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On the structure of certain valued fields ☆

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ABSTRACT

In this article, we study the structure of finitely ramified mixed characteristic valued fields. For any two complete discrete valued fields K_1 and K_2 of mixed characteristic with perfect residue fields, we show that if the n-th residue rings are isomorphic for each $n \geq 1$, then K_1 and K_2 are isometric and isomorphic. More generally, for $n_1 \geq 1$, there is n_2 depending only on the ramification indices of K_1 and K_2 such that any homomorphism from the n_1-th residue ring of K_1 to the n_2-th residue ring of K_2 can be lifted to a homomorphism between the valuation rings. Moreover, we get a functor from the category of certain principal Artinian local rings of length n to the category of certain complete discrete valuation rings of mixed characteristic with perfect residue fields, which naturally generalizes the functorial property of unramified complete discrete valuation rings. Our lifting result improves Basarab's relative completeness theorem for finitely ramified henselian valued fields, which solves a question posed by Basarab, in the case of perfect residue fields.
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1. Introduction

In this paper, we are interested in finitely ramified mixed characteristic valued fields (see Definition 2.3). In model theory of valued fields, one of the most important theorems is the AKE-principle, proved by Ax

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and Kochen in [1,2], and independently by Ershov in [7,8]. The AKE-principle says that the theory of an unramified henselian valued field of characteristic 0 is determined by the theory of the residue field and the theory of the value group.

Fact 1.1 *(The Ax-Kochen-Ershov principle). [1,2,7,8] Let (K_i, k_i, Γ_i) be an unramified henselian valued field of characteristic zero, where k_i is the residue field and Γ_i is the valuation group respectively, for $i = 1, 2$.*

$$K_1 \equiv K_2 \textit{ if and only if } k_1 \equiv k_2 \textit{ and } \Gamma_1 \equiv \Gamma_2.$$

Basarab in [4] generalized the AKE-principle to the finitely ramified case. Actually, he showed that the theory of a finitely ramified henselian valued fields of mixed characteristic is determined by the theory of each n-th residue ring (see Definition 2.8), the quotient of the valuation ring by the n-th power of the maximal ideal and the theory of the valuation group.

Fact 1.2. *[4] Let $(K_i, R_{i,(n)}, \Gamma_i)$ be finitely ramified henselian valued fields of mixed characteristic, where $R_{i,(n)}$ is the n-th residue ring and Γ_i is the valuation group respectively for $i = 1, 2$. The following are equivalent:*

(1) $K_1 \equiv K_2$.
(2) $R_{1,(n)} \equiv R_{2,(n)}$ *for each* $n \geq 1$ *and* $\Gamma_1 \equiv \Gamma_2$.

Motivated by Fact 1.2, we ask the following related question on isomorphisms.

Question 1.3. Given two complete discrete valued fields K_1 and K_2 of mixed characteristic with perfect residue fields, if the n-th residue rings of K_1 and K_2 are isomorphic for each $n \geq 1$, then are K_1 and K_2 isomorphic? Moreover, is there $N > 0$ such that K_1 and K_2 are isomorphic if the N-th residue rings of K_1 and K_2 are isomorphic?

We give a comment on Question 1.3. Macintyre in [16] raised the following question on the problem of lifting of homomorphisms of the n-th residue rings for more general rings.

Question 1.4. Are two complete local noetherian rings A and B isomorphic if the n-th residue rings of A and B are isomorphic for each $n \geq 1$?

In [16], van den Dries gave a positive answer to Question 1.4 in the case that the residue fields are algebraic over their prime fields. Furthermore, given complete local noetherian rings A and B, it is enough to check whether the N-th residue rings of A and B are isomorphic for some $N = N(A, B)$ depending on A and B. Note that van den Dries showed the existence of a non explicit bound N, and in general, there is a counter example by Gabber in [16] for Question 1.4.

Next we recall the following well-known fact on unramified complete discrete valuation rings.

