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Uniqueness of bubbling solutions with collapsing singularities

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A B S T R A C T

The seminal work [7] by Brezis and Merle showed that the bubbling solutions of the mean field equation have the property of mass concentration. Recently, Lin and Tarantello in [30] found that the "bubbling implies mass concentration" phenomena might not hold if there is a collapse of singularities. Furthermore, a sharp estimate [23] for the bubbling solutions has been obtained. In this paper, we prove that there exists at most one sequence of bubbling solutions if the collapsing singularity occurs. The main difficulty comes from that after re-scaling, the difference of two solutions locally converges to an element in the kernel space of the linearized operator. It is well-known that the kernel space is three dimensional. So the main technical ingredient of the proof is to show that the limit after re-scaling is orthogonal to the kernel space.

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1. Introduction

We are concerned with the following mean field type equation:

$$\begin{cases} \Delta_M u + \rho\left(\frac{h_* e^u}{\int_M h_* e^u dv_g} - 1\right) = 4\pi \sum_{q_i \in S} \alpha_i(\delta_{q_i} - 1) \text{ in } M, \\ \int_M u dv_g = 0, \end{cases} \tag{1.1}$$

where (M, ds) is a compact Riemann surface, $\rho > 0$, dv_g is the volume form, Δ_M is the Laplace–Beltrami operator on (M, ds), $S \subseteq M$ is a finite set of distinct points q_i, $\alpha_{q_i} > -1$, and δ_{q_i} is the Dirac measure at q_i. The point q_i with Dirac measure is called vortex point or singular source. Throughout this paper, we always assume that $|M| = 1$, $h_* > 0$ and $h_* \in C^3(M)$. The equation (1.1) arises in various different fields. In conformal geometry, (1.1) is related to the Nirenberg problem of finding prescribed Gaussian curvature if $S = \varnothing$, and the existence of a positive constant curvature metric with conic singularities if $S \neq \varnothing$ (see [42] and the references therein). The equation (1.1) is also related to the self-dual equation of the relativistic Chern–Simons–Higgs model. For the recent developments related to (1.1), we refer to the readers to [6,4,2,7,15,11–14, 29,33,31,32,35–37,39,42,44,43,5,8,17–21,28,38,41,45] and references therein.

In the seminal work [7] by Brezis and Merle, the blow up behavior of solutions for (1.1) has been studied as follows:

Theorem A. [7,27,4] *Given fixed each vortex point $q_i \in S$, suppose $\alpha_i \in \mathbb{N}$, $i = 1, \cdots, N$. We assume that h^* is a positive smooth function. Let u_k^* be a sequence of blow up solutions for (1.1), that is: $\max_M u_k^* \to +\infty$ as $k \to +\infty$. Then there is a non-empty finite set $\mathcal{B}$ (blow up set) such that,*

$$\rho \frac{h^* e^{u_k^*}}{\int_M h^* e^{u_k^*} dv_g} \to \sum_{p \in \mathcal{B}} \beta_p \delta_p, \quad \textit{where} \quad \beta_p \in 8\pi\mathbb{N}.$$

For equation (1.1), we call $\rho \frac{h^* e^{u^*}}{\int_M h^* e^{u^*} dv_g}$ the *mass distribution* of the solution u^*. Following this terminology, Theorem A states that: When the vortex points are not collapsing, the mean field equation possesses the property of the so-called "blow up solutions has the mass concentration property". The version of Theorem A for the following Chern–Simons–Higgs (CSH) equation was also proved in [15, Theorem 3.1]:

$$\Delta_{\mathbb{T}} u + \frac{1}{\varepsilon^2} e^u (1 - e^u) = 4\pi \sum_{j=1}^{N} \delta_{p_j} \quad \text{in} \quad \mathbb{T}, \tag{1.2}$$

where $\varepsilon > 0$ is a small parameter and $\mathbb{T}$ is a flat torus. The equation (1.2) was derived from the CSH model to describe vortices in high temperature superconductivity, and has been extensively studied during few decades. We refer the readers to [22,40,44,15, 16,33,34,39] and references therein. Among them, Lin and Yan in [34] proved the local uniqueness result of bubbling solutions for (1.2), that is, if $u_{\varepsilon,1}$ and $u_{\varepsilon,2}$ are two sequence of bubbling solutions blowing up at the same points under some non-degenerate condition, then $u_{\varepsilon,1} = u_{2,\varepsilon}$ for small $\varepsilon > 0$. By applying the idea in [34], the local uniqueness result of bubbling solutions of (1.1) was obtained in [3]. We note that the works [34,3] are concerned with the local uniqueness of bubbling solutions when the vortex points are not collapsing.

However, there is a big difference when the collapsing singularities are considered. First, Lin and Tarantello in [30] observed a new phenomena such that blow-up solutions with collapsing singularities might not have the "concentration" property of its mass distribution. The general version was studied in [23]. To describe the results, let us consider the following equation:

$$\begin{cases} \Delta_M u_t^* + \rho\left(\frac{h^* e^{u_t^*}}{\int_M h^* e^{u_t^*}\,dv_g} - 1\right) \\ = 4\pi\sum_{i=1}^d \alpha_i(\delta_{q_i(t)} - 1) + 4\pi\sum_{i=d+1}^N \alpha_i(\delta_{q_i} - 1) \text{ in } M, \\ \int_M u_t^* dv_g = 0, \end{cases} \tag{1.3}$$

where $\lim_{t\to 0} q_i(t) = \mathfrak{q} \notin \{q_{d+1}, \cdots, q_N\}$, $\forall\ i = 1, \cdots, d$ and $q_i(t) \neq q_j(t)$ for $i \neq j \in \{1, \cdots, d\}$. Then the following holds:

Theorem B. [30,23] *Assume $\alpha_i \in \mathbb{N}$, $1 \le i \le N$. Let u_t^* be a sequence of blow up solutions of (1.3) with $\rho \notin 8\pi\mathbb{N}$. Then u_t^* blows up only at $\mathfrak{q} \in M$. Furthermore, there exists a function w^* such that*

$$u_t^* \to w^* \text{ in } C^2_{\text{loc}}(M \setminus \{\mathfrak{q}\})$$

as $t \to 0$, and w^* satisfies:

$$\begin{cases} \Delta_M w^* + (\rho - 8m\pi)\left(\frac{h^* e^{w^*}}{\int_M h^* e^{w^*}} - 1\right) \\ = 4\pi\left(\sum_{i=1}^d \alpha_i - 2m\right)(\delta_{\mathfrak{q}} - 1) + 4\pi\sum\limits_{i=d+1}^{N} \alpha_i(\delta_{q_i} - 1) \quad in \quad M, \\ \int_M w^* dv_g = 0, \end{cases} \tag{1.4}$$

for some $m \in \mathbb{N}$ with $1 \le m \le [\frac{1}{2}\sum_{i=1}^d \alpha_i]$[1] *and $\rho > 8m\pi$.*

So Theorem B tells us that the mass concentration does no longer hold if the collapsing singularity occurs. Indeed, we have $\lim_{t\to 0}\int_M h^* e^{u_t^*} dv_g < +\infty$, which is different from the situation described in Theorem A. We note that Theorem B could be improved provided that the following nondegeneracy condition holds:

[1] $[x]$ stands for the integer part of x.

Definition 1.1. A solution w^* of (1.4) is said non-degenerate, if the linearized problem

$$\Delta_M \phi + (\rho - 8m\pi)\frac{h^* e^{w^*}}{\int_M h^* e^{w^*} dv_g}\left(\phi - \frac{\int_M h^* e^{w^*}\phi dv_g}{\int_M h^* e^{w^*} dv_g}\right) = 0, \quad \int\limits_M \phi dv_g = 0 \quad (1.5)$$

only admits the trivial solution.

Using the transversality theorem, we can always choose a positive smooth function h^* such that w^* is non-degenerate. See Theorem 4.1 in [24]. Based on the non-degeneracy assumption for (1.4), some sharper estimates on the bubbling solutions of (1.3) were obtained in [23] (see also section 2 below).

For the simplicity, throughout this paper, we focus on the case where the collapsing vortices are only *two*, that is,

$$d = 2, \alpha_1 = \alpha_2 = 1, \alpha_i \in \mathbb{N} \ \text{ for } \ i = 3, \cdots, N. \quad (1.6)$$

Our main goal is to prove the local uniqueness of blow up solutions of (1.3) with collapsing singularities.

Theorem 1.2. *Assume that* (1.6) *holds and* $\rho \notin 8\pi\mathbb{N}$. *Suppose that* $u^*_{t,1}$ *and* $u^*_{t,2}$ *are two blow up solutions of* (1.3). *Assume that* $u^*_{t,1}, u^*_{t,2} \to w^*$ *in* $C_{loc}(M \setminus \{\mathfrak{q}\})$, *where* w^* *is a non-degenerate solution of (1.4) with* $m = 1$. *Then* $u^*_{t,1} = u^*_{t,2}$ *for sufficiently small* $t > 0$.

We remark that the study of blow up solutions of (1.3) with collapsing singularities is also important to compute the topological degree for the Toda system as noticed in [24,26], where the degree counting of the whole system is reduced to computing the degree of the corresponding shadow system (see [24, Theorem 1.4]). Thus it is inevitable to encounter with the phenomena of collapsing singularities when we want to find a priori bound for solutions of an associated shadow system. For the details, we refer to the readers to [24,26].

In order to prove Theorem 1.2, we need to analyze the asymptotic behaviour of $\zeta_t = \frac{u^{(1)}_{t,*} - u^{(2)}_{t,*}}{\|u^{(1)}_{t,*} - u^{(2)}_{t,*}\|_{L^\infty(M)}}$. After a suitable scaling on small domain of $\mathfrak{q}$, ζ_t converges to an entire solution of the linearized problem associated to the Liouville equation:

$$\Delta v + e^v = 0 \quad \text{in } \mathbb{R}^2, \quad (1.7)$$

where $\Delta = \sum_{i=1}^2 \frac{\partial^2}{\partial x_i^2}$ denotes the standard Laplacian in $\mathbb{R}^2$. A solution v of (1.7) is completely classified [10] such that

$$v(z) = v_{\mu,a}(z) = \ln \frac{8e^\mu}{(1 + e^\mu|z + a|^2)^2}, \quad \mu \in \mathbb{R}, \ a = (a_1, a_2) \in \mathbb{R}^2. \quad (1.8)$$

The freedom in the choice of μ and a is due to the invariance of equation (1.7) under dilations and translations. The linearized operator L relative to $v_{0,0}$ is defined by,

$$L\phi := \Delta\phi + \frac{8}{(1+|z|^2)^2}\phi \quad \text{in } \mathbb{R}^2. \tag{1.9}$$

In [1, Proposition 1], it has been proved that any kernel of L is a linear combination of Y_0, Y_1, Y_2, where

$$\begin{cases} Y_0(z) := \frac{1-|z|^2}{1+|z|^2} = -1 + \frac{2}{1+|z|^2} = \frac{\partial v_{\mu,a}}{\partial \mu}\Big|_{(\mu,a)=(0,0)}, \\ Y_1(z) := \frac{z_1}{1+|z|^2} = -\frac{1}{4}\frac{\partial v_{\mu,a}}{\partial a_1}\Big|_{(\mu,a)=(0,0)}, \\ Y_2(z) := \frac{z_2}{1+|z|^2} = -\frac{1}{4}\frac{\partial v_{\mu,a}}{\partial a_2}\Big|_{(\mu,a)=(0,0)}. \end{cases} \tag{1.10}$$

The orthogonality to Y_1, Y_2 can be obtained by applying a suitable Pohozaev-type identities as in [34]. However, due to the non-concentration of mass, we meet a new difficulty to show the orthogonality with Y_0. In order to overcome this obstacle, we apply the Green's representation formula with some delicate analysis. This idea comes from the recent work [25]. In [25], it was proved that if w^* is a non-degenerate solution of (1.4), and the assumptions (1.6) and $\rho \notin 8\pi\mathbb{N}$ hold, then there is a blow up solution u_t^* of (1.3) such that $u_t^* \to w^*$ in $C_{\text{loc}}(M \setminus \{\mathfrak{q}\})$.

This paper is organized as follows. In section 2, we review some known sharp estimates for blow up solutions of (1.3). In section 3, we analyze the limit behavior of ζ_t in $M \setminus \{\mathfrak{q}\}$ and a small neighborhood of $\mathfrak{q}$ respectively. Finally, we prove Theorem 1.2 by using suitable Pohozaev-type identities and Green's representation formula.

2. Preliminary

Let $G(x,p)$ denote the Green's function for the Laplace Beltrami operator Δ_M on M, that is

$$\Delta_M G(x,p) + (\delta_p - 1) = 0, \qquad \int_M G(x,p)d\sigma(x) = 0. \tag{2.1}$$

We recall the following assumption:

$$d = 2, \alpha_1 = \alpha_2 = 1, \alpha_i \in \mathbb{N} \ \text{ for } \ i = 3, \cdots, N.$$

Let u_t^* be a sequence of blow up solutions of (1.3) and w^* be the limit of u_t^* in Theorem B. Set

$$u_t(x) = u_t^*(x) + 4\pi\sum_{i=1}^{2} G(x, q_i(t)) + 4\pi\sum_{i=3}^{N}\alpha_i G(x, q_i), \tag{2.2}$$

and

$$w(x) = w^*(x) + 4\pi \sum_{i=3}^{N} \alpha_i G(x, q_i). \tag{2.3}$$

We may choose a suitable coordinate centered at $\mathfrak{q}$ and

$$\mathfrak{q} = 0, \quad q_1(t) = t\underline{e}, \quad q_2(t) = -t\underline{e}, \quad \text{where } \ \underline{e} \ \text{ is a fixed unit vector in } \ \mathbb{S}^1.$$

We can rewrite equation (1.3) as follows

$$\begin{cases} \Delta_M u_t + \rho \left(\frac{h(x)e^{u_t(x)-G_t(x)}}{\int_M he^{u_t-G_t}dv_g} - 1 \right) = 0, \\ \int_M u_t dv_g = 0, \end{cases} \tag{2.4}$$

where

$$G_t(x) := 4\pi G(x, t\underline{e}) + 4\pi G(x, -t\underline{e}), \quad \text{and} \tag{2.5}$$

$$h(x) := h_*(x) \exp(-4\pi \sum_{i=3}^{N} \alpha_i G(x, q_i)) \geq 0, \quad h \in C^3(M), \quad h(0) > 0. \tag{2.6}$$

From Theorem B, we have that $u_t(x) \to w(x) + 8\pi G(x, 0)$ in $C^2_{\text{loc}}(M \setminus \{0\})$ and w satisfies

$$\begin{cases} \Delta_M w + (\rho - 8\pi) \left(\frac{h(x)e^{w(x)}}{\int_M he^w dv_g} - 1 \right) = 0, \\ \int_M w dv_g = 0, \quad w \in C^2(M). \end{cases} \tag{2.7}$$

We assume that the local isothermal coordinate system satisfies

$$ds^2 = e^{2\varphi(x)}|dx|^2, \ \varphi(0) = \nabla\varphi(0) = 0, \tag{2.8}$$

that is, $e^{2\varphi}\Delta_M = \Delta$, where $\Delta = \sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2}$ denotes the standard Laplacian in $\mathbb{R}^2$. Fix a small constant $r_0 \in (0, \frac{1}{2})$. It is well known that the conformal factor φ is a solution of

$$-\Delta\varphi = e^{2\varphi}K \ \text{ in } \ B_{r_0}(0), \tag{2.9}$$

where $K(p)$ is the Gaussian curvature at $p \in M$.

