\mathfrak{X}$ be a reflexive and separable Banach space. Then there are $\{e_n\} \subseteq \mathfrak{X}$ and $\{f_n^*\} \subseteq \mathfrak{X}^*$ such that

$$\mathfrak{X} = \overline{\operatorname{span}\{e_n : n = 1, 2, \cdots\}}, \quad \mathfrak{X}^* = \overline{\operatorname{span}\{f_n^* : n = 1, 2, \cdots\}},$$

and

$$\langle f_i^*, e_j \rangle = \begin{cases} 1 \text{ if } & i = j, \\ 0 \text{ if } & i \ne j. \end{cases}$$

Let us denote $\mathfrak{X}_k = \operatorname{span}\{e_k\}$, $\mathfrak{Y}_k = \bigoplus_{m=1}^k \mathfrak{X}_m$, and $\mathfrak{Z}_k = \overline{\bigoplus_{m=k}^\infty \mathfrak{X}_m}$ for $k \in \mathbb{N}$ (see [15]).

Definition 3.3. Let $\mathfrak{X}$ be a real separable and reflexive Banach space and $I \in C^1(\mathfrak{X}, \mathbb{R})$. For every $c \in \mathbb{R}$, we say that I satisfies the $(PS)_c^*$-condition (with respect to $\mathfrak{Y}_n$) if any sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathfrak{X}$ for which $u_n \in \mathfrak{Y}_n$, for any $n \in \mathbb{N}$,

$$I(u_n) \to c \quad \text{and} \quad \|(I|_{\mathfrak{Y}_n})'(u_n)\|_{\mathfrak{X}^*} \to 0 \text{ as } n \to \infty,$$

contains a subsequence converging to a critical point of I.

Proposition 3.2 ((Dual Fountain Theorem)Theorem 3.18, [36])**.** *Assume that* $\mathfrak{X}$ *is a Banach space,* $I \in C^1(\mathfrak{X}, \mathbb{R})$ *is an even functional. If there exists* $k_0 > 0$ *such that, for each* $k \ge k_0$*, there are* $\rho_k > \delta_k > 0$ *such that*

(D1) $\inf\{I(u) : u \in \mathfrak{Z}_k, \|u\|_{\mathfrak{X}} = \rho_k\} \ge 0$.

(D2) $b_k := \max\{I(u) : u \in \mathfrak{Y}_k, \|u\|_{\mathfrak{X}} = \delta_k\} < 0$.

(D3) $d_k := \inf\{I(u) : u \in \mathfrak{Z}_k, \|u\|_{\mathfrak{X}} \le \rho_k\} \to 0$ *as* $k \to \infty$.

(D4) I satisfies the $(PS)_c^*$-condition for every $c \in [d_{k_0}, 0)$,

then I has a sequence of negative critical values converging to 0.

With the help of Lemmas 3.7, 3.8 and Proposition 3.2, we are in a position to derive our second major result.

Theorem 3.2. *Suppose that (h1), (h2), (h7) and (h8) hold and we fix*

$$\lambda \in \left(0, \gamma / \left(q\|\varrho\|_{L^\infty(\Omega)} C_{1,imb}^\gamma\right)\right)$$

where $C_{1,imb}$ is the imbedding constant for the imbedding $X \hookrightarrow L^\gamma(\Omega)$. Then there exists an interval Γ such that problem (1.2) has a sequence of nontrivial solutions $\{v_n\}$ in X whose $\varphi(v_n) \to 0$ and $\|v_n\|_{L^\infty(\Omega)} \to 0$ as $n \to \infty$ for every $\mu \in \Gamma$.

Proof. Consider the modified energy functional $\widetilde{\varphi} : X \to \mathbb{R}$ given by

$$\widetilde{\varphi}(v) := \Phi(v) - \widetilde{\Psi}(v),$$

where

$$\widetilde{\Psi}(v) = \frac{\lambda}{\gamma} \int_\Omega \varrho(x)|v|^\gamma \, dx + \mu \int_\Omega \widetilde{H}(x, v) \, dx.$$

Then it is clear by Lemma 3.8 that $\widetilde{\varphi} \in C^1(X, \mathbb{R})$ is an even functional. Now we will show that conditions (D1)–(D4) of Proposition 3.2 are satisfied.