Fact 1.5. *[15]*

(1) Let k be a perfect field of characteristic p. Then there exists a complete discrete valuation ring of characteristic 0 which is unramified and has k as its residue field. Such a ring is unique up to isomorphism. This unique ring is called the ring of Witt vectors of k, denoted by $W(k)$.
(2) Let R_1 and R_2 be complete discrete valuation rings of mixed characteristic with perfect residue fields k_1 and k_2 respectively. Suppose R_1 is unramified. Then for every homomorphism $\phi : k_1 \longrightarrow k_2$, there exists a unique homomorphism $g : R_1 \longrightarrow R_2$ making the following diagram commutative:

$$\begin{tikzcd} R_1 \arrow[r, "g"] \arrow[d, "\mathrm{pr}_{1,1}"'] & R_2 \arrow[d, "\mathrm{pr}_{2,1}"'] \\ k_1 \arrow[r, "\phi"] & k_2, \end{tikzcd}$$

where two vertical maps are the canonical epimorphisms.

In categorical setting, Fact 1.5 is equivalent to the following statement.

Fact 1.6. *Let $\mathcal{C}_p$ be the category of complete unramified discrete valuation rings of mixed characteristic $(0, p)$ with perfect residue fields and $\mathcal{R}_p$ the category of perfect fields of characteristic p. Then $\mathcal{C}_p$ is equivalent to $\mathcal{R}_p$. More precisely, there is a functor $\mathrm{L}' : \mathcal{R}_p \to \mathcal{C}_p$ which satisfies:*

- $\mathrm{Pr} \circ \mathrm{L}'$ *is equivalent to the identity functor* $\mathrm{Id}_{\mathcal{R}_p}$ *where* $\mathrm{Pr} : \mathcal{C}_p \longrightarrow \mathcal{R}_p$ *is the natural projection functor.*
- $\mathrm{L}' \circ \mathrm{Pr}$ *is equivalent to* $\mathrm{Id}_{\mathcal{C}_p}$.

Based on Question 1.3 and Fact 1.6, we ask the following generalized questions for the finitely ramified case.

Question 1.7.

(1) For a principal Artinian local ring $\overline{R}$ of length n with a perfect residue field, is there a unique complete discrete valuation ring R which has $\overline{R}$ as its n-th residue ring? Moreover, if it has a positive answer, can a lower bound for such n be effectively computed in terms of the ramification index of $\overline{R}$?
(2) Given complete discrete valuation rings R_1 and R_2 of mixed characteristic with perfect residue fields, let $R_{1,(n_1)}$ and $R_{2,(n_2)}$ be the n_1-th residue ring of R_1 and the n_2-th residue ring of R_2 respectively. If n_1 and n_2 are large enough, is there a unique lifting homomorphism $g : R_1 \longrightarrow R_2$ such that g induces a given homomorphism $\phi : R_{1,(n_1)} \longrightarrow R_{2,(n_2)}$? Moreover, can such lower bounds on n_1 and n_2 be effectively computed in terms of the ramification indices of R_1 and R_2?

Question 1.8. Let $\mathcal{C}_{p,e}$ be the category of complete discrete valuation rings of mixed characteristic $(0, p)$ with perfect residue fields and ramification index e. For $n > e$, let $\mathcal{R}^n_{p,e}$ be the category of principal Artinian local rings of length n having ramification index e and perfect residue fields (see at the beginning of *Section 4* for the precise definition). Let $\mathrm{Pr}_n : \mathcal{C}_{p,e} \longrightarrow \mathcal{R}^n_{p,e}$ be the natural projection functor. Is there a lifting functor $\mathrm{L} : \mathcal{R}^n_{p,e} \longrightarrow \mathcal{C}_{p,e}$ which satisfies:

- $\mathrm{Pr}_n \circ \mathrm{L}$ is equivalent to $\mathrm{Id}_{\mathcal{R}^n_{p,e}}$.
- $\mathrm{L} \circ \mathrm{Pr}_n$ is equivalent to $\mathrm{Id}_{\mathcal{C}_{p,e}}$.

In general, the answer for Question 1.7.(2) is not positive, that is, there is a homomorphism $\phi : R_{1,n_1} \longrightarrow R_{2,n_2}$ such that no homomorphism from R_1 into R_2 induces ϕ (see Example 3.5). Instead of finding a 'usual' lifting in the sense of Question 1.8, we will show that for sufficiently large n_2, if there is a given homomorphism $\phi : R_{1,(n_1)} \longrightarrow R_{2,(n_2)}$, then there is an 'approximate' lifting $g : R_1 \longrightarrow R_2$ of ϕ (see Definition 3.4).