Let $\bar{\varphi}(x)$ satisfy the following local problem:

$$\Delta\bar{\varphi} - e^{2\varphi}\rho = 0 \ \text{ in } \ B_{r_0}(0), \ \ \bar{\varphi}(0) = \nabla\bar{\varphi}(0) = 0. \tag{2.10}$$

We denote

$$\psi = 2\varphi + \bar{\varphi}. \tag{2.11}$$

In view of (2.8) and (2.10), we note that

$$\nabla_x\psi(x) = \nabla\psi(0) + O(|x|) = O(|x|),\ \nabla_x\bar{\varphi}(x) = O(|x|),$$
$$\nabla_x\varphi(x) = O(|x|) \text{ for } \ x \in B_{r_0}(0). \tag{2.12}$$

By using the local coordinate, we also set the regular part of Green function $G(x, q_i(t))$ to be

$$R(x, q_i(t)) = G(x, q_i(t)) + \frac{1}{2\pi}\ln|x - q_i(t)|. \tag{2.13}$$

Let

$$R_t(x) := 4\pi R(x, t\underline{e}) + 4\pi R(x, -t\underline{e}). \tag{2.14}$$

Therefore we can formulate the local version of (2.4) around 0 as follows:

$$\Delta\bar{u}_t + h_1(x)|x - t\underline{e}|^2|x + t\underline{e}|^2 e^{\bar{u}_t(x)} = 0 \text{ in } B_{r_0}(0), \tag{2.15}$$

where

$$\bar{u}_t(x) = u_t(x) - \ln\int\limits_M he^{u_t - G_t}dv_g - \bar{\varphi}(x),$$
$$h_1(x) = \rho h(x)e^{\psi(x) - R_t(x)}, \quad h_1(x) > 0 \text{ in } B_{r_0}(0). \tag{2.16}$$

In order to study the behaviour of $\bar{u}_t$ near the origin, we consider the scaled sequence

$$v_t(y) = \bar{u}_t(ty) + 6\ln\ t,\ x \in B_{\frac{r_0}{t}}(0), \tag{2.17}$$

which satisfies:

$$\begin{cases} \Delta v_t + h_t(y)e^{v_t(y)} = 0 \ \text{ in } B_{\frac{r_0}{t}}(0), \\ \int_{B_{\frac{r_0}{t}}(0)} h_t(y)e^{v_t(y)}dy \le C, \end{cases} \tag{2.18}$$

with

$$h_t(y) = h_1(ty)|y - \underline{e}|^2|y + \underline{e}|^2 = \rho h(ty)e^{\psi(ty) - R_t(ty)}|y - \underline{e}|^2|y + \underline{e}|^2. \tag{2.19}$$

In [23], the following result was obtained.

Theorem C. [23, Theorem 1.2, Section 5] *Assume that* (1.6) *holds and* $\rho \notin 8\pi\mathbb{N}$. *Suppose that* u_t *be a sequence of blow up solutions of* (2.4). *Then the scaled function* v_t *defined by (2.17) blows up at* 0.

Now we are going to give refined estimates than those provided in Theorem B and Theorem C under the non-degeneracy assumption for (2.7). To state our result, we fix a constant $R_0 > 2$, and define the following notations:

$$\lambda_t = \max_{B_{r_0}(0)} v_t = v_t(p_t), \tag{2.20}$$

$$\rho_t = \frac{\int_{B_{tR_0}(tp_t)} \rho h e^{u_t - G_t^{(2)}} dv_g}{\int_M h e^{u_t - G_t^{(2)}} dv_g}, \tag{2.21}$$

$$C_t = \frac{1}{8} h_1(tp_t)|p_t - \underline{e}|^2 |p_t + \underline{e}|^2, \tag{2.22}$$

$$\widetilde{\phi}_t(x) = u_t(x) - w(x) - \rho_t G(x, tp_t). \tag{2.23}$$

Let $\|\widetilde{\phi}_t\|_* = \|\widetilde{\phi}_t\|_{C^1(M \setminus B_{2R_0 t}(tp_t))}$. Then we have the following result.

Theorem D. [23, Theorem 1.4, Section 5] *Assume that* (1.6) *holds and* $\rho \notin 8\pi\mathbb{N}$. *Let* u_t *be the sequence of blow up solutions of (2.4) and* $w + 8\pi G(x, 0)$ *be its limit in* $M \setminus \{0\}$. *If* w *is a non-degenerate solution of (2.7), then*

(i) $\|\widetilde{\phi}_t\|_* = O(t \ln t)$,

(ii)

$$\lambda_t + 2\ln t - \ln\left(\frac{\rho}{\rho - 8\pi} \int_M h e^w\right) + w(tp_t) + 2\ln C_t + 8\pi R(tp_t, tp_t)$$
$$= O(t \ln t),$$

(iii) $\rho_t - 8\pi = O(t^2 \ln t)$,

(iv) $\left| \int_M h e^{u_t - G_t} dv_g - \frac{\rho}{\rho - 8\pi} \int_M h e^w dv_g \right| = O(t)$,

(v) $|p_t| = O(t)$.

In order to prove Theorem D, the authors in [23] analyzed the scaled function v_t with the following ingredients: Set

$$I_t(y) = \ln \frac{e^{\lambda_t}}{(1 + C_t e^{\lambda_t} |y - q_t|^2)^2}, \tag{2.24}$$

where q_t is chosen such that $|q_t - p_t| \ll 1$ and

$$\nabla_y I_t(y)\Big|_{y = p_t} = -t\rho_t \nabla_x R(x, tp_t)\Big|_{x = tp_t} - t\nabla_x w(x)\Big|_{x = tp_t}.$$

By direct computation, we have

$$|q_t - p_t| = O(te^{-\lambda_t}). \tag{2.25}$$

For $y \in B_{2r_0}(p_t)$, we set

$$\eta_t(y) = v_t(y) - I_t(y) - (G_{*,t}(ty) - G_{*,t}(tp_t)), \tag{2.26}$$

where

$$G_{*,t}(x) = \rho_t R(x, tp_t) + w(x). \tag{2.27}$$

It is easy to see that

$$\eta_t(p_t) = v_t(p_t) - I_t(p_t) = O(t^2 e^{-\lambda_t}), \ \nabla\eta_t(p_t) = 0. \tag{2.28}$$

Let

$$\Lambda_{t,+} = \sqrt{C_t} e^{\frac{\lambda_t}{2}}, \quad \text{and} \quad \Lambda_{t,-} = (\Lambda_{t,+})^{-1} = \frac{e^{-\frac{\lambda_t}{2}}}{\sqrt{C_t}} \tag{2.29}$$

and $\widetilde{\eta}_t$ be the scaled function of η_t, that is

$$\widetilde{\eta}_t(z) = \eta_t((\Lambda_{t,-})z + p_t) \quad \text{for} \quad |z| \le 2R_0\Lambda_{t,+}.$$

For $\widetilde{\eta}_t(z)$, we have the following estimate

Theorem E. [23, Lemma 7.1] *Suppose that the assumptions of Theorem D hold. Then for any $\varepsilon \in (0,1)$, there exists a constant $C_\varepsilon > 0$, independent of $t > 0$ and $z \in B_{2R_0\Lambda_{t,+}}(0)$ such that*

$$|\widetilde{\eta}_t(z)| \le C_\varepsilon(t\|\widetilde{\phi}_t\|_* + t^2)(1 + |z|)^\varepsilon.$$

3. Uniqueness of the blow up solutions with mass concentration

To prove Theorem 1.2 is equivalent to prove the local uniqueness of blow up solutions of the equation (2.4). To show it, we argue by contradiction and suppose that (2.4) has two different blow up solutions $u_t^{(1)}$ and $u_t^{(2)}$, which satisfy $u_t^{(1)}, u_t^{(2)} \to w$ in $C_{\rm loc}(M \setminus \{0\})$, where w is a non-degenerate solution of (2.7). We will use $p_t^{(i)}$, $\lambda_t^{(i)}$, $\bar{u}_t^{(i)}$, $I_t^{(i)}$, $\widetilde{\phi}_t^{(i)}$, $C_t^{(i)}$, $q_t^{(i)}$, $v_t^{(i)}$, $\rho_t^{(i)}$, $\eta_t^{(i)}$, $\widetilde{\eta}_t^{(i)}$, $G_{*,t}^{(i)}$, $\Lambda_{t,+}^{(i)}$, $\Lambda_{t,-}^{(i)}$ to denote p_t, λ_t, $\bar{u}_t$, I_t, $\widetilde{\phi}_t$, C_t, q_t, v_t, ρ_t, η_t, $\widetilde{\eta}_t$, $G_{*,t}$, $\Lambda_{t,+}$, $\Lambda_{t,-}$ in section 2 corresponding to $u_t^{(i)}$, $i = 1, 2$, respectively.

From Theorem D, we have $|p_t^{(i)}| = O(t)$ for $i = 1, 2$. In the following lemma, we shall improve the estimation for $|p_t^{(1)} - p_t^{(2)}|$.

Lemma 3.1. $|p_t^{(1)} - p_t^{(2)}| = O\left(t^2 \ln t\right)$.

Proof. Recall that $v_t^{(i)}(y) = u_t^{(i)}(ty) - \ln \int_M he^{u_t^{(i)}-G_t}dv_g - \bar{\varphi}(ty) + 6\ln\ t$ satisfies

$$\Delta v_t^{(i)} + h_t(y)e^{v_t^{(i)}(y)} = 0, \tag{3.1}$$

where $h_t(y) = \rho h(ty)|y - \underline{e}|^2|y + \underline{e}|^2 e^{-R_t(ty)+\psi(ty)}$.

On $\partial B_{2R_0}(p_t^{(i)})$, we have

$$\begin{aligned} v_t^{(i)}(y) = &-\frac{\rho_t^{(i)}}{2\pi}\ln|y - p_t^{(i)}| + \left(6 - \frac{\rho_t^{(i)}}{2\pi}\right)\ln t + G_{*,t}^{(i)}(ty) \\ &- \ln\int_M he^{u_t^{(i)}-G_t}dv_g + \widetilde{\phi}_t^{(i)}(ty) - \bar{\varphi}(ty), \end{aligned} \tag{3.2}$$

where $G_{*,t}^{(i)}(x) = \rho_t^{(i)}R(x, tp_t^{(i)}) + w(x)$.

For any unit vector $\xi \in \mathbb{R}^2$, we apply the Pohozaev identity to (3.1) by multiplying $\xi \cdot \nabla v_t^{(i)}$, and obtain

$$\begin{aligned} &\sum_{i=1}^{2}(-1)^{i+1}\int_{B_{2R_0}(p_t^{(i)})}(\xi\cdot\nabla h_t)e^{v_t^{(i)}(y)} \\ &= \sum_{i=1}^{2}(-1)^{i+1}\int_{\partial B_{2R_0}(p_t^{(i)})}\left\{(\nu\cdot\nabla v_t^{(i)})(\xi\cdot\nabla v_t^{(i)}) - \frac{1}{2}(\nu\cdot\xi)|\nabla v_t^{(i)}|^2\right\} \\ &\quad + \sum_{i=1}^{2}(-1)^{i+1}\int_{\partial B_{2R_0}(p_t^{(i)})}(\nu\cdot\xi)h_t e^{v_t^{(i)}}, \end{aligned} \tag{3.3}$$

where ν denotes the unit normal of $\partial B_{2R_0}(p_t^{(i)})$. From (2.12), we have

$$|\nabla_y\bar{\varphi}(ty)| = t|\nabla_{ty}\bar{\varphi}(ty)| = O(t^2|y|) \ \text{ for } \ |y| \le \frac{r_0}{t}. \tag{3.4}$$

For the right hand side of (3.3), we can use (3.2), Theorem D, and Theorem E to get

$$\begin{aligned} &\text{(RHS) of (3.3)} \\ &= \sum_{i=1}^{2}(-1)^{i+1}\int_{\partial B_{2R_0}(p_t^{(i)})}\left[(\nu\cdot\nabla v_t^{(i)})(\xi\cdot\nabla v_t^{(i)}) - \frac{1}{2}(\nu\cdot\xi)|\nabla v_t^{(i)}|^2 dy + O(\sum_{i=1}^{2}e^{-\lambda_t^{(i)}})\right] \\ &= \sum_{i=1}^{2}(-1)^{i}\left[t\rho_t^{(i)}\xi\cdot\nabla_x G_{*,t}^{(i)}(x)\Big|_{x=tp_t^{(i)}} + O(t\|\widetilde{\phi}_t^{(i)}\|_* + t^2)\right] = O(t^2\ln t). \end{aligned} \tag{3.5}$$