(D1): Let us denote

$$\theta_{t,k} = \sup \left\{ \int_\Omega |v|^t \, dx : v \in \mathfrak{Z}_k, \|v\|_X \leq 1 \right\} \text{ for } t > 1$$

and

$$\vartheta_k = \max\{\theta_{r,k}, \theta_{p,k}, \theta_{\gamma,k}\}. \tag{3.31}$$

Then, it is easy to verify that $\theta_{r,k} \to 0$, $\theta_{p,k} \to 0$ and $\theta_{\gamma,k} \to 0$ as $k \to \infty$(see [27]). From Lemmas 2.1 and 2.2, it follows that

$$\begin{aligned}
\widetilde{\varphi}(v) &= \Phi(v) - \widetilde{\Psi}(v) \\
&= \int_\Omega \left(\frac{1}{p}|\nabla v|^p + \frac{a(x)}{q}|\nabla v|^q\right) dx - \frac{\lambda}{\gamma} \int_\Omega \varrho(x)|v|^\gamma \, dx - \mu \int_\Omega \widetilde{H}(x, v) \, dx \\
&\geq \frac{1}{q} \int_\Omega (|\nabla v|^p + a(x)|\nabla v|^q) \, dx - \frac{\lambda}{\gamma}\|\varrho\|_{L^\infty(\Omega)} \int_\Omega |v|^\gamma \, dx \\
&\quad - \mu \int_\Omega (H(x, v) + \xi|v|^p) \, dx \\
&\geq \frac{1}{q} \min\{\|\nabla v\|_{\mathcal{H}}^p, \|\nabla v\|_{\mathcal{H}}^q\} - \frac{\lambda}{\gamma}\|\varrho\|_{L^\infty(\Omega)}\|v\|_{L^\gamma(\Omega)}^\gamma \\
&\quad - \frac{\mu\|\sigma\|_{L^\infty(\Omega)}}{r} \int_\Omega |v|^r dx - \mu\xi \int_\Omega |v|^p dx \\
&\geq \frac{1}{q} \min\{\|\nabla v\|_{\mathcal{H}}^p, \|\nabla v\|_{\mathcal{H}}^q\} - \frac{\lambda}{\gamma}\|\varrho\|_{L^\infty(\Omega)}\|v\|_{L^\gamma(\Omega)}^\gamma \\
&\quad - \frac{\mu\|\sigma\|_{L^\infty(\Omega)}}{r}\vartheta_k\|v\|_X^r - \mu\xi\vartheta_k\|v\|_X^p.
\end{aligned}$$

Let us choose $\rho_k = (\xi_1 \vartheta_k)^{\frac{1}{p-2r}}$ and let $v \in \mathfrak{Z}_k$ with $\|v\|_X = \rho_k > 1$ for sufficiently large k where $\xi_1 = r^{-1}\|\sigma\|_{L^\infty(\Omega)} + \xi$. If we set

$$\mu \in \Gamma_1 := \left(0, \frac{1}{q} - \frac{\lambda}{\gamma}\|\varrho\|_{L^\infty(\Omega)} C_{1,imb}^{\gamma}\right),$$

then there exists $k_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\widetilde{\varphi}(v) &\geq \frac{1}{q}\|v\|_X^p - \frac{\lambda}{\gamma}\|\varrho\|_{L^\infty(\Omega)}\|v\|_{L^\gamma(\Omega)}^{\gamma} - \frac{\mu\|\sigma\|_{L^\infty(\Omega)}}{r}\vartheta_k\|v\|_X^r - \mu\xi\vartheta_k\|v\|_X^p \\
&\geq \frac{1}{q}\|v\|_X^p - \frac{\lambda}{\gamma}\|\varrho\|_{L^\infty(\Omega)} C_{1,imb}^{\gamma}\|v\|_X^p - \mu\xi_1\vartheta_k\|v\|_X^{2r} \\
&\geq \left(\frac{1}{q} - \frac{\lambda}{\gamma}\|\varrho\|_{L^\infty(\Omega)} C_{1,imb}^{\gamma}\right)(\xi_1\vartheta_k)^{\frac{p}{p-2r}} - \mu\xi_1\vartheta_k\,(\xi_1\vartheta_k)^{\frac{2r}{p-2r}} \\
&= \left(\frac{1}{q} - \frac{\lambda}{\gamma}\|\varrho\|_{L^\infty(\Omega)} C_{1,imb}^{\gamma} - \mu\right)(\xi_1\vartheta_k)^{\frac{p}{p-2r}} \geq 0
\end{aligned}$$