Let us come back to the question of elementary equivalence. In [4], Basarab posed the following question (see [4, pp. 23–24]):

Question 1.9. For a finitely ramified henselian valued field K of ramification index e, is there a finite integer $N' \geq 1$ depending on K such that any finitely ramified henselian valued field of the same ramification index

e is elementarily equivalent to K if their N'-th residue rings are elementarily equivalent and their value groups are elementarily equivalent?

Given a finitely ramified henselian valued field K, Basarab in [4] denoted the minimal number N', which satisfies the condition in Question 1.9, by $\lambda(T)$ for the complete theory T of K. He showed that $\lambda(T)$ for a local field K is finite but did not give any explicit value of $\lambda(T)$.

The goal of this paper is to answer these questions when the residue fields are perfect. Its organization is as follows. In Section 2, we recall basic definitions and facts. In Section 3, we answer Question 1.3 positively for the perfect residue field case in Theorem 3.7. Our main result shows that if n_2 is sufficiently large, then for a given homomorphism $\phi : R_{1,(n_1)} \longrightarrow R_{2,(n_2)}$, there is a homomorphism $\mathrm{L}(\phi) : R_1 \longrightarrow R_2$ satisfying a lifting property similar to that of the unramified case. This provides an answer for Question 1.3. Also, the lifting map L provides an answer for Question 1.7.(2) and Question 1.7.(1). In Section 4, we concentrate on Question 1.8. We can show that L is compatible with the composition of homomorphisms between residue rings. More precisely, $\mathrm{L}(\phi_2 \circ \phi_1) = \mathrm{L}(\phi_2) \circ \mathrm{L}(\phi_1)$ for any $\phi_1 : R_{1,(n_1)} \longrightarrow R_{2,(n_2)}$ and $\phi_2 : R_{2,(n_2)} \longrightarrow R_{3,(n_3)}$. This defines a functor $\mathrm{L} : \mathcal{R}^n_{p,e} \longrightarrow \mathcal{C}_{p,e}$ for sufficiently large n. We prove that a lower bound for n depends only on the ramification index e and the prime number p. Even though L does not give an equivalence between $\mathcal{R}^n_{p,e}$ and $\mathcal{C}_{p,e}$, it turns out that L satisfies a similar functorial property to that of $\mathrm{L}' : \mathcal{R}_p \to \mathcal{C}_p$. This provides an answer for Question 1.8. In Section 5, we reduce the problem on elementary equivalence between finitely ramified henselian valued fields of mixed characteristic to the problem on isometricity between complete discrete valued fields of mixed characteristic. Using results in Section 3, we improve Basarab's result on the AKE-principle which gives a positive answer to Question 1.9 when the residue fields are perfect. Under certain conditions, we calculate $\lambda(T)$ explicitly for the tame case and get a lower bound for $\lambda(T)$ for the wild case. Surprisingly we show that $\lambda(T)$ can be 1 even when K is not unramified. As a special case, we conclude that $\lambda(T)$ is 1 or $e + 1$ if $p \nmid e$, and $\lambda(T) \geq e + 1$ if $p \mid e$ when K is a finitely ramified henselian subfield of $\mathbb{C}_p$ with the ramification index e.

2. Preliminaries

In this section, we introduce basic notations, terminologies, and several preliminary facts which will be used in this paper. We denote a valued field by a tuple $(K, R, \mathfrak{m}, \nu, k, \Gamma)$ consisting of the following data: K is the underlying field, R is the valuation ring, $\mathfrak{m}$ is the maximal ideal of R, ν is the valuation, k is the residue field, and Γ is the value group. Hereafter, the full tuple $(K, R, \mathfrak{m}, \nu, k, \Gamma)$ will be abbreviated in accordance with the situational need for the components. For any field L, L^{alg} denotes a fixed algebraic closure of L.

Notation 2.1. Let (L, ν) be a valued field of mixed characteristic $(0, p)$ whose value group is contained in $\mathbb{R}$. We define a normalized valuation $\overline{\nu}$ on L of ν by the property $\overline{\nu}(p) = 1$, that is, $\nu(p)\overline{\nu} = \nu$. We denote an extended valuation of $\overline{\nu}$ on L^{alg} by $\widetilde{\nu}$. Note that $\widetilde{\nu}$ is unique when L is henselian.