For the left hand side of (3.3), by using Theorem D, we get that

$$
\begin{aligned}
(LHS) &= \sum_{i=1}^{2}(-1)^{i+1}\int\limits_{B_{2R_0}(p_t^{(i)})}\left(\xi\cdot\frac{\nabla h_t(p_t^{(i)})}{h_t(p_t^{(i)})}\right)h_t(y)e^{v_t^{(i)}(y)}dy\\
&+\sum_{i=1}^{2}(-1)^{i+1}\int\limits_{B_{2R_0}(p_t^{(i)})}\xi\cdot\left(\frac{\nabla h_t(y)}{h_t(y)}-\frac{\nabla h_t(p_t^{(i)})}{h_t(p_t^{(i)})}\right)h_t(y)e^{v_t^{(i)}(y)}dy\\
&=\sum_{i=1}^{2}(-1)^{i+1}\rho_t^{(1)}\left(\xi\cdot\frac{\nabla h_t(p_t^{(i)})}{h_t(p_t^{(i)})}\right)\\
&+\sum_{i=1}^{2}(-1)^{i+1}\int\limits_{B_{2R_0}(p_t^{(i)})}\xi\cdot\left(\frac{\nabla h_t(y)}{h_t(y)}-\frac{\nabla h_t(p_t^{(i)})}{h_t(p_t^{(i)})}\right)\\
&\times\frac{h_t(y)e^{\lambda_t^{(i)}+\eta_t^{(i)}+G_{*,t}^{(i)}(ty)-G_{*,t}^{(i)}(tp_t^{(i)})}}{(1+C_t^{(i)}e^{\lambda_t^{(i)}}|y-q_t^{(i)}|^2)^2}dy+O(t^2\ln t).
\end{aligned}
\tag{3.6}
$$

By the change of variable $z=\Lambda_{t,+}^{(i)}(y-p_t^{(i)})$ and $\int_{B_{2R_0\Lambda_{t,+}^{(i)}}(0)}\frac{z_k}{(1+|z|^2)^2}dz=0$ for $k=1,2$, we see that

$$
\begin{aligned}
&\int\limits_{B_{2R_0}(p_t^{(i)})}\xi\cdot\left(\frac{\nabla h_t(y)}{h_t(y)}-\frac{\nabla h_t(p_t^{(i)})}{h_t(p_t^{(i)})}\right)\frac{h_t(y)e^{\lambda_t^{(i)}+\eta_t^{(i)}+G_{*,t}^{(i)}(ty)-G_{*,t}^{(i)}(tp_t^{(i)})}}{(1+C_t^{(i)}e^{\lambda_t^{(i)}}|y-q_t^{(i)}|^2)^2}dy\\
&=\int\limits_{B_{2R_0}(p_t^{(i)})}\xi\cdot\left(\nabla\left(\frac{\nabla h_t(p_t^{(i)})}{h_t(p_t^{(i)})}\right)\cdot(y-p_t^{(i)})+O(|y-p_t^{(i)}|^2)\right)\\
&\quad\times\frac{h_t(y)e^{\lambda_t^{(i)}+\eta_t^{(i)}+G_{*,t}^{(i)}(ty)-G_{*,t}^{(i)}(tp_t^{(i)})}}{(1+C_t^{(i)}e^{\lambda_t^{(i)}}|y-q_t^{(i)}|^2)^2}dy\\
&=\int\limits_{B_{2R_0\Lambda_{t,+}^{(i)}}(0)}\xi\cdot\left(\nabla\left(\frac{\nabla h_t(p_t^{(i)})}{h_t(p_t^{(i)})}\right)\cdot(\Lambda_{t,-}^{(i)})z+O(t^2|z|^2)\right)\\
&\quad\times\frac{h_t((\Lambda_{t,-}^{(i)})z+p_t^{(i)})e^{\widetilde{\eta}_t^{(i)}+G_{*,t}^{(i)}(t(\Lambda_{t,-}^{(i)})z+tp_t^{(i)})-G_{*,t}^{(i)}(tp_t^{(i)})}}{C_t^{(i)}(1+|z+\Lambda_{t,+}^{(i)}(p_t^{(i)}-q_t^{(i)})|^2)^2}dz\\
&=\int\limits_{B_{2R_0\Lambda_{t,+}^{(i)}}(0)}\xi\cdot\left(\nabla\left(\frac{\nabla h_t(p_t^{(i)})}{h_t(p_t^{(i)})}\right)\cdot(\Lambda_{t,-}^{(i)})z+O(t^2|z|^2)\right)\\
&\quad\times\frac{h_t(p_t^{(i)})(1+O(t|z|)+O(|\widetilde{\eta}_t^{(i)}|)+O(t^2))}{C_t^{(i)}(1+|z|^2)^2}dz
\end{aligned}
$$

$$= O(t^2 \ln t) \quad \text{for } i = 1, 2, \tag{3.7}$$

here we used Theorem E in the last line.

From (3.5)–(3.7), we have

$$\frac{\nabla h_t(p_t^{(1)})}{h_t(p_t^{(1)})} - \frac{\nabla h_t(p_t^{(2)})}{h_t(p_t^{(2)})} = O(t^2 \ln t). \tag{3.8}$$

By using the expression (2.19) of h_t, we see that

$$\begin{aligned} &\frac{\nabla h_t(p_t^{(1)})}{h_t(p_t^{(1)})} - \frac{\nabla h_t(p_t^{(2)})}{h_t(p_t^{(2)})} \\ &= \nabla(\ln|y-\underline{e}|^2|y+\underline{e}|^2)|_{y=p_t^{(1)}} - \nabla(\ln|y-\underline{e}|^2|y+\underline{e}|^2)|_{y=p_t^{(2)}} + O(t|p_t^{(1)} - p_t^{(2)}|). \end{aligned} \tag{3.9}$$

Note that $|p_t^{(1)} - p_t^{(2)}| = O(t)$ from Theorem D, and $\nabla^2(\ln|y-\underline{e}|^2|y+\underline{e}|^2)|_{y=0}$ is invertible. So (3.8) and (3.9) yield that $|p_t^{(1)} - p_t^{(2)}| = O(t^2 \ln t)$, and thus we complete the proof of Lemma 3.1. □

Now we are going to estimate $\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}$.

Lemma 3.2.

$$\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)} = O(t \ln t).$$

Proof. We note that $M \setminus B_{2R_0 t}(tp_t^{(2)}) \subseteq M \setminus B_{2R_1 t}(tp_t^{(1)})$ for some $R_1 > 0$. For $x \in M \setminus B_{2R_0 t}(tp_t^{(1)})$, we see from Theorem D that

$$\begin{aligned} &u_t^{(1)}(x) - u_t^{(2)}(x) \\ &= (w(x) + \rho_t^{(1)} G(x, tp_t^{(1)}) + \widetilde{\phi}_t^{(1)}(x)) - (w(x) + \rho_t^{(2)} G(x, tp_t^{(2)}) + \widetilde{\phi}_t^{(2)}(x)) \\ &= \rho_t^{(1)} G(x, tp_t^{(1)}) - \rho_t^{(2)} G(x, tp_t^{(2)}) + O(t \ln t). \end{aligned} \tag{3.10}$$

Together with Theorem D and Theorem E, we have for some $\theta \in (0, 1)$,

$$\begin{aligned} u_t^{(1)}(x) - u_t^{(2)}(x) &= -\frac{\rho_t^{(1)}}{2\pi}(\ln|x - tp_t^{(1)}| - \ln|x - tp_t^{(2)}|) + O(t \ln t) \\ &= \frac{O(t|p_t^{(1)} - p_t^{(2)}|)}{\theta|x - tp_t^{(1)}| + (1-\theta)|x - tp_t^{(2)}|} + O(t \ln t) \\ &= O(|p_t^{(1)} - p_t^{(2)}|) + O(t \ln t) = O(t \ln t) \quad \text{for} \quad x \in M \setminus B_{2R_0 t}(tp_t^{(1)}). \end{aligned} \tag{3.11}$$

We note that $B_{2R_0 t}(tp_t^{(2)}) \subseteq B_{2R_2 t}(tp_t^{(1)})$ for some $R_2 > 0$. For $y \in B_{2R_0}(p_t^{(1)})$, we see that

$$
\begin{aligned}
&\eta_t^{(1)}(y) - \eta_t^{(2)}(y) \\
&= \Big(v_t^{(1)}(y) - I_t^{(1)}(y) - (G_{*,t}^{(1)}(ty) - G_{*,t}^{(1)}(tp_t^{(1)}))\Big) \\
&\quad - \Big(v_t^{(2)}(y) - I_t^{(2)}(y) - (G_{*,t}^{(2)}(ty) - G_{*,t}^{(2)}(tp_t^{(2)}))\Big) \\
&= u_t^{(1)}(ty) - \ln \int_M he^{u_t^{(1)} - G_t} dv_g - \lambda_t^{(1)} + 2\ln(1 + C_t^{(1)} e^{\lambda_t^{(1)}} |y - q_t^{(1)}|^2) \\
&\quad - \Big(u_t^{(2)}(ty) - \ln \int_M he^{u_t^{(2)} - G_t} dv_g - \lambda_t^{(2)} + 2\ln(1 + C_t^{(2)} e^{\lambda_t^{(2)}} |y - q_t^{(2)}|^2)\Big) \\
&\quad + O(t).
\end{aligned} \tag{3.12}
$$

By Theorem D, we have

$$
\int_M he^{u_t^{(1)} - G_t} dv_g - \int_M he^{u_t^{(2)} - G_t} dv_g = O(t), \tag{3.13}
$$

$$
\begin{aligned}
&\lambda_t^{(i)} + 2\ln t + 2\ln C_t^{(i)} + 8\pi R(tp_t^{(i)}, tp_t^{(i)}) \\
&\quad - \ln\Big(\frac{\rho}{\rho - 8\pi} \int_M he^w\Big) + w(tp_t^{(i)}) = O(t\ln t),
\end{aligned} \tag{3.14}
$$

and

$$
\begin{aligned}
C_t^{(1)} - C_t^{(2)} &= \frac{\rho h(tp_t^{(1)})|p_t^{(1)} - \underline{e}|^2 |p_t^{(1)} + \underline{e}|^2 e^{-R_t(tp_t^{(1)}) + \psi(tp_t^{(1)})}}{8} \\
&\quad - \frac{\rho h(tp_t^{(2)})|p_t^{(2)} - \underline{e}|^2 |p_t^{(2)} + \underline{e}|^2 e^{-R_t(tp_t^{(2)}) + \psi(tp_t^{(2)})}}{8} \\
&= O(|p_t^{(1)} - p_t^{(2)}|) = O(t^2 \ln t),
\end{aligned} \tag{3.15}
$$

which imply

$$
\lambda_t^{(1)} - \lambda_t^{(2)} = O(t\ln t). \tag{3.16}
$$

For $y \in B_{2R_0}(p_t^{(1)})$, we want to estimate

$$
2\ln(1 + C_t^{(1)} e^{\lambda_t^{(1)}} |y - q_t^{(1)}|^2) - 2\ln(1 + C_t^{(2)} e^{\lambda_t^{(2)}} |y - q_t^{(2)}|^2).
$$

In view of (2.25) and Lemma 3.1, we have

$$
|q_t^{(1)} - q_t^{(2)}| \le \sum_{i=1}^{2} |p_t^{(i)} - q_t^{(i)}| + |p_t^{(1)} - p_t^{(2)}| = O(t^2 \ln t). \tag{3.17}
$$

Let $y = q_t^{(1)} + \Lambda_{t,-}^{(1)} z$. Then we have for $y \in B_{2R_0}(p_t^{(1)})$,

$$\begin{aligned}
&2\ln(1+C_t^{(1)}e^{\lambda_t^{(1)}}|y-q_t^{(1)}|^2)-2\ln(1+C_t^{(2)}e^{\lambda_t^{(2)}}|y-q_t^{(2)}|^2)\\
&=2\ln(1+|z|^2)-2\ln(1+\frac{C_t^{(2)}e^{\lambda_t^{(2)}}}{C_t^{(1)}e^{\lambda_t^{(1)}}}|\Lambda_{t,+}^{(1)}(y-q_t^{(1)})+\Lambda_{t,+}^{(1)}(q_t^{(1)}-q_t^{(2)})|^2) \quad (3.18)\\
&=2\ln(1+|z|^2)-2\ln(1+(1+O(t\ln t))|z+O(t\ln t)|^2)=O(t\ln t).
\end{aligned}$$

By Theorem E, we have

$$\eta_t^{(1)}(y)-\eta_t^{(2)}(y)=O(t\ln t) \quad \text{for} \quad y\in B_{2R_0}(p_t^{(1)}). \tag{3.19}$$

From (3.12)–(3.19), we have

$$u_t^{(1)}(x)-u_t^{(2)}(x)=O(t\ln t) \quad \text{for} \quad x\in B_{2R_0 t}(tp_t^{(1)}). \tag{3.20}$$

By (3.11) and (3.20), we complete the proof of Lemma 3.2. □

Let

$$\zeta_t(x)=\frac{u_t^{(1)}(x)-u_t^{(2)}(x)}{\|u_t^{(1)}-u_t^{(2)}\|_{L^\infty(M)}}, \tag{3.21}$$

and

$$\widetilde{\zeta}_t(z)=\zeta_t(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)})-\frac{\int_M he^{u_t^{(1)}-G_t}\zeta_t dv_g}{\int_M he^{u_t^{(1)}-G_t}dv_g}. \tag{3.22}$$

Now we have the following estimation for the scaled function $\widetilde{\zeta}_t$.

Lemma 3.3. *There are constants b_0, b_1, and b_2 satisfying*

$$\widetilde{\zeta}_t(z)\to\widetilde{\zeta}_0(z)=b_0Y_0(z)+b_1Y_1(z)+b_2Y_2(z) \quad \text{in} \quad C^0_{loc}(\mathbb{R}^2),$$

where $Y_0(z)=\frac{1-|z|^2}{1+|z|^2}$, $Y_1(z)=\frac{z_1}{1+|z|^2}$, $Y_2(z)=\frac{z_2}{1+|z|^2}$.