for all $k \in \mathbb{N}$ with $k \geq k_0$, by being

$$\lim_{k\to\infty}\left(\frac{1}{q} - \frac{\lambda}{\gamma}\|\varrho\|_{L^\infty(\Omega)} C_{1,imb}^{\gamma} - \mu\right)(\xi_1\vartheta_k)^{\frac{p}{p-2r}} = \infty.$$

Then one has

$$\inf\{\widetilde{\varphi}(v) : v \in \mathfrak{Z}_k, \|v\|_X = \rho_k\} \geq 0.$$

(D2): Observe that $\|\cdot\|_{L^\infty(\Omega)}$, $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_X$ are equivalent on $\mathfrak{Y}_k$. Then there are positive constants $\varsigma_{1,k}$ and $\varsigma_{2,k}$ such that

$$\varsigma_{1,k}\|v\|_{L^\infty(\Omega)} \leq \|v\|_X \leq \varsigma_{2,k}\|v\|_{L^p(\Omega)} \tag{3.32}$$

for any $v \in \mathfrak{Y}_k$. From $(h7)$ and $(h8)$, for any $\mathcal{K} > 0$ there exists $s_3 \in (0, s_2)$ such that

$$H(x,t) \geq \frac{\mathcal{K}\varsigma_{2,k}^p}{p}|t|^p$$

for almost all $x \in \Omega$ and all $|t| \leq s_3$. Choose $\delta_k := \min\{\frac{1}{2}, s_3\varsigma_{1,k}\}$ for all $k \in \mathbb{N}$. Then we know that $\|v\|_{L^\infty(\Omega)} \leq s_3$ for $v \in \mathfrak{Y}_k$ with $\|v\|_X = \delta_k$, and so $\widetilde{H}(x,v) = H(x,v)$. Hence we derive by (3.32) that

$$\begin{aligned}
\widetilde{\varphi}(v) &= \int_\Omega \left(\frac{1}{p}|\nabla v|^p + \frac{a(x)}{q}|\nabla v|^q\right)dx - \frac{\lambda}{\gamma}\int_\Omega \varrho(x)|v|^\gamma\,dx - \mu\int_\Omega \widetilde{H}(x,v)\,dx \\
&\leq \frac{1}{p}\int_\Omega \left(|\nabla v|^p + a(x)\,|\nabla v|^q\right)dx - \mu\int_\Omega \frac{\mathcal{K}\varsigma_{2,k}^p}{p}\,|v|^p\,dx \\
&\leq \frac{1}{p}\|\nabla v\|_{\mathcal{H}}^p - \frac{\mu\mathcal{K}\varsigma_{2,k}^p}{p}\|v\|_{L^p(\Omega)}^p \\
&\leq \frac{1}{p}\|v\|_X^p - \frac{\mu\mathcal{K}}{p}\|v\|_X^p \\
&= \frac{1-\mu\mathcal{K}}{p}\delta_k^p
\end{aligned}$$

for any $v \in \mathfrak{Y}_k$ with $\|v\|_X = \delta_k$. If we choose $\mathcal{K}$ large enough such that $1 < \mu\mathcal{K}$, we obtain that

$$b_k = \max\{\widetilde{\varphi}(v) : v \in \mathfrak{Y}_k, \|v\|_X = \delta_k\} < 0.$$

If necessary, we can change k_0 to a larger value, so that $\rho_k > \delta_k > 0$ for all $k \geq k_0$.