Definition 2.2. Let (K, ν, k, Γ) be a valued field of characteristic zero. We say (K, ν) is *unramified* if $char(k) = 0$, or $char(k) = p$ and $\nu(p)$ is the minimal positive element in Γ for $p > 0$. We say (K, ν) is *ramified* if it is not unramified.

Definition 2.3. Let (K, R, ν, k, Γ) be a valued field whose residue field has prime characteristic p.

(1) We say (K, R, ν, k, Γ) is *finitely ramified* if K is ramified and the set $\{\gamma \in \Gamma |\ 0 < \gamma \leq \nu(p)\}$ is finite. For $x \in R$, we write $e_\nu(x) := |\{\gamma \in \Gamma |\ 0 < \gamma \leq \nu(x)\}|$. If there is no confusion, we write $e(x)$ for $e_\nu(x)$.

The number $e_\nu(p)$, which is the cardinality of $\{\gamma \in \Gamma | \; 0 < \gamma \le \nu(p)\}$, is called *the ramification index* of (K, ν).

(2) Let (K, R, ν, k, Γ) be finitely ramified. If p does not divide $e_\nu(p)$, we say (K, ν) is *tamely ramified.* Otherwise, we say (K, ν) is *wildly ramified.*

Note that if a valued field of mixed characteristic has a finite ramification index, then its value group has a minimum positive element.

Definition 2.4. Let (R, ν, k) be a complete discrete valuation ring of mixed characteristic with a perfect residue field. Let (R', ν', k') be a finite extension of R. Let K and K' be fraction fields of R and R' respectively. If $k = k'$, we say that R' is a *totally ramified extension* of R, or K' is a *totally ramified extension* of K.

Definition 2.5. Let (K_1, ν_1) and (K_2, ν_2) be valued fields. Let R'_1 and R'_2 be subrings of K_1 and K_2 respectively. Let $f : R'_1 \to R'_2$ be an injective ring homomorphism. We say f is an *isometry* if for $a, b \in R'_1$,

$$\nu_1(a) > \nu_1(b) \Leftrightarrow \nu_2(f(a)) > \nu_2(f(b)).$$

Fact 2.6. *Let (R_1, ν_1) and (R_2, ν_2) be finitely ramified valuation rings of mixed characteristic $(0, p)$ whose value groups are isomorphic to $\mathbb{Z}$. Let $f : R_1 \to R_2$ be a ring homomorphism. Then we have the following.*

(1) $f : R_1 \to R_2$ is an isometry.

(2) Let K_1 and K_2 be the fraction fields of R_1 and R_2 respectively. Then the homomorphism $K_1 \longrightarrow K_2$ induced by f is an isometry.

(3) If both of valuation groups of R_1 and R_2 are contained in a common ordered abelian group and $\nu_1(p) = \nu_2(p)$, then $\nu_1(x) = \nu_2(f(x))$ for any $x \in R_1$.

Proof. (1) We have $f(n) = n$ for all $n \in \mathbb{Z}$. Take $a \in R_1$. Since f sends units to units, $\nu_2(f(a)) = 0$ if $\nu_1(a) = 0$. To show that f is an isometry, it is enough to show that $\nu_2(f(a)) > 0$ if $\nu_1(a) > 0$. Suppose $\nu_1(a) > 0$. Then there is $k \in R_1^\times$ such that $ka^n = p^m$ for some $n, m > 0$ since R_1 is finitely ramified. Since $f(p) = p$, we have that $p^m = f(p^m) = f(k)f(a)^n$. Therefore, we have that

$$\nu_1(a) = \frac{m}{n}\nu_1(p), \;\; \nu_2(f(a)) = \frac{m}{n}\nu_2(p) \tag{$*$}$$

and f is injective. Thus, f is an isometry.

(2) This follows directly from (1).