Proof. First, we see that

$$\begin{aligned}
0&=\Delta_M\zeta_t+\frac{1}{\|u_t^{(1)}-u_t^{(2)}\|_{L^\infty(M)}}\left(\frac{\rho h(x)e^{u_t^{(1)}(x)-G_t(x)}}{\int_M he^{u_t^{(1)}-G_t}dv_g}-\frac{\rho h(x)e^{u_t^{(2)}(x)-G_t(x)}}{\int_M he^{u_t^{(2)}-G_t}dv_g}\right)\\
&=\Delta_M\zeta_t+\frac{\rho h(x)e^{u_t^{(1)}(x)-G_t(x)}}{\|u_t^{(1)}-u_t^{(2)}\|_{L^\infty(M)}\int_M he^{u_t^{(1)}-G_t}dv_g}\left(1-\frac{e^{u_t^{(2)}(x)-u_t^{(1)}(x)}\int_M he^{u_t^{(1)}-G_t}}{\int_M he^{u_t^{(2)}-G_t}dv_g}\right)\\
&=\Delta_M\zeta_t+\frac{\rho h(x)e^{u_t^{(1)}(x)-G_t(x)}}{\|u_t^{(1)}-u_t^{(2)}\|_{L^\infty(M)}\int_M he^{u_t^{(1)}-G_t}dv_g}\Bigg(1
\end{aligned}$$

$$-\frac{(1+u_t^{(2)}(x)-u_t^{(1)}(x)+O(\|u_t^{(1)}-u_t^{(2)}\|^2_{L^\infty(M)}))\int_M he^{u_t^{(1)}-G_t}dv_g}{\int_M he^{u_t^{(1)}-G_t}(1+u_t^{(2)}(x)-u_t^{(1)}(x)+O(\|u_t^{(1)}-u_t^{(2)}\|^2_{L^\infty(M)}))dv_g}\Bigg)$$

$$=\Delta_M\zeta_t+\frac{\rho h(x)e^{u_t^{(1)}(x)-G_t(x)}}{\int_M he^{u_t^{(1)}-G_t}dv_g}\left(\zeta_t-\frac{\int_M he^{u_t^{(1)}-G_t}\zeta_t dv_g}{\int_M he^{u_t^{(1)}-G_t}dv_g}+O(\|u_t^{(1)}-u_t^{(2)}\|_{L^\infty(M)})\right). \tag{3.23}$$

By using the change of variables $y=t\Lambda_{t,-}^{(1)}z+tp_t^{(1)}$, (2.17), (2.26), we have

$$\begin{aligned}
\Delta_z\widetilde{\zeta}_t(z)&=-\frac{t^6(\Lambda_{t,-}^{(1)})^2\rho h(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)})e^{u_t^{(1)}(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)})-R_t(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)})}}{\int_M he^{u_t^{(1)}-G_t}dv_g}\\
&\quad\times\left|\Lambda_{t,-}^{(1)}z+p_t^{(1)}-\underline{e}\right|^2\left|\Lambda_{t,-}^{(1)}z+p_t^{(1)}+\underline{e}\right|^2(\widetilde{\zeta}_t(z)+O(\|u_t^{(1)}-u_t^{(2)}\|_{L^\infty(M)}))\\
&=-(\Lambda_{t,-}^{(1)})^2h_1(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)})e^{v_t^{(1)}(\Lambda_{t,-}^{(1)}z+p_t^{(1)})}\\
&\quad\times\left|\Lambda_{t,-}^{(1)}z+p_t^{(1)}-\underline{e}\right|^2\left|\Lambda_{t,-}^{(1)}z+p_t^{(1)}+\underline{e}\right|^2(\widetilde{\zeta}_t(z)+O(\|u_t^{(1)}-u_t^{(2)}\|_{L^\infty(M)}))\\
&=-\left(\frac{1}{C_t^{(1)}}\right)\frac{h_1(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)})e^{G_{*,t}^{(1)}(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)})-G_{*,t}^{(1)}(tp_t^{(1)})+\widetilde{\eta}_t^{(1)}(z)}}{(1+|z+\Lambda_{t,+}^{(1)}(p_t^{(1)}-q_t^{(1)})|^2)^2}\\
&\quad\times\left|\Lambda_{t,-}^{(1)}z+p_t^{(1)}-\underline{e}\right|^2\left|\Lambda_{t,-}^{(1)}z+p_t^{(1)}+\underline{e}\right|^2(\widetilde{\zeta}_t(z)+O(\|u_t^{(1)}-u_t^{(2)}\|_{L^\infty(M)}))\\
&=\frac{-8h_1(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)})}{h_1(tp_t^{(1)})}\frac{\left|\Lambda_{t,-}^{(1)}z+p_t^{(1)}-\underline{e}\right|^2\left|\Lambda_{t,-}^{(1)}z+p_t^{(1)}+\underline{e}\right|^2}{|p_t^{(1)}-\underline{e}|^2|p_t^{(1)}+\underline{e}|^2}\\
&\quad\times\frac{(\widetilde{\zeta}_t(z)+O(\|u_t^{(1)}-u_t^{(2)}\|_{L^\infty(M)}))(1+O(|\widetilde{\eta}_t(z)|)+O(t^2|z|))}{(1+|z+\Lambda_{t,+}^{(1)}(p_t^{(1)}-q_t^{(1)})|^2)^2}.
\end{aligned} \tag{3.24}$$

Together with (2.25), Lemma 3.2, and Theorem E, we have for $z\in B_{2\Lambda_{t,+}R_0}(0)$,

$$\Delta_z\widetilde{\zeta}_t(z)+\frac{8\widetilde{\zeta}_t(z)}{(1+|z|^2)^2}=-\frac{8\widetilde{\zeta}_t\nabla\ln H_t(p_t^{(1)})\cdot(\Lambda_{t,-}^{(1)}z)+O(t\ln t)+O(t^2|z|^2)}{(1+|z|^2)^2}, \tag{3.25}$$

where

$$H_t(x)=h_1(tx)|x-\underline{e}|^2|x+\underline{e}|^2. \tag{3.26}$$

Since $\widetilde{\zeta}_t$ is uniformly bounded, there is a function $\widetilde{\zeta}_0$ such that $\widetilde{\zeta}_t\to\widetilde{\zeta}_0$ in $C_{\mathrm{loc}}(\mathbb{R}^2)$, where

$$\begin{cases}\Delta\widetilde{\zeta}_0+\frac{8\widetilde{\zeta}_0}{(1+|z|^2)^2}=0\ \text{ in }\ \mathbb{R}^2,\\ \|\widetilde{\zeta}_0\|_{L^\infty(\mathbb{R}^2)}\le c\ \text{ for some constant }\ c>0.\end{cases} \tag{3.27}$$

By [1, Proposition 1], we see that $\widetilde{\zeta}_0(z) = \sum_{i=0}^{2} b_i Y_i(z)$ for some constants $b_i \in \mathbb{R}^2$, $i = 0, 1, 2$. This completes the proof of Lemma 3.3. □

In the following lemma, we observe the behavior of ζ_t in $M \setminus \{0\}$.

Lemma 3.4. *(i)* $\zeta_t \to 0$ *in* $C^0_{loc}(M \setminus \{0\})$,
(ii) $\lim_{t\to 0}\left(\int_M he^{u_t^{(1)}-G_t}\zeta_t dv_g\right) = 0$.

Proof. We recall from (3.23) that in M,

$$\Delta_M \zeta_t + \frac{\rho h(x) e^{u_t^{(1)}-G_t}}{\int_M he^{u_t^{(1)}-G_t}dv_g}\left(\zeta_t - \frac{\int_M he^{u_t^{(1)}-G_t}\zeta_t dv_g}{\int_M he^{u_t^{(1)}-G_t}dv_g} + O(\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}\right) = 0.$$

Since $\|\zeta_t\|_{L^\infty(M)} \leq 1$, we see that there is a function ζ_* satisfying

$$\zeta_t \to \zeta_* \ \text{ in } \ C_{\text{loc}}(M \setminus \{0\}). \tag{3.28}$$

From Theorem D, we have

$$\lim_{t\to 0}\int_M he^{u_t^{(1)}-G_t}dv_g = \frac{\rho}{\rho - 8\pi}\int_M he^w dv_g. \tag{3.29}$$

For any small fixed $r \in (0, 1)$, we see from Theorem D that

$$\begin{aligned}
\int_M he^{u_t^{(1)}-G_t}\zeta_t dv_g &= \left[\int_{M\setminus B_r(0)} + \int_{B_r(0)\setminus B_{2R_0t}(tp_t^{(1)})}\right] he^{u_t^{(1)}-G_t}\zeta_t dv_g \\
&\quad + \int_{B_{2R_0t}(tp_t^{(1)})} he^{u_t^{(1)}-G_t}\left(\zeta_t - \frac{\int_M he^{u_t^{(1)}-G_t}\zeta_t dv_g}{\int_M he^{u_t^{(1)}-G_t}dv_g}\right)dv_g \\
&\quad + \int_{B_{2R_0t}(tp_t^{(1)})} he^{u_t^{(1)}-G_t}dv_g \frac{\int_M he^{u_t^{(1)}-G_t}\zeta_t dv_g}{\int_M he^{u_t^{(1)}-G_t}dv_g} \\
&= \int_M he^w \zeta_* dv_g + \frac{1}{\rho}\left(\int_M he^{u_t^{(1)}-G_t}dv_g\right)\int_{B_{2R_0t}(tp_t^{(1)})}(-\Delta_M\zeta_t)dv_g \\
&\quad + \frac{8\pi}{\rho}\int_M he^{u_t^{(1)}-G_t}\zeta_t dv_g + o(1) + O(r^2) + O(\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}).
\end{aligned} \tag{3.30}$$

By using the change of variable $x = t(\Lambda_{t,-}^{(1)}z + p_t^{(1)})$, we note that as $t \to 0$,

$$\int_{B_{2R_0t}(tp_t^{(1)})} -\Delta_x\zeta_t(x)dx = \int_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)} -\Delta_z\widetilde{\zeta}_t(z)dz$$

$$= \int_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)} \frac{8\widetilde{\zeta}_t(z)+O(t|z|)+o(1)}{(1+|z|^2)^2}dz = o(1), \tag{3.31}$$

since $\widetilde{\zeta}_t \to \sum_{j=0}^2 b_jY_j$ in $C_{\text{loc}}(\mathbb{R}^2)$ and $\int_{\mathbb{R}^2}\frac{Y_i}{(1+|z|^2)^2}=0$ for $i=0,1,2$. So we obtain from (3.30) and (3.31) that

$$\left(1-\frac{8\pi}{\rho}\right)\int_M he^{u_t^{(1)}-G_t}\zeta_t dv_g = \int_M he^w\zeta_* dv_g + o(1),$$

which implies

$$\int_M he^{u_t^{(1)}-G_t}\zeta_t dv_g = \left(\frac{\rho}{\rho-8\pi}\right)\int_M he^w\zeta_* dv_g + o(1). \tag{3.32}$$

Then we have

$$\Delta_M\zeta_* + \frac{(\rho-8\pi)he^w}{\int_M he^w dv_g}\left(\zeta_* - \frac{\int_M he^w\zeta_* dv_g}{\int_M he^w dv_g}\right) = 0 \ \text{ in } \ M\setminus\{0\}. \tag{3.33}$$

Since $\|\zeta_*\|_{L^\infty(M)} \le 1$, the above equation (3.33) holds in M. Moreover, we note that

$$\int_M \zeta_t dv_g = \frac{\int_M (u_t^{(1)}-u_t^{(2)})dv_g}{\|u_t^{(1)}-u_t^{(2)}\|_{L^\infty(M)}} = 0, \tag{3.34}$$

and thus $\int_M \zeta_* dv_g = 0$. Together with non-degeneracy condition for w, we obtain $\zeta_* \equiv 0$. In view of (3.28) and (3.32), we complete the proof of Lemma 3.4. □

To connect the behavior of ζ_t in $M\setminus\{0\}$ and in a small neighborhood of 0, we need the following result.

Lemma 3.5. [25] *(i) If* $\frac{tR_0}{2} \le |x_2 - tp_t^{(1)}| \le |x_1 - tp_t^{(1)}| \le r_0$, *then*

$$\zeta_t(x_1)-\zeta_t(x_2) = O\left(\ln\frac{|x_1-tp_t^{(1)}|}{|x_2-tp_t^{(1)}|}\int_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)} \Delta\widetilde{\zeta}_t dz\right)$$

$$+ O(|x_1-tp_t^{(1)}|\ln|x_1-tp_t^{(1)}|) + O(t^{\frac{\alpha}{2}}\ln t), \tag{3.35}$$

(ii) If $t^2R_0 \le |x - tp_t^{(1)}| \le \frac{tR_0}{2}$, then

$$\zeta_t(x) - \zeta_t(tp_t^{(1)}) = O\left(\int\limits_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)} (\ln|z|)\Delta\widetilde{\zeta}_t dz\right) + O\left(\ln\frac{|x-tp_t^{(1)}|}{t^2}\int\limits_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)}\Delta\widetilde{\zeta}_t dz\right)$$
$$+ O\left(\left(\frac{|x-tp_t^{(1)}|}{t^2}\right)^{-\frac{\alpha}{2}}\ln\left(\frac{|x-tp_t^{(1)}|}{t^2}\right)\right) + O(t\ln t). \quad (3.36)$$

Proof. For any function g satisfying $g(z)(1+|z|)^{1+\frac{\alpha}{2}} \in L^2(\mathbb{R}^2)$, we recall the following estimation (see [9]): there is a constant $c > 0$, independent of $x \in \mathbb{R}^2 \setminus B_2(0)$ and g, such that

$$\left|\int\limits_{\mathbb{R}^2}(\ln|x-z| - \ln|x|)g(z)\mathrm{d}z\right| \le c|x|^{-\frac{\alpha}{2}}(\ln|x|+1)\|g(z)(1+|z|)^{1+\frac{\alpha}{2}}\|_{L^2(\mathbb{R}^2)}.$$

Together with the Green representation formula, Lemma 3.5 can be obtained. See [25] for the detail. □

Let χ_t be a cut-off function satisfying $0 \le \chi_t \le 1$, $|\nabla\chi_t| = O(t)$, $|\nabla^2\chi_t| = O(t^2)$, and

$$\chi_t(z) = \chi_t(|z|) = \begin{cases} 1 & \text{if} \quad |z| \le R_0\Lambda_{t,+}^{(1)}, \\ 0 & \text{if} \quad |z| \ge 2R_0\Lambda_{t,+}^{(1)}. \end{cases} \quad (3.37)$$

Then we have the following result.

Lemma 3.6.

(i) $\int_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)} \frac{\widetilde{\zeta}_t(z)\chi_t(z)}{(1+|z|^2)^2}dz = O(t\ln t)$,

(ii) $\int_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)} \Delta\widetilde{\zeta}_t dz = \int_{B_{2R_0t}(tp_t^{(1)})} \Delta\zeta_t dx = O(t\ln t)$,

(iii) $\lim_{t\to 0}\|\zeta_t\|_{L^\infty(M\setminus B_{\frac{tR_0}{2}}(tp_t^{(1)}))} = 0$,

(iv) $\lim_{t\to 0}\int_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)} \frac{\widetilde{\zeta}_t(z)Y_0(z)\chi_t(z)}{(1+|z|^2)^2}dz = o(1)$,

(v) $b_0 = 0$.