(D3): Because $\mathfrak{Y}_k \cap \mathfrak{Z}_k \neq \phi$ and $0 < \delta_k < \rho_k$, we have $d_k \leq b_k < 0$ for all $k \geq k_0$. For any $v \in \mathfrak{Z}_k$ with $\|v\|_X = 1$ and $0 < t < \rho_k$, we have

$$\begin{aligned}
\widetilde{\varphi}(tv) &= \int_\Omega \left(\frac{1}{p}|\nabla tv|^p + \frac{a(x)}{q}|\nabla tv|^q\right) dx - \frac{\lambda}{\gamma}\int_\Omega \varrho(x)|tv|^\gamma\, dx - \mu\int_\Omega \widetilde{H}(x,tv)\, dx \\
&\geq -\frac{\lambda}{\gamma}\int_\Omega \varrho(x)|tv|^\gamma\, dx - \mu\int_\Omega \left(H(x,tv) + \xi|tv|^p\right) dx \\
&\geq -\frac{\lambda}{\gamma}\|\varrho\|_{L^\infty(\Omega)}\int_\Omega |tv|^\gamma\, dx - \frac{\mu\|\sigma\|_{L^\infty(\Omega)}}{r}\int_\Omega |tv|^r dx - \mu\int_\Omega \xi|tv|^p dx \\
&\geq -\frac{\lambda}{\gamma}\|\varrho\|_{L^\infty(\Omega)}\rho_k^\gamma\int_\Omega |v|^\gamma\, dx - \frac{\mu\|\sigma\|_{L^\infty(\Omega)}}{r}\rho_k^r\int_\Omega |v|^r dx - \mu\rho_k^p\int_\Omega \xi|v|^p dx \\
&\geq -\frac{\lambda}{\gamma}\|\varrho\|_{L^\infty(\Omega)}\rho_k^\gamma\vartheta_k - \frac{\mu\|\sigma\|_{L^\infty(\Omega)}}{r}\rho_k^r\vartheta_k - \mu\xi\rho_k^p\vartheta_k,
\end{aligned}$$

where ϑ_k is given in (3.31). Hence, we achieve

$$\begin{aligned}
d_k &\geq -\frac{\lambda}{\gamma}\|\varrho\|_{L^\infty(\Omega)}\rho_k^\gamma\vartheta_k - \frac{\mu\|\sigma\|_{L^\infty(\Omega)}}{r}\rho_k^r\vartheta_k - \mu\xi\rho_k^p\vartheta_k \\
&\geq -\frac{\lambda}{\gamma}\|\varrho\|_{L^\infty(\Omega)}\xi_1^{\frac{\gamma}{p-2r}}\vartheta_k^{\frac{\gamma+p-2r}{p-2r}} - \frac{\mu\|\sigma\|_{L^\infty(\Omega)}}{r}\xi_1^{\frac{r}{p-2r}}\vartheta_k^{\frac{p-r}{p-2r}} - \mu\xi\xi_1^{\frac{p}{p-2r}}\vartheta_k^{\frac{2p-2r}{p-2r}}.
\end{aligned}$$

Because $\gamma + p < 2r$, $p < r$ and $\vartheta_k \to 0$ as $k \to \infty$, we conclude that $\lim_{k\to\infty} d_k = 0$.

(D4): Let $v \in X$ with $\|v\|_X \geq 1$. By $(h2)$, (3.30) and the definition of χ, we deduce that there exist positive constants C_{11}, C_{12} such that

$$\widetilde{H}(x,t) \leq C_{11} + \xi|t|^p \text{ and } \tilde{h}(x,t) \leq C_{12}(1 + |t|^{p-1}) \tag{3.33}$$

for almost all $x \in \Omega$ and all $t \in \mathbb{R}$. Using Lemmas 2.1, 2.2 and (3.33), we arrive at