(3) This follows from ($*$). □

Fact 2.7. *Let (K_1, ν_1) and (K_2, ν_2) be valued fields whose value groups are contained in $\mathbb{R}$. Let $f : K_1 \longrightarrow K_2$ be an isometry. Suppose K_1 is henselian. Let $\widetilde{f} : K_1^{alg} \longrightarrow K_2^{alg}$ be an extended homomorphism of f. Then $\widetilde{f}$ is an isometry.*

Proof. There are two valuations on $\widetilde{f}(K_1^{alg})$, $\widetilde{\nu_1} \circ \widetilde{f}^{-1}$ and $\widetilde{\nu_2}|_{\widetilde{f}(K_1^{alg})}$ where $\widetilde{\nu_2}|_{\widetilde{f}(K_1^{alg})}$ is the restriction of $\widetilde{\nu_2}$ to $\widetilde{f}(K_1^{alg})$. Since f is an isometry, the restrictions of $\widetilde{\nu_1} \circ \widetilde{f}^{-1}$ and $\widetilde{\nu_2}|_{\widetilde{f}(K_1^{alg})}$ to $f(K_1)$ are equivalent, in fact, they are equal since $(\widetilde{\nu_1} \circ \widetilde{f}^{-1})(p) = \widetilde{\nu_2}|_{\widetilde{f}(K_1^{alg})}(p) = 1$. Since K_1 is henselian, $f(K_1)$ is henselian. Hence, $\widetilde{\nu_1} \circ \widetilde{f}^{-1}$ is equal to $\widetilde{\nu_2}|_{\widetilde{f}(K_1^{alg})}$ by the henselian property. This shows that $\widetilde{f}$ is an isometry. □

Definition 2.8. For a local ring R with maximal ideal $\mathfrak{m}$, we denote $R/\mathfrak{m}^n$ by $R_{(n)}$, and we call $R_{(n)}$ the *n-th residue ring* of R. In particular, $R_{(1)}$ is the residue field of R. For each $m > n$, we write $\mathrm{pr}_n : R \to R_{(n)}$ and $\mathrm{pr}_n^m : R_{(m)} \to R_{(n)}$ for the canonical epimorphisms respectively.

For R-algebras S_1 and S_2, we denote the set of R-algebra homomorphisms from S_1 to S_2 by $\mathrm{Hom}_R(S_1, S_2)$, and we write $\mathrm{Hom}(S_1, S_2)$ for $\mathrm{Hom}_{\mathbb{Z}}(S_1, S_2)$.

We recall some facts on the structure of finite extensions of unramified complete valued fields.

Fact 2.9. *Let (R, ν) be a complete discrete valuation ring of mixed characteristic $(0, p)$ with perfect residue field k whose valuation group is $\mathbb{Z}$. Then $W(k)$ can be embedded as a subring of R and R is a free $W(k)$-module of rank $\nu(p)$. Moreover, R is a $W(k)$-algebra generated by π, denoted by $W(k)[\pi]$, where π is a uniformizer of R.*

Proof. Chapter 2, Section 5 of [15]. □

Fact 2.10. *Let A be a ring that is Hausdorff and complete for a topology defined by a decreasing sequence $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset ...$ of ideals such that $\mathfrak{a}_n \cdot \mathfrak{a}_m \subset \mathfrak{a}_{n+m}$. Assume that the residue ring $A_1 = A/\mathfrak{a}_1$ is a perfect field of characteristic p. Then:*

(1) There exists a unique system of representatives $h : A_1 \longrightarrow A$ which commute with p-th powers: $h(\lambda^p) = h(\lambda)^p$. This system of representatives is called the set of Teichmüller representatives.
(2) In order for $a \in A$ to belong to $S = h(A_1)$, it is necessary and sufficient that a be a p^n-th power for all $n \geq 0$.
(3) This system of representatives is multiplicative which means

$$h(\lambda\mu) = h(\lambda)h(\mu)$$

for all $\lambda, \mu \in A_1$.
(4) S contains 0 and 1.
(5) $S \setminus \{0\}$ is a subgroup of the unit group of A.

Proof. (1)(2)(3): Chapter 2, Section 4 of [15].
(4): 0 and 1 satisfy (2).
(5): (3) and (4) show that $S \setminus \{0\}$ is a subgroup of the unit group of A. □

Remark 2.11. Let $(R, \mathfrak{m})$ be a complete discrete valuation ring of mixed characteristic $(0, p)$ with perfect residue field. By Fact 2.10, R and $R_{(n)}$ have the sets S and S_n of Teichmüller