Proof. (i) We note that $\eta_1(z) = -\frac{2}{(1+|z|^2)}$ satisfies

$$\Delta\eta_1 + \frac{8\eta_1}{(1+|z|^2)^2} = -\frac{8}{(1+|z|^2)^2} \quad \text{in} \quad \mathbb{R}^2. \quad (3.38)$$

From (3.25), we recall the following equation:

$$\Delta_z\widetilde{\zeta}_t(z)+\frac{8\widetilde{\zeta}_t(z)}{(1+|z|^2)^2}=-\frac{8(\Lambda_{t,-}^{(1)})\widetilde{\zeta}_t\nabla\ln H_t(p_t^{(1)})\cdot z+O(t\ln t)+O(t^2|z|^2)}{(1+|z|^2)^2}.$$

Multiplying both sides of (3.25) by $\eta_1\chi_t$ and using the integration by parts, we have

$$\begin{aligned}0&=\int\limits_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)}\widetilde{\zeta}_t\left(\Delta(\eta_1\chi_t)+\frac{8\eta_1\chi_t}{(1+|z|^2)^2}\right)dz+O(t\ln t)\\&=\int\limits_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)}\widetilde{\zeta}_t\left[\left(\Delta\eta_1+\frac{8\eta_1}{(1+|z|^2)^2}\right)\chi_t+2\nabla\eta_1\cdot\nabla\chi_t+\eta_1\Delta\chi_t\right]dz\\&\quad+O(t\ln t).\end{aligned}\tag{3.39}$$

Together with (3.38), we obtain

$$\int\limits_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)}\frac{8\widetilde{\zeta}_t\chi_t}{(1+|z|^2)^2}dz=O(t\ln t).\tag{3.40}$$

(ii) By integrating (3.25) and using (3.40), we have Lemma 3.6-(ii).

(iii) By Lemma 3.5-(i) and Lemma 3.4-(i), we see that if $\frac{tR_0}{2}\le|x-tp_t^{(1)}|\le r_0$ and $|x'-tp_t^{(1)}|=r$, then

$$\zeta_t(x)=\zeta_t(x')+O(\ln t\int\limits_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)}\Delta\widetilde{\zeta}_t dz)+O(r\ln r)+o(1),$$

for any small $r>0$. Together with Lemma 3.6-(ii), we can get that Lemma 3.6-(iii).

(iv) We note that $\eta_2(z)=\frac{4}{3}\ln(1+|z|^2)\left(\frac{1-|z|^2}{1+|z|^2}\right)+\frac{8}{3(1+|z|^2)}$ satisfies

$$\Delta\eta_2+\frac{8\eta_2}{(1+|z|^2)^2}=\frac{16Y_0(z)}{(1+|z|^2)^2}\quad\text{in }\ \mathbb{R}^2.\tag{3.41}$$

Multiplying both sides of (3.25) by $\eta_2\chi_t$ and using the integration by parts, we have

$$0=\int\limits_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)}\widetilde{\zeta}_t\left[\left(\Delta\eta_2+\frac{8\eta_2}{(1+|z|^2)^2}\right)\chi_t+2\nabla\eta_2\cdot\nabla\chi_t+\eta_2\Delta\chi_t\right]dz+O(t\ln t).\tag{3.42}$$

Fix a point $e_t\in\partial B_{R_0\Lambda_{t,+}^{(1)}}(0)$. Then (3.42) implies

$$\begin{aligned}\int_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)} \frac{16Y_0\widetilde{\zeta}_t\chi_t}{(1+|z|^2)^2}dz = &-\int_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)} (\widetilde{\zeta}_t(z)-\widetilde{\zeta}_t(e_t))(2\nabla\eta_2\cdot\nabla\chi_t+\eta_2\Delta\chi_t)dz \\ &-\widetilde{\zeta}_t(e_t)\int_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)} (2\nabla\eta_2\cdot\nabla\chi_t+\eta_2\Delta\chi_t)dz + O(t\ln t).\end{aligned} \tag{3.43}$$

Together with Lemma 3.5-(i) and Lemma 3.6-(ii), we have

$$\begin{aligned}&\int_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)} \frac{16Y_0\widetilde{\zeta}_t\chi_t}{(1+|z|^2)^2}dz \\ &= \int_{R_0\Lambda_{t,+}^{(1)}\le|z|\le 2R_0\Lambda_{t,+}^{(1)}} O(t^{\frac{\alpha}{2}}\ln t)(\frac{t}{|z|}+|\ln|z||t^2)dz + O(\widetilde{\zeta}_t(e_t)) + O(t\ln t),\end{aligned} \tag{3.44}$$

here we used $\int_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)}\eta_2\Delta\chi_t dz = -\int_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)}\nabla\eta_2\cdot\nabla\chi_t dz$.

By applying Lemma 3.4-(ii) and Lemma 3.6-(iii), we obtain $\lim_{t\to 0}\widetilde{\zeta}_t(e_t) = 0$, and thus

$$\int_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)} \frac{16Y_0\widetilde{\zeta}_t\chi_t}{(1+|z|^2)^2}dz = o(1) \quad \text{as} \quad t\to 0. \tag{3.45}$$

So we obtain Lemma 3.6-(iv).

(v) By Lemma 3.3 and Lemma 3.6-(iv), we have

$$b_0 \equiv 0. \tag{3.46}$$

So we complete the proof of Lemma 3.6. □

Let

$$\widetilde{u}_t^{(i)} = u_t^{(i)} - \ln\int_M he^{u_t^{(i)}-G_t}dv_g \quad \text{for} \quad i=1,2. \tag{3.47}$$

We note that

$$\begin{aligned}&\widetilde{u}_t^{(1)} - \widetilde{u}_t^{(2)} \\ &= u_t^{(1)} - u_t^{(2)} - \ln\int_M he^{u_t^{(1)}-G_t}dv_g + \ln\int_M he^{u_t^{(2)}-G_t}dv_g\end{aligned}$$

$$
\begin{aligned}
&= u_t^{(1)} - u_t^{(2)} - \ln \int\limits_M h e^{u_t^{(1)} - G_t} dv_g \\
&\quad + \ln \int\limits_M h e^{u_t^{(1)} - G_t}(1 + u_t^{(2)} - u_t^{(1)} + O(|u_t^{(1)} - u_t^{(2)}|^2) dv_g \\
&= u_t^{(1)} - u_t^{(2)} - \frac{\int_M h e^{u_t^{(1)} - G_t}(u_t^{(1)} - u_t^{(2)}) dv_g}{\int_M h e^{u_t^{(1)} - G_t} dv_g} + O(\|u_t^{(1)} - u_t^{(2)}\|^2_{L^\infty(M)}). \quad (3.48)
\end{aligned}
$$

Let

$$
A_t := \int\limits_{B_{2R_0 t}(tp_t^{(1)})} \frac{\rho h e^{-G_t}(e^{\widetilde{u}_t^{(1)}} - e^{\widetilde{u}_t^{(2)}})}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}} dx = \int\limits_{B_{2R_0 t}(tp_t^{(1)})} -\Delta \zeta_t dx. \quad (3.49)
$$

Without loss of generality, from now on, we assume that

$$
\nabla_x (8\pi R(x,0) + w(x))\Big|_{x=0} = 0. \quad (3.50)
$$

Indeed, we can change the regular part of $G(x,0)$ locally such that

$$
8\pi R_{\text{new}}(x,0) = 8\pi R_{\text{old}}(x,0) - \nabla_x (8\pi R_{\text{old}}(x,0) + w(x))\Big|_{x=0} \cdot x.
$$

Now we shall improve Lemma 3.6-(ii) by applying the arguments in [34].

Lemma 3.7.

$$
A_t = \int\limits_{B_{2R_0 t}(tp_t^{(1)})} -\Delta \zeta_t dx = O(t).
$$

Proof. Recall that

$$
\begin{aligned}
v_t^{(i)}(y) &= \widetilde{u}_t^{(i)}(ty) + 6\ln t - \bar{\varphi}(ty) \\
&= \eta_t^{(i)}(y) + I_t^{(i)}(y) + G_{*,t}^{(i)}(ty) - G_{*,t}^{(i)}(tp_t^{(i)}) \quad \text{for} \quad i = 1, 2.
\end{aligned} \quad (3.51)
$$

Set

$$
\widetilde{v}_t^{(i)}(z) = v_t^{(i)}(\Lambda_{t,-}^{(1)} z + p_t^{(1)}) \quad \text{for} \quad i = 1, 2. \quad (3.52)
$$

Then

$$
\begin{aligned}
\widetilde{v}_t^{(i)}(z) &= \eta_t^{(i)}(\Lambda_{t,-}^{(1)} z + p_t^{(1)}) + \ln \frac{e^{\lambda_t^{(i)}}}{(1 + (\Lambda_{t,+}^{(i)})^2 |\Lambda_{t,-}^{(1)} z + p_t^{(1)} - q_t^{(i)}|^2)^2} \\
&\quad + G_{*,t}^{(i)}(t\Lambda_{t,-}^{(1)} z + tp_t^{(1)}) - G_{*,t}^{(i)}(tp_t^{(i)}).
\end{aligned} \quad (3.53)
$$

We also see from (3.48) that

$$\frac{\widetilde{v}_t^{(1)}(z)-\widetilde{v}_t^{(2)}(z)}{\|u_t^{(1)}-u_t^{(2)}\|_{L\infty(M)}}=\frac{\widetilde{u}_t^{(1)}(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)})-\widetilde{u}_t^{(2)}(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)})}{\|u_t^{(1)}-u_t^{(2)}\|_{L\infty(M)}} \\ =\widetilde{\zeta}_t(z)+O(\|u_t^{(1)}-u_t^{(2)}\|_{L^\infty(M)}), \tag{3.54}$$

which implies

$$\frac{1-e^{\widetilde{v}_t^{(2)}-\widetilde{v}_t^{(1)}}}{\|u_t^{(1)}-u_t^{(2)}\|_{L\infty(M)}}=\frac{\widetilde{v}_t^{(1)}(z)-\widetilde{v}_t^{(2)}(z)+O(|\widetilde{v}_t^{(1)}-\widetilde{v}_t^{(2)}|^2)}{\|u_t^{(1)}-u_t^{(2)}\|_{L\infty(M)}} \\ =\widetilde{\zeta}_t(z)+O(\|u_t^{(1)}-u_t^{(2)}\|_{L^\infty(M)}). \tag{3.55}$$

We have

$$\Delta_z\widetilde{v}_t^{(i)}(z)+(\Lambda_{t,-}^{(1)})^2h_t(\Lambda_{t,-}^{(1)}z+p_t^{(1)})e^{\widetilde{v}_t^{(i)}(z)}=0, \tag{3.56}$$

where $h_t(y)=\rho h(ty)|y-\underline{e}|^2|y+\underline{e}|^2e^{-R_t(ty)+\psi(ty)}$. We see that

$$(\Delta(\widetilde{v}_t^{(1)}-\widetilde{v}_t^{(2)}))(\nabla(\widetilde{v}_t^{(1)}+\widetilde{v}_t^{(2)})\cdot z)+(\Delta(\widetilde{v}_t^{(1)}+\widetilde{v}_t^{(2)}))(\nabla(\widetilde{v}_t^{(1)}-\widetilde{v}_t^{(2)})\cdot z) \\ =\mathrm{div}\Big\{(\nabla(\widetilde{v}_t^{(1)}-\widetilde{v}_t^{(2)}))(\nabla(\widetilde{v}_t^{(1)}+\widetilde{v}_t^{(2)})\cdot z) \\ +(\nabla(\widetilde{v}_t^{(1)}+\widetilde{v}_t^{(2)}))(\nabla(\widetilde{v}_t^{(1)}-\widetilde{v}_t^{(2)})\cdot z)-\nabla(\widetilde{v}_t^{(1)}-\widetilde{v}_t^{(2)})\cdot\nabla(\widetilde{v}_t^{(1)}+\widetilde{v}_t^{(2)})z\Big\}, \tag{3.57}$$

and

$$(\Delta(\widetilde{v}_t^{(1)}-\widetilde{v}_t^{(2)}))(\nabla(\widetilde{v}_t^{(1)}+\widetilde{v}_t^{(2)})\cdot z)+(\Delta(\widetilde{v}_t^{(1)}+\widetilde{v}_t^{(2)}))(\nabla(\widetilde{v}_t^{(1)}-\widetilde{v}_t^{(2)})\cdot z) \\ =-(\Lambda_{t,-}^{(1)})^2h_t(\Lambda_{t,-}^{(1)}z+p_t^{(1)})(e^{\widetilde{v}_t^{(1)}(z)}-e^{\widetilde{v}_t^{(2)}(z)})(\nabla(\widetilde{v}_t^{(1)}+\widetilde{v}_t^{(2)})\cdot z) \\ -(\Lambda_{t,-}^{(1)})^2h_t(\Lambda_{t,-}^{(1)}z+p_t^{(1)})(e^{\widetilde{v}_t^{(1)}(z)}+e^{\widetilde{v}_t^{(2)}(z)})(\nabla(\widetilde{v}_t^{(1)}-\widetilde{v}_t^{(2)})\cdot z) \\ =-\mathrm{div}\left(2(\Lambda_{t,-}^{(1)})^2h_t(\Lambda_{t,-}^{(1)}z+p_t^{(1)})(e^{\widetilde{v}_t^{(1)}(z)}-e^{\widetilde{v}_t^{(2)}(z)})z\right) \\ +4(\Lambda_{t,-}^{(1)})^2h_t(\Lambda_{t,-}^{(1)}z+p_t^{(1)})(e^{\widetilde{v}_t^{(1)}(z)}-e^{\widetilde{v}_t^{(2)}(z)}) \\ +2(\Lambda_{t,-}^{(1)})^2h_t(\Lambda_{t,-}^{(1)}z+p_t^{(1)})(e^{\widetilde{v}_t^{(1)}(z)}-e^{\widetilde{v}_t^{(2)}(z)})\left(\nabla_z\ln h_t(\Lambda_{t,-}^{(1)}z+p_t^{(1)})\cdot z\right). \tag{3.58}$$