$$\begin{aligned}
\widetilde{\varphi}(v) &= \int_\Omega \left(\frac{1}{p}|\nabla v|^p + \frac{a(x)}{q}|\nabla v|^q\right) dx - \frac{\lambda}{\gamma}\int_\Omega \varrho(x)|v|^\gamma\, dx - \mu\int_\Omega \widetilde{H}(x,v)\, dx \\
&\geq \frac{1}{q}\int_\Omega \left(|\nabla v|^p + a(x)|\nabla v|^q\right) dx - \frac{\lambda}{\gamma}\|\varrho\|_{L^\infty(\Omega)}\int_\Omega |v|^\gamma\, dx - \mu\int_\Omega (C_{11} + \xi|v|^p)\, dx \\
&\geq \frac{1}{q}\|\nabla v\|_{\mathcal{H}}^p - \frac{\lambda}{\gamma}\|\varrho\|_{L^\infty(\Omega)}\int_\Omega |v|^\gamma\, dx - C_{11}\mu|\Omega| - \mu\xi\int_\Omega |v|^p dx \\
&\geq \frac{1}{q}\|v\|_X^p - \frac{\lambda}{\gamma}\|\varrho\|_{L^\infty(\Omega)}C_{1,imb}^\gamma\|v\|_X^\gamma - C_{11}\mu|\Omega| - \mu\xi C_{2,imb}^p\|v\|_X^p \\
&\geq \left(\frac{1}{q} - \mu\xi C_{2,imb}^p\right)\|v\|_X^p - \frac{\lambda}{\gamma}\|\varrho\|_{L^\infty(\Omega)}C_{1,imb}^\gamma\|v\|_X^\gamma - C_{11}\mu|\Omega|,
\end{aligned}$$

where $C_{2,imb}$ is an imbedding constant of $X \hookrightarrow L^p(\Omega)$. Therefore we deduce that for any

$$\mu \in \Gamma_2 := \left(0, \frac{1}{q\xi C_{2,imb}^p}\right),$$

the functional $\widetilde{\varphi}$ is coercive, that is, $\widetilde{\varphi}(v) \to \infty$ as $\|v\|_X \to \infty$ and thus is bounded from below on X.

Now we show that $\widetilde{\Psi}'$ is weakly strongly continuous on X for any $\lambda > 0$. Let $\{v_n\}$ be a sequence in X such that $v_n \rightharpoonup v$ in X as $n \to \infty$. Since $\{v_n\}$ is bounded in X, Lemma 2.2 guarantees that there exists a subsequence $\{v_{n_k}\}$ such that

$$v_{n_k}(x) \to v(x) \text{ a.e. in } \Omega \text{ and } v_{n_k} \to v \text{ in } L^p(\Omega) \text{ as } k \to \infty. \tag{3.34}$$

First of all, by the Young inequality, we infer that

$$\begin{aligned}
&\int_\Omega \left|\varrho(x)\left|v_{n_k}\right|^{\gamma-2} v_{n_k} - \varrho(x)|v|^{\gamma-2}v\right|^{\gamma'} dx \\
&\leq C_{13} \int_\Omega |\varrho(x)|^{\frac{1}{\gamma-1}} |\varrho(x)| \left(\left|v_{n_k}\right|^\gamma + |v|^\gamma\right) dx \\
&\leq C_{14} \int_\Omega |\varrho(x)| \left(\left|v_{n_k}\right|^\gamma + |v|^\gamma\right) dx \\
&\leq C_{15} \int_\Omega \left(\frac{2(p-\gamma)}{p} |\varrho(x)|^{\frac{p}{p-\gamma}} + \frac{\gamma}{p}\left|v_{n_k}\right|^p + \frac{\gamma}{p}|v|^p\right) dx
\end{aligned} \tag{3.35}$$

for some positive constants C_{13}, C_{14} and C_{15}. In this manner, due to (3.33), we obtain

$$\begin{aligned}
\int_\Omega \left|\widetilde{h}(x, v_{n_k}) - \widetilde{h}(x,v)\right|^{p'} dx &\leq C_{16} \int_\Omega \left|\widetilde{h}(x, v_{n_k})\right|^{p'} + \left|\widetilde{h}(x,v)\right|^{p'} dx \\
&\leq C_{17} \int_\Omega \left(1 + \left|v_{n_k}\right|^{p-1}\right)^{p'} + \left(1 + |v|^{p-1}\right)^{p'} dx \\
&\leq C_{18} \int_\Omega \left|v_{n_k}\right|^p + |v|^p dx
\end{aligned} \tag{3.36}$$

for some positive constants C_{16}, C_{17} and C_{18}. Invoking (3.34)-(3.36), and the convergence principle, one has