Therefore, we obtain for any $r>0$,

$$\begin{aligned}
&\frac{1}{2}\int_{\partial B_r(0)} \nabla(\widetilde{v}_t^{(1)}-\widetilde{v}_t^{(2)})\cdot\nabla(\widetilde{v}_t^{(1)}+\widetilde{v}_t^{(2)})|z|d\sigma \\
&\quad -\int_{\partial B_r(0)} \frac{(\nabla(\widetilde{v}_t^{(1)}-\widetilde{v}_t^{(2)})\cdot z)(\nabla(\widetilde{v}_t^{(1)}+\widetilde{v}_t^{(2)})\cdot z)}{|z|}d\sigma \\
&=\int_{\partial B_r(0)} (\Lambda_{t,-}^{(1)})^2 h_t(\Lambda_{t,-}^{(1)}z+p_t^{(1)})e^{\widetilde{v}_t^{(1)}(z)}(1-e^{\widetilde{v}_t^{(2)}(z)-\widetilde{v}_t^{(1)}(z)})|z|d\sigma \qquad (3.59)\\
&-\int_{B_r(0)} (\Lambda_{t,-}^{(1)})^2 h_t(\Lambda_{t,-}^{(1)}z+p_t^{(1)})e^{\widetilde{v}_t^{(1)}(z)}(1-e^{\widetilde{v}_t^{(2)}(z)-\widetilde{v}_t^{(1)}(z)}) \\
&\quad\times\left(2+\nabla_z \ln h_t(\Lambda_{t,-}^{(1)}z+p_t^{(1)})\cdot z\right)dz.
\end{aligned}$$

Let $2R_0\Lambda_{t,+}^{(1)}\le |z|\le \frac{r_0}{t}\Lambda_{t,+}^{(1)}$. By (2.12) and Theorem D, we have

$$\nabla_z\bar{\varphi}(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)}) = t^2 O(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)}) = O(t^4(|z|+1)),$$

and

$$\begin{aligned}
\nabla_z\widetilde{v}_t^{(i)}(z) &= \nabla_z\left(\widetilde{u}_t^{(i)}(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)})-\bar{\varphi}(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)})\right)\\
&=\nabla_z\widetilde{\phi}_t^{(i)}(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)})\\
&\quad+\nabla_z\left(\rho_t^{(i)}G(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)},tp_t^{(i)})+w(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)})\right)+O(t^4|z|)\\
&=\nabla_z\widetilde{\phi}_t^{(i)}(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)})-\frac{\rho_t^{(i)}t\Lambda_{t,-}^{(1)}}{2\pi}\frac{(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)}-tp_t^{(i)})}{|t\Lambda_{t,-}^{(1)}z+tp_t^{(1)}-tp_t^{(i)}|^2}\\
&+\nabla_z\left(\rho_t^{(i)}R(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)},tp_t^{(i)})+w(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)})\right)+O(t^4|z|) \qquad (3.60)\\
&=-\frac{\rho_t^{(i)}}{2\pi}\frac{(z+\Lambda_{t,+}^{(1)}(p_t^{(1)}-p_t^{(i)}))}{|z+\Lambda_{t,+}^{(1)}(p_t^{(1)}-p_t^{(i)})|^2}+O(t^2\|\nabla\widetilde{\phi}_t\|_{L^\infty(M\setminus B_{2R_0t}(tp_t^{(1)}))})+O(t^4|z|)\\
&+O(t^2)\nabla_x\left(\rho_t^{(i)}R(x,tp_t^{(i)})+w(x)\right)\Big|_{x=t\Lambda_{t,-}^{(1)}z+tp_t^{(1)}}.
\end{aligned}$$

In view of Lemma 3.1 and Theorem D, we see that there are $a_t^{(i)}\in\mathbb{R}^2$ such that $a_t^{(1)}=0$, $|a_t^{(2)}|=O(t\ln t)$, and

$$\begin{aligned}
\nabla_z\widetilde{v}_t^{(i)}(z) &= -4\frac{z+a_t^{(i)}}{|z+a_t^{(i)}|^2}+O\left(t^2\nabla_x\left(\rho_t^{(1)}R(x,tp_t^{(1)})+w(x)\right)\Big|_{x=tp_t^{(1)}}\right)\\
&\quad+O(t^3\ln t)+O(t^4|z|) \qquad \text{for} \quad 2R_0\Lambda_{t,+}^{(1)}\le|z|\le\frac{r_0}{t}\Lambda_{t,+}^{(1)}.
\end{aligned} \qquad (3.61)$$

In view of (3.50) and Theorem D, we have $\nabla_x(\rho_t^{(1)}R(x,tp_t^{(1)})+w(x))\Big|_{x=tp_t^{(1)}} = O(t^2\ln t)$, and get that if $2R_0\Lambda_{t,+}^{(1)} \le |z| \le \frac{r_0}{t}\Lambda_{t,+}^{(1)}$, then

$$\begin{aligned}\nabla_z\widetilde{v}_t^{(i)}(z) &= -4\frac{(z+a_t^{(i)})}{|z+a_t^{(i)}|^2} + O(t^3\ln t) + O(t^4|z|)\\ &= -4\frac{z}{|z|^2} + O(\frac{|a_t^{(i)}|}{|z|^2}) + O(t^3\ln t) + O(t^4|z|)\\ &= -4\frac{z}{|z|^2} + O(t^3\ln t) + O(t^4|z|).\end{aligned} \tag{3.62}$$

From (3.34), we recall $\int_M \zeta_t dv_g = 0$. Together with Green's representation formula, we have

$$\zeta_t(x) = \int\limits_M \rho h e^{-G_t}\frac{(e^{\widetilde{u}_t^{(1)}} - e^{\widetilde{u}_t^{(2)}})}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}}G(x,y)dy,$$

and thus

$$\begin{aligned}\nabla_x\zeta_t(x) = &\int\limits_{M\setminus B_{2R_0t}(tp_t^{(1)})} \rho h e^{-G_t}\frac{(e^{\widetilde{u}_t^{(1)}} - e^{\widetilde{u}_t^{(2)}})}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}}\nabla_x G(x,y)dy\\ &+ \nabla_x G(x,tp_t^{(1)})\int\limits_{B_{2R_0t}(tp_t^{(1)})} \rho h e^{-G_t}\frac{(e^{\widetilde{u}_t^{(1)}} - e^{\widetilde{u}_t^{(2)}})}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}}dy\\ &+ \int\limits_{B_{2R_0t}(tp_t^{(1)})} \rho h e^{-G_t}\frac{(e^{\widetilde{u}_t^{(1)}} - e^{\widetilde{u}_t^{(2)}})}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}}(\nabla_x G(x,y) - \nabla_x G(x,tp_t^{(1)}))dy\\ &:= I + II + III.\end{aligned} \tag{3.63}$$

From Lemma 3.4-(ii) and Lemma 3.6-(iii), we see that if $x \in M \setminus B_{2R_0t}(tp_t^{(1)})$, then

$$\begin{aligned}I = &\int\limits_{M\setminus B_{2R_0t}(tp_t^{(1)})} \nabla_x G(x,y)\frac{\rho h e^{-G_t+\widetilde{u}_t^{(1)}}}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}}\\ &\times\Big(u_t^{(1)} - u_t^{(2)} - \frac{\int_M h e^{u_t^{(1)}-G_t}(u_t^{(1)} - u_t^{(2)})dv_g}{\int_M h e^{u_t^{(1)}-G_t}dv_g} + O(\|u_t^{(1)} - u_t^{(2)}\|^2_{L^\infty(M)})\Big)dy\\ = &\int\limits_{M\setminus B_{2R_0t}(tp_t^{(1)})} \nabla_x G(x,y)\rho h e^{-G_t+\widetilde{u}_t^{(1)}}\end{aligned}$$

$$\times\Big(\zeta_t - \frac{\int_M he^{u_t^{(1)}-G_t}\zeta_t dv_g}{\int_M he^{u_t^{(1)}-G_t}dv_g} + O(\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)})\Big)dy$$
$$= o(1) \quad \text{as} \quad t \to 0. \tag{3.64}$$

From Lemma 3.6-(ii), we have

$$A_t = -\int_{B_{2R_0t}(tp_t^{(1)})} \Delta\zeta_t dx = -\int_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)} \Delta\widetilde{\zeta}_t dz = O(t\ln t).$$

Then we see that if $x \in M \setminus B_{2R_0t}(tp_t^{(1)})$, then

$$II = \nabla_x G(x, tp_t^{(1)})A_t = \left\{-\frac{1}{2\pi}\frac{(x - tp_t^{(1)})1_{B_{r_0}(tp_t^{(1)})}(x)}{|x - tp_t^{(1)}|^2} + O(1)\right\}A_t, \tag{3.65}$$

where

$$1_{B_{r_0}(tp_t^{(1)})}(x) = \begin{cases} 1 & \text{if} \;\; x \in B_{r_0}(tp_t^{(1)}), \\ 0 & \text{if} \;\; x \in M \setminus B_{r_0}(tp_t^{(1)}). \end{cases} \tag{3.66}$$

Now we also see that if $x \in M \setminus B_{2R_0t}(tp_t^{(1)})$, then

$$III = \int_{B_{2R_0t}(tp_t^{(1)})} \frac{\rho he^{-G_t}(e^{\widetilde{u}_t^{(1)}} - e^{\widetilde{u}_t^{(2)}})}{2\pi\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}}\Big(\frac{x - tp_t^{(1)}}{|x - tp_t^{(1)}|^2} - \frac{x-y}{|x-y|^2}\Big)1_{B_{r_0}(tp_t^{(1)})}(x)dy + o(1)$$
$$= -\frac{1}{2\pi}\int_{B_{2R_0t}(tp_t^{(1)})} \Delta\zeta_t\Big(\frac{x - tp_t^{(1)}}{|x - tp_t^{(1)}|^2} - \frac{x-y}{|x-y|^2}\Big)1_{B_{r_0}(tp_t^{(1)})}(x)dy + o(1), \tag{3.67}$$

and

$$\int_{B_{2R_0t}(tp_t^{(1)})} \Delta\zeta_t\Big(\frac{x - tp_t^{(1)}}{|x - tp_t^{(1)}|^2} - \frac{x-y}{|x-y|^2}\Big)1_{B_{r_0}(tp_t^{(1)})}(x)dy$$
$$= \left(\int_{B_{R_0\Lambda_{t,+}^{(1)}}(0)} + \int_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)\setminus B_{R_0\Lambda_{t,+}^{(1)}}(0)}\right)\Delta_z\widetilde{\zeta}_t(z)$$
$$\times\Big(\frac{x - tp_t^{(1)}}{|x - tp_t^{(1)}|^2} - \frac{x - tp_t^{(1)} - t\Lambda_{t,-}^{(1)}z}{|x - tp_t^{(1)} - t\Lambda_{t,-}^{(1)}z|^2}\Big)1_{B_{r_0}(tp_t^{(1)})}(x)dz$$

$$
\begin{aligned}
&= \int\limits_{B_{R_0\Lambda_{t,+}^{(1)}}(0)} |\Delta_z\widetilde{\zeta}_t(z)| O\Big(\frac{t\Lambda_{t,-}^{(1)}|z|}{|x-tp_t^{(1)}|^2}\Big) 1_{B_{r_0}(tp_t^{(1)})}(x)dz \\
&+ \int\limits_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)\setminus B_{R_0\Lambda_{t,+}^{(1)}}(0)} \frac{O(1)}{|z|^4}\left(\frac{1}{|x-tp_t^{(1)}|}+\frac{1}{|x-tp_t^{(1)}-t\Lambda_{t,-}^{(1)}z|}\right) 1_{B_{r_0}(tp_t^{(1)})}(x)dz \\
&= \int\limits_{B_{R_0\Lambda_{t,+}^{(1)}}(0)} O(\frac{1}{(1+|z|^2)^2})\Big(\frac{t^2|z|}{|x-tp_t^{(1)}|^2}\Big) 1_{B_{r_0}(tp_t^{(1)})}(x)dz \\
&+ \int\limits_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)\setminus B_{R_0\Lambda_{t,+}^{(1)}}(0)} \frac{O(1)}{|z|^4}\left(O(t^{-1})+\frac{O(t^{-2})}{\left|\frac{x-tp_t^{(1)}}{t\Lambda_{t,-}^{(1)}}-z\right|}\right) 1_{B_{r_0}(tp_t^{(1)})}(x)dz \\
&= O(\frac{t^2 1_{B_{r_0}(tp_t^{(1)})}(x)}{|x-tp_t^{(1)}|^2}) + o(1) \quad \text{as} \quad t\to 0.
\end{aligned}
\tag{3.68}
$$

From (3.64)–(3.68), we see that if $x \in M \setminus B_{2R_0t}(tp_t^{(1)})$,

$$
\nabla_x\zeta_t(x) = -\frac{A_t}{2\pi}\frac{(x-tp_t^{(1)})1_{B_{r_0}(tp_t^{(1)})}(x)}{|x-tp_t^{(1)}|^2} + O(\frac{t^2 1_{B_{r_0}(tp_t^{(1)})}(x)}{|x-tp_t^{(1)}|^2}) + o(1) \quad \text{as} \quad t\to 0. \tag{3.69}
$$

Here we also note that if $2R_0\Lambda_{t,+}^{(1)} \le |z| \le \frac{2r_0\Lambda_{t,+}^{(1)}}{t}$, then

$$
\begin{aligned}
&\frac{\nabla_z(\widetilde{v}_t^{(1)}(z)-\widetilde{v}_t^{(2)}(z))}{\|u_t^{(1)}-u_t^{(2)}\|_{L^\infty(M)}} \\
&\quad = \frac{\nabla_z(\widetilde{u}_t^{(1)}(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)})-\widetilde{u}_t^{(2)}(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)}))}{\|u_t^{(1)}-u_t^{(2)}\|_{L^\infty(M)}} \\
&\quad = \nabla_z\zeta_t(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)}) \\
&\quad = \nabla_z\widetilde{\zeta}_t(z) = t\Lambda_{t,-}^{(1)}\left(-\frac{A_t}{2\pi}\frac{1}{t\Lambda_{t,-}^{(1)}}\frac{z}{|z|^2}+O(1)\right) = -\frac{A_t}{2\pi}\frac{z}{|z|^2}+O(t^2),
\end{aligned}
\tag{3.70}
$$

and

$$
\nabla_z\widetilde{\zeta}_t(z)\Big|_{z\in\partial B_{2R_0\Lambda_{t,+}^{(1)}}(0)} = t\Lambda_{t,-}^{(1)}\nabla_x\zeta_t(x)\Big|_{x\in\partial B_{2R_0t}(tp_t^{(1)})}. \tag{3.71}
$$