$$\left|\varrho(x)\left|v_{n_k}\right|^{\gamma-2} v_{n_k} - \varrho(x)|v|^{\gamma-2}v\right|^{\gamma'} \leq k_1(x) \text{ and } \left|\widetilde{h}\left(x, v_{n_k}\right) - \widetilde{h}(x,v)\right|^{p'} \leq k_2(x)$$

for almost all $x \in \Omega$ and for some $k_1, k_2 \in L^1(\Omega)$, and also $\varrho(x)\left|v_{n_k}\right|^{\gamma-2} v_{n_k} \to \varrho(x)|v|^{\gamma-2} v$ and $\widetilde{h}\left(x, v_{n_k}\right) \to \widetilde{h}(x,v)$ as $k \to \infty$ for almost all $x \in \Omega$. This together with the Lebesgue dominated convergence theorem yields that

$$\begin{aligned}
&\|\widetilde{\Psi}'\left(v_{n_k}\right) - \widetilde{\Psi}'(v)\|_{X^*} \\
&= \sup_{\|u\|_X \leq 1} \left|\left\langle \widetilde{\Psi}'\left(v_{n_k}\right) - \widetilde{\Psi}'(v), u\right\rangle\right| \\
&= \sup_{\|u\|_X \leq 1} \left|\lambda \int_\Omega \left(\varrho(x)\left|v_{n_k}\right|^{\gamma-2} v_{n_k} - \varrho(x)|v|^{\gamma-2} v\right) u\, dx \right. \\
&\quad \left. + \mu \int_\Omega \left(\widetilde{h}\left(x, v_{n_k}\right) - \widetilde{h}(x,v)\right) u\, dx\right| \\
&\leq C_{19} \left(\lambda \|\varrho(x)\left|v_{n_k}\right|^{\gamma-2} v_{n_k} - \varrho(x)|v|^{\gamma-2}v\|_{L^{\gamma'}(\Omega)} + \|\widetilde{h}\left(x, v_{n_k}\right) - \widetilde{h}(x,v)\|_{L^{p'}(\Omega)}\right) \to 0
\end{aligned}$$

as $k \to \infty$ and for some positive constant C_{19}. Therefore, we derive that $\widetilde{\Psi}'\left(v_{n_k}\right) \to \widetilde{\Psi}'(v)$ in X^* as $k \to \infty$.

Consequently, by setting $\Gamma := \Gamma_1 \cap \Gamma_2$, all conditions of Proposition 3.2 are fulfilled, and hence for $\mu \in \Gamma$ we have a sequence $c_n < 0$ for $\widetilde{\varphi}$ satisfying $c_n \to 0$ when n goes to ∞. Then for any $v_n \in X$ satisfying $\widetilde{\varphi}(v_n) = c_n$ and $\widetilde{\varphi}'(v_n) = 0$, the sequence $\{v_n\}$ is a (PS)-sequence of $\widetilde{\varphi}(v)$ and $\{v_n\}$ admits a convergent subsequence. Thus, up to a subsequence, still denoted by $\{v_n\}$, one has $v_n \to v$ in X as $n \to \infty$. Lemmas 3.7 and 3.8 imply that 0 is the only critical point with 0 energy and the subsequence $\{v_n\}$ has to converge to 0 in X; so $\|v_n\|_{L^t(\Omega)} \to 0$ as $n \to \infty$ for any t with $1 \le t < p^*$. According to Proposition 3.1, any weak solution u of our problem belongs to the space $L^\infty(\Omega)$ and there exist positive constants C, η independent of u such that

$$\|u\|_{L^\infty(\Omega)} \le C\|u\|^\eta_{L^r(\Omega)}.$$

From this fact, we know $\|v_n\|_{L^\infty(\Omega)} \to 0$ and thus by Lemma 3.8 again, we have $\|v_n\|_{L^\infty(\Omega)} \le s_3$ for large n. Thus $\{v_n\}$ with large enough n is a sequence of weak solutions of the problem (1.2). The proof is complete. □

Acknowledgements

The authors gratefully thank to the Referee for the constructive comments and recommendations which definitely help to improve the readability and quality of the paper.

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