By (3.59), we obtain for $r_{t,R_0} = 2R_0\Lambda_{t,+}^{(1)}$,

$$\frac{1}{2}\int_{\partial B_{r_{t,R_0}}(0)} \nabla(\widetilde{\zeta}_t)\cdot\nabla(\widetilde{v}_t^{(1)}+\widetilde{v}_t^{(2)})|z|d\sigma - \int_{\partial B_{r_{t,R_0}}(0)} \frac{(\nabla(\widetilde{\zeta}_t)\cdot z)(\nabla(\widetilde{v}_t^{(1)}+\widetilde{v}_t^{(2)})\cdot z)}{|z|}d\sigma$$

$$= \int_{\partial B_{r_{t,R_0}}(0)} \frac{(\Lambda_{t,-}^{(1)})^2 h_t(\Lambda_{t,-}^{(1)}z+p_t^{(1)})}{\|u_t^{(1)}-u_t^{(2)}\|_{L^\infty(M)}} e^{\widetilde{v}_t^{(1)}(z)}(1-e^{\widetilde{v}_t^{(2)}(z)-\widetilde{v}_t^{(1)}(z)})|z|d\sigma$$

$$- \int_{B_{r_{t,R_0}}(0)} \frac{(\Lambda_{t,-}^{(1)})^2 h_t(\Lambda_{t,-}^{(1)}z+p_t^{(1)})}{\|u_t^{(1)}-u_t^{(2)}\|_{L^\infty(M)}} e^{\widetilde{v}_t^{(1)}(z)}(1-e^{\widetilde{v}_t^{(2)}(z)-\widetilde{v}_t^{(1)}(z)})$$

$$\times\left(2+\nabla_z \ln h_t(\Lambda_{t,-}^{(1)}z+p_t^{(1)})\cdot z\right)dz. \tag{3.72}$$

We see from (3.62) and (3.70) that

$$\begin{aligned}
\text{(LHS) of (3.72)} &= \frac{1}{2}\int_{\partial B_{r_{t,R_0}}(0)} |z|\left(-8\frac{z}{|z|^2}+O(t^3\ln t)\right)\cdot\left(-\frac{A_t}{2\pi}\frac{z}{|z|^2}+O(t^2)\right)d\sigma \\
&- \int_{\partial B_{r_{t,R_0}}(0)} \frac{1}{|z|}\left(-\frac{A_t}{2\pi}+O(t)\right)\left(-8+O(t^2\ln t)\right)d\sigma \\
&= \frac{1}{2}\int_{\partial B_{r_{t,R_0}}(0)} |z|\left(-8\frac{z}{|z|^2}+O(t^3\ln t)\right)\cdot\left(-\frac{A_t}{2\pi}\frac{z}{|z|^2}+O(t^2)\right)d\sigma \\
&- \int_{\partial B_{r_{t,R_0}}(0)} \frac{1}{|z|}\left(\frac{4A_t}{\pi}+O(t)\right)d\sigma \\
&= \frac{1}{2}\int_{\partial B_{r_{t,R_0}}(0)} |z|\left(\frac{4A_t}{\pi|z|^2}+O(t^3)\right)d\sigma - \int_{\partial B_{r_{t,R_0}}(0)} \frac{1}{|z|}\left(\frac{4A_t}{\pi}\right)d\sigma + O(t) \\
&= -4A_t+O(t).
\end{aligned} \tag{3.73}$$

We also see from (3.55)–(3.56) and (3.53) that

$$\begin{aligned}
\text{(RHS) of (3.72)} &= \int_{B_{r_{t,R_0}}(0)} \frac{2(\Delta\widetilde{v}_t^{(1)}-\Delta\widetilde{v}_t^{(2)})}{\|u_t^{(1)}-u_t^{(2)}\|_{L^\infty(M)}} + \frac{O(t|z|)}{(1+|z|^2)^2}dz + \int_{\partial B_{r_{t,R_0}}(0)} O(\frac{1}{|z|^3})d\sigma \\
&= 2\int_{B_{r_{t,R_0}}(0)} \Delta_z\widetilde{\zeta}_t(z)dz + O(t)
\end{aligned}$$

$$= 2 \int\limits_{B_{2R_0 t}(tp_t^{(1)})} \Delta_x \zeta_t(x)dx + O(t) = -2A_t + O(t). \tag{3.74}$$

By (3.73)–(3.74), we obtain $A_t = O(t)$, and complete the proof of Lemma 3.7. □

For any function f, we denote

$$D_l f(z) = \frac{\partial f(z)}{\partial z_l} \quad \text{for} \quad l = 1, 2. \tag{3.75}$$

Lemma 3.8. *(i)* $b_1 = b_2 = 0$,
(ii) $\widetilde{\zeta}_t(z) \to 0$, $\zeta_t(t\Lambda_{t,-}^{(1)} z + tp_t^{(1)}) \to 0$ *in* $C^0_{loc}(\mathbb{R}^2)$ *as* $t \to 0$,
(iii) $\lim_{t\to 0} \left(\int_{B_{2R_0 \Lambda_{t,+}^{(1)}}(0)} (\ln |z|) \Delta \widetilde{\zeta}_t dz \right) = 0$.

Proof. (i) We have

$$\begin{aligned}
&\operatorname{div}\left(\nabla\widetilde{\zeta}_t D_l \widetilde{v}_t^{(i)} + \nabla\widetilde{v}_t^{(i)} D_l\widetilde{\zeta}_t - \nabla\widetilde{\zeta}_t \cdot \nabla\widetilde{v}_t^{(i)} e_l\right) = \Delta\widetilde{\zeta}_t D_l \widetilde{v}_t^{(i)} + \Delta\widetilde{v}_t^{(i)} D_l\widetilde{\zeta}_t \\
&= \frac{\Delta(\widetilde{v}_t^{(1)} - \widetilde{v}_t^{(2)})}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}} D_l \widetilde{v}_t^{(i)} + \frac{\Delta\widetilde{v}_t^{(i)} D_l(\widetilde{v}_t^{(1)} - \widetilde{v}_t^{(2)})}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}} \\
&= -\frac{(\Lambda_{t,-}^{(1)})^2 h_t(\Lambda_{t,-}^{(1)} z + p_t^{(1)})(e^{\widetilde{v}_t^{(1)}} - e^{\widetilde{v}_t^{(2)}}) D_l \widetilde{v}_t^{(i)}}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}} \\
&\quad - \frac{(\Lambda_{t,-}^{(1)})^2 h_t(\Lambda_{t,-}^{(1)} z + p_t^{(1)}) e^{\widetilde{v}_t^{(i)}} D_l(\widetilde{v}_t^{(1)} - \widetilde{v}_t^{(2)})}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}} \\
&= -\operatorname{div}\left(\frac{(\Lambda_{t,-}^{(1)})^2 h_t(\Lambda_{t,-}^{(1)} z + p_t^{(1)})(e^{\widetilde{v}_t^{(1)}} - e^{\widetilde{v}_t^{(2)}}) e_l}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}} \right) \\
&+ \left[\frac{(\Lambda_{t,-}^{(1)})^2 h_t(\Lambda_{t,-}^{(1)} z + p_t^{(1)})(e^{\widetilde{v}_t^{(1)}} - e^{\widetilde{v}_t^{(2)}})}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}} D_l \left((-1)^i (\widetilde{v}_t^{(1)} - \widetilde{v}_t^{(2)}) + \ln h_t(\Lambda_{t,-}^{(1)} z + p_t^{(1)}) \right) \right].
\end{aligned} \tag{3.76}$$

For any constant $R \geq R_0$, let $r_{t,R} = 2R\Lambda_{t,+}^{(1)}$. Then (3.76) implies

$$\begin{aligned}
&\int\limits_{\partial B_{r_{t,R}}(0)} \left(2\nabla\widetilde{\zeta}_t D_l \widetilde{v}_t^{(1)} + 2\nabla\widetilde{v}_t^{(2)} D_l\widetilde{\zeta}_t - \nabla\widetilde{\zeta}_t \cdot (\nabla\widetilde{v}_t^{(1)} + \nabla\widetilde{v}_t^{(2)}) e_l\right) \cdot \frac{z}{|z|} d\sigma \\
&= -2 \int\limits_{\partial B_{r_{t,R}}(0)} \left(\frac{(\Lambda_{t,-}^{(1)})^2 h_t(\Lambda_{t,-}^{(1)} z + p_t^{(1)})(e^{\widetilde{v}_t^{(1)}} - e^{\widetilde{v}_t^{(2)}}) e_l}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}} \right) \cdot \frac{z}{|z|} d\sigma
\end{aligned}$$

$$+2\Bigg[\int\limits_{B_{r_{t,R}}(0)} \frac{(\Lambda_{t,-}^{(1)})^2 h_t(\Lambda_{t,-}^{(1)} z + p_t^{(1)})(e^{\widetilde{v}_t^{(1)}} - e^{\widetilde{v}_t^{(2)}})}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}} D_l\left(\ln h_t(\Lambda_{t,-}^{(1)} z + p_t^{(1)})\right) dz\Bigg]. \tag{3.77}$$

By (3.69) and Lemma 3.7, we have if $x \in M \setminus B_{2R_0 t}(tp_t^{(1)})$, then

$$\begin{aligned}\nabla_x \zeta_t(x) &= -\frac{A_t(x - tp_t^{(1)}) 1_{B_{r_0}(tp_t^{(1)})}(x)}{2\pi |x - tp_t^{(1)}|^2} + \frac{O(t^2) 1_{B_{r_0}(tp_t^{(1)})}(x)}{|x - tp_t^{(1)}|^2} + o(1)\\ &= \frac{O(t) 1_{B_{r_0}(tp_t^{(1)})}(x)}{|x - tp_t^{(1)}|} + o(1),\end{aligned} \tag{3.78}$$

which implies

$$\begin{aligned}\nabla_z \widetilde{\zeta}_t(z) &= t\Lambda_{t,-}^{(1)} \nabla_x \zeta_t(x)\Big|_{x = t\Lambda_{t,-}^{(1)} z + tp_t^{(1)}}\\ &= \frac{O(t)}{|z|} 1_{B_{\frac{r_0 \Lambda_{t,+}^{(1)}}{t}}(0)}(z) + o(t^2),\end{aligned} \tag{3.79}$$

for $2R_0 \Lambda_{t,+}^{(1)} \le |z| \le \frac{r_0 \Lambda_{t,+}^{(1)}}{t}$.

Therefore, in view of (3.62) and (3.79) we get that

$$\begin{aligned}\text{(LHS) of (3.77)} &= |z| \left(O(t^3 \ln t) + O(t^4 |z|) + \frac{O(1)}{|z|}\right)\left(O(\frac{t}{|z|}) + o(t^2)\right)\Big|_{z \in \partial B_{2R\Lambda_{t,+}^{(1)}}(0)}.\\ &= O(t^4 (\ln t) R) + O(\frac{t^2}{R}) + o(t^4) R^2 + o(t^2).\end{aligned} \tag{3.80}$$

To estimate (RHS) of (3.77), by the change of variables $x = t\Lambda_{t,-}^{(1)} z + tp_t^{(1)}$, we see that if $|z| = 2R\Lambda_{t,+}^{(1)} \ge 2R_0 \Lambda_{t,+}^{(1)}$, then Theorem D implies

$$\begin{aligned}&-2 \int\limits_{\partial B_{r_{t,R}}(0)} \left(\frac{(\Lambda_{t,-}^{(1)})^2 h_t(\Lambda_{t,-}^{(1)} z + p_t^{(1)})(e^{\widetilde{v}_t^{(1)}} - e^{\widetilde{v}_t^{(2)}}) e_l}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}}\right) \cdot \frac{z}{|z|} d\sigma(z)\\ &= \frac{-2\Lambda_{t,+}^{(1)}}{t} \int\limits_{\partial B_{2Rt}(tp_t^{(1)})} \frac{\rho(\Lambda_{t,-}^{(1)})^2 \left(h(ty)|y - \underline{e}|^2 |y + \underline{e}|^2 e^{-R_t(ty)}\right)\Big|_{y = \Lambda_{t,-}^{(1)} z + p_t^{(1)}}}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}}\\ &\times \frac{(e^{\widetilde{u}_t^{(1)}} - e^{\widetilde{u}_t^{(2)}}) t^6 e_l \cdot (x - tp_t^{(1)})}{|x - tp_t^{(1)}|} d\sigma(x)\end{aligned}$$

$$= O(1)\left(\int\limits_{\partial B_{2Rt}(tp_t^{(1)})} \frac{t^2 h(x)e^{-G_t(x)}(e^{\widetilde{u}_t^{(1)}} - e^{\widetilde{u}_t^{(2)}})e_l}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}} \cdot \frac{x - tp_t^{(1)}}{|x - tp_t^{(1)}|} d\sigma(x)\right) = O(t^3 R). \tag{3.81}$$

Let $x = t\Lambda_{t,-}^{(1)} z + tp_t^{(1)}$ and $y = \Lambda_{t,-}^{(1)} z + p_t^{(1)}$.

Then $r_0 \geq |x - tp_t^{(1)}| = t\Lambda_{t,-}^{(1)}|z| = 2Rt \geq 2R_0 t$ implies $\frac{r_0}{t} \geq |y - p_t^{(1)}| = \Lambda_{t,-}^{(1)}|z| = 2R \geq 2R_0$. So we see that if $2R_0 \leq |y - p_t^{(1)}| \leq \frac{r_0}{t}$, then

$$\begin{aligned} \nabla_y \ln h_t(y) &= \nabla_y(\ln \rho h(ty)e^{-R_t(ty)+\psi(ty)} + 2\ln|y - \underline{e}| + 2\ln|y + \underline{e}|) \\ &= t\nabla_{ty}\ln(\rho h(ty)e^{-R_t(ty)+\psi(ty)}) + O(\frac{1}{|y|}) = O(1). \end{aligned} \tag{3.82}$$

In view of Lemma 3.4-(ii) and Lemma 3.6-(iii), we have

$$\zeta_t - \frac{\int_M he^{u_t^{(1)}-G_t}\zeta_t dv_g}{\int_M he^{u_t^{(1)}-G_t} dv_g} = o(1) \quad \text{in} \quad M \setminus B_{\frac{tR_0}{2}}(tp_t^{(1)}). \tag{3.83}$$

Together with (3.54), we see that

$$\begin{aligned} &\frac{1}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}} \\ &\times \int\limits_{B_{2R\Lambda_{t,+}^{(1)}}(0)\setminus B_{2R_0\Lambda_{t,+}^{(1)}}(0)} (\Lambda_{t,-}^{(1)})^2 h_t(\Lambda_{t,-}^{(1)} z + p_t^{(1)})(e^{\widetilde{v}_t^{(1)}} - e^{\widetilde{v}_t^{(2)}}) D_l\left(\ln h_t(\Lambda_{t,-}^{(1)} z + p_t^{(1)})\right) dz \\ &= O(\frac{t^3}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}}) \\ &\times \int\limits_{B_{2R\Lambda_{t,+}^{(1)}}(0)\setminus B_{2R_0\Lambda_{t,+}^{(1)}}(0)} h_t(\Lambda_{t,-}^{(1)} z + p_t^{(1)})(e^{\widetilde{v}_t^{(1)}} - e^{\widetilde{v}_t^{(2)}}) D_{y_l} \ln h_t(y)\Big|_{y=\Lambda_{t,-}^{(1)} z + p_t^{(1)}} dz \\ &= O(t^3) \int\limits_{B_{2R\Lambda_{t,+}^{(1)}}(0)\setminus B_{2R_0\Lambda_{t,+}^{(1)}}(0)} h(t\Lambda_{t,-}^{(1)} z + tp_t^{(1)}) e^{-R_t(t\Lambda_{t,-}^{(1)} z + tp_t^{(1)})} \\ &\times|\Lambda_{t,-}^{(1)} z + p_t^{(1)} - \underline{e}|^2 |\Lambda_{t,-}^{(1)} z + p_t^{(1)} + \underline{e}|^2 O(t^6) e^{\widetilde{u}_t^{(1)}(t\Lambda_{t,-}^{(1)} z + tp_t^{(1)})} \\ &\times \frac{(\widetilde{u}_t^{(1)}(t\Lambda_{t,-}^{(1)} z + tp_t^{(1)}) - \widetilde{u}_t^{(2)}(t\Lambda_{t,-}^{(1)} z + tp_t^{(1)}) + O(\|u_t^{(1)} - u_t^{(2)}\|^2_{L^\infty(M)})}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}} dz \\ &= O(t) \int\limits_{B_{2Rt}(tp_t^{(1)})\setminus B_{2R_0 t}(tp_t^{(1)})} h(x)|x - t\underline{e}|^2 |x + t\underline{e}|^2 e^{-R_t(x)} e^{\widetilde{u}_t^{(1)}(x)} \end{aligned}$$

$$\times(\zeta_t(x) - \frac{\int_M he^{u_t^{(1)}-G_t}\zeta_t dv_g}{\int_M he^{u_t^{(1)}-G_t}dv_g} + O(\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}))dx$$
$$= O(t) \int_{B_{2Rt}(tp_t^{(1)})\setminus B_{2R_0t}(tp_t^{(1)})} o(1)dx = o(t^3)R^2. \tag{3.84}$$

To estimate $2\int_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)} \frac{(\Lambda_{t,-}^{(1)})^2 h_t(\Lambda_{t,-}^{(1)}z+p_t^{(1)})(e^{\tilde{v}_t^{(1)}}-e^{\tilde{v}_t^{(2)}})D_l\left(\ln h_t(\Lambda_{t,-}^{(1)}z+tp_t^{(1)})\right)}{\|u_t^{(1)}-u_t^{(2)}\|_{L^\infty(M)}}dz$, we note that if $|z| \le 2R_0\Lambda_{t,+}^{(1)}$, then there is $\theta \in (0,1)$ such that

$$D_{z_l}[\ln h_t(\Lambda_{t,-}^{(1)}z + p_t^{(1)})] = [D_{y_l}\ln h_t(y)]\Big|_{y=\Lambda_{t,-}^{(1)}z+p_t^{(1)}}\Lambda_{t,-}^{(1)}$$
$$= \left[D_{y_l}\ln h_t(y)\Big|_{y=p_t^{(1)}} + \sum_{k=1}^{2} D_{y_k}D_{y_l}\ln h_t(y)\Big|_{y=p_t^{(1)}}\Lambda_{t,-}^{(1)}z_k\right.$$
$$\left. + O(D^3\ln h_t\Big|_{y=\theta\Lambda_{t,-}^{(1)}z+p_t^{(1)}} t^2|z|^2)\right]\Lambda_{t,-}^{(1)}$$
$$= \left[D_{y_l}\ln h_t(y)\Big|_{y=p_t^{(1)}} + \sum_{k=1}^{2} D_{y_k}D_{y_l}\ln h_t(y)\Big|_{y=p_t^{(1)}}\Lambda_{t,-}^{(1)}z_k + O(t^2|z|^2)\right]\Lambda_{t,-}^{(1)}. \tag{3.85}$$

Moreover, by using the proof of Lemma 3.1 and (3.50), we get that

$$\nabla_y \ln h_t(y)\Big|_{y=p_t^{(1)}} = -t\nabla_x G_{*,t}^{(1)}(x)\Big|_{x=tp_t^{(1)}} + O(t\|\widetilde{\phi}_t\|_* + t^2\ln t) = O(t^2\ln t). \tag{3.86}$$

Now we obtain

$$D_{z_l}[\ln h_t(\Lambda_{t,-}^{(1)}z + p_t^{(1)})]$$
$$= \left[\sum_{k=1}^{2} D_{y_k}D_{y_l}\ln h_t(y)\Big|_{y=p_t^{(1)}}\Lambda_{t,-}^{(1)}z_k + O(t^2\ln t) + O(t^2|z|^2)\right]\Lambda_{t,-}^{(1)}. \tag{3.87}$$

Together with (3.53) and (3.55), we see that

$$2\int_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)} \frac{(\Lambda_{t,-}^{(1)})^2 h_t(\Lambda_{t,-}^{(1)}z + p_t^{(1)})(e^{\tilde{v}_t^{(1)}} - e^{\tilde{v}_t^{(2)}})}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}} D_l\left(\ln h_t(\Lambda_{t,-}^{(1)}z + p_t^{(1)})\right)dz$$
$$= 2\int_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)} \frac{(\Lambda_{t,-}^{(1)})^2 h_t(\Lambda_{t,-}^{(1)}z + p_t^{(1)})e^{\tilde{v}_t^{(1)}}(1 - e^{\tilde{v}_t^{(2)}-\tilde{v}_t^{(1)}})}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}}$$

$$\times[\sum_{k=1}^{2} D_{y_k} D_{y_l} \ln h_t(y)\Big|_{y=p_t^{(1)}} \Lambda_{t,-}^{(1)} z_k + O(t^2 \ln t) + O(t^2|z|^2)]\Lambda_{t,-}^{(1)} dz$$

$$= 2\Bigg\{ \int\limits_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)} \frac{h_t(\Lambda_{t,-}^{(1)} z + p_t^{(1)})(1 + |\widetilde{\eta}_t^{(1)}| + t^2|z|)}{C_t^{(1)}(1 + |z + O(t^2)|^2)^2} (\widetilde{\zeta}_t(z) + O(\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}))$$

$$\times[\sum_{k=1}^{2} D_{y_k} D_{y_l} \ln h_t(y)\Big|_{y=p_t^{(1)}} \Lambda_{t,-}^{(1)} z_k + O(t^2 \ln t) + O(t^2|z|^2)]\Lambda_{t,-}^{(1)} dz\Bigg\}. \tag{3.88}$$

We note from (3.86) that

$$\frac{h_t(\Lambda_{t,-}^{(1)} z + p_t^{(1)})}{C_t^{(1)}} = \frac{8h_t(\Lambda_{t,-}^{(1)} z + p_t^{(1)})}{h_t(p_t^{(1)})} = 8\left(1 + \frac{\nabla h_t(p_t^{(1)})}{h_t(p_t^{(1)})} \cdot \Lambda_{t,-}^{(1)} z + O(t^2|z|^2)\right) \tag{3.89}$$
$$= 8 + O(t^2 \ln t) + O(t^2|z|^2).$$

By using Lemma 3.3 and Lemma 3.6-(v), we have

$$\widetilde{\zeta}_t \to \sum_{i=1}^{2} \frac{b_i z_i}{1 + |z|^2} \quad \text{in} \quad C^0_{\text{loc}}(\mathbb{R}^2). \tag{3.90}$$

Together with Theorem E, we have for any $R > 1$,

$$2 \int\limits_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)} \frac{(\Lambda_{t,-}^{(1)})^2 h_t(\Lambda_{t,-}^{(1)} z + p_t^{(1)})(e^{\widetilde{v}_t^{(1)}} - e^{\widetilde{v}_t^{(2)}})}{\|u_t^{(1)} - u_t^{(2)}\|_{L^\infty(M)}} D_l\left(\ln h_t(\Lambda_{t,-}^{(1)} z + tp_t^{(1)})\right) dz$$

$$= 16 \int\limits_{B_R(0)} \frac{(1 + O(t^2 \ln t(|z|+1)^\varepsilon) + O(t^2|z|^2))}{(1 + |z|^2)^2} (\sum_{i=1}^{2} \frac{b_i z_i}{(1 + |z|^2)} + o(1))$$

$$\times [\sum_{k=1}^{2} D_{y_k} D_{y_l} \ln h_t(y)\Big|_{y=p_t^{(1)}} \Lambda_{t,-}^{(1)} z_k + O(t^2 \ln t) + O(t^2|z|^2)]\Lambda_{t,-}^{(1)} dz \tag{3.91}$$

$$+ \int\limits_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)\setminus B_R(0)} \frac{t}{|z|^4}(O(t|z|) + O(t^2 \ln t))dz$$

$$= 8(\Lambda_{t,-}^{(1)})^2 \sum_{k=1}^{2} D_{y_k} D_{y_l} \ln h_t(y)\Big|_{y=p_t^{(1)}} \int\limits_{B_R(0)} \frac{b_k |z|^2}{(1 + |z|^2)^3}$$

$$+ \frac{O(t^2)}{R} + O(t^3 \ln R) + O(t^4 R) + o(t^2),$$

for some $\varepsilon \in (0, 1)$. Therefore, we obtain from (3.81), (3.84), and (3.91) that

$$\begin{aligned}&\text{(RHS) of (3.77)}\\&=8(\Lambda_{t,-}^{(1)})^2\sum_{k=1}^{2}D_{y_k}D_{y_l}\ln h_t(y)\Big|_{y=p_t^{(1)}}\left(\int_{\mathbb{R}^2}\frac{b_k|z|^2}{(1+|z|^2)^3}dz+O(R^{-2})\right)\\&\quad+\frac{O(t^2)}{R}+O(t^3R^2)+o(t^2).\end{aligned}\tag{3.92}$$

Since $\det[\nabla^2(\ln h_t(p_t^{(1)}))]=-16+O(t)$, (3.80) and (3.92) imply for $i=1,2$,

$$|b_i|=O(\frac{1}{R})+o(1)\ \text{ for any large }\quad R\gg 1.\tag{3.93}$$

So we obtain $b_1=b_2=0$, and prove Lemma 3.8-(i).

(ii) By Lemma 3.3, Lemma 3.6-(v), and $b_1=b_2=0$, we have

$$\widetilde{\zeta}_t(z)=\zeta_t(t\Lambda_{t,-}^{(1)}z+tp_t^{(1)})-\frac{\int_M he^{u_t^{(1)}-G_t}\zeta_t dv_g}{\int_M he^{u_t^{(1)}-G_t}dv_g}\to 0\ \text{ in }\quad C^0_{\mathrm{loc}}(\mathbb{R}^2).$$

Together with Lemma 3.4-(ii), we obtain Lemma 3.8-(ii).

(iii) In view of Lemma 3.3, (3.25), and $b_0=b_1=b_2=0$, it is easy to see that Lemma 3.8-(iii) holds. □

Now we are going to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Let x_t^* be a maximum point of ζ_t. So we have

$$|\zeta_t(x_t^*)|=1.\tag{3.94}$$

Then from Lemma 3.4, we have

$$\lim_{t\to 0}x_t^*=0.\tag{3.95}$$

Moreover, by Lemma 3.6-(iii) and Lemma 3.8, we see that

$$\Lambda_{t,-}^{(1)}t\ll s_t:=|x_t^*-tp_t^{(1)}|\le\frac{tR_0}{2}.\tag{3.96}$$

Let $\hat{\zeta}_t(\xi)=\zeta_t(s_t\xi+tp_t^{(1)})=\widetilde{\zeta}_t(\Lambda_{t,+}^{(1)}\frac{s_t}{t}\xi)+\frac{\int_M he^{u_t^{(1)}-G_t}\zeta_t dv_g}{\int_M he^{u_t^{(1)}-G_t}dv_g}$. By (3.25), $\hat{\zeta}_t$ satisfies

$$0=\Delta\hat{\zeta}_t+O((\Lambda_{t,+}^{(1)})^2\frac{s_t^2}{t^2})\frac{(1+\frac{s_t^2}{t^2}|\xi|^2)}{(1+(\Lambda_{t,+}^{(1)})^2\frac{s_t^2}{t^2}|\xi|^2)^2}\quad\text{for }\ |\xi|\le\frac{r_0}{s_t}.\tag{3.97}$$

By (3.94), we have

$$\left|\hat{\zeta}_t\Big(\frac{x_t^* - tp_t^{(1)}}{s_t}\Big)\right| = |\zeta_t(x_t^*)| = 1. \tag{3.98}$$

By (3.96) and $|\hat{\zeta}_t| \le 1$, we see that $\hat{\zeta}_t \to \hat{\zeta}_0$ in any compact subset of $\mathbb{R}^2 \setminus \{0\}$, where $\hat{\zeta}_0$ satisfies $\Delta\hat{\zeta}_0 = 0$ in $\mathbb{R}^2 \setminus \{0\}$. Since $|\hat{\zeta}_0| \le 1$, we have $\Delta\hat{\zeta}_0 = 0$ in $\mathbb{R}^2$. So $\hat{\zeta}_0$ is a constant. From $\frac{|x_t^* - tp_t^{(1)}|}{s_t} = 1$ and (3.98), we have $\hat{\zeta}_0 \equiv 1$ or $\hat{\zeta}_0 \equiv -1$. So we have

$$|\zeta_t(x)| \ge \frac{1}{2} \ \text{ if } \ \frac{s_t}{2} \le |x - tp_t^{(1)}| \le s_t. \tag{3.99}$$

By Lemma 3.5-(ii), Lemma 3.8, and Lemma 3.7, we see that if $\frac{s_t}{2} \le |x - tp_t^{(1)}| \le s_t$, then

$$\zeta_t(x) = O(\ln t) \int\limits_{B_{2R_0\Lambda_{t,+}^{(1)}}(0)} \Delta\widetilde{\zeta}_t dz + o(1) = o(1) \ \text{ as } \ t \to 0,$$

which contradicts (3.99). So we complete the proof of Theorem 1.2. □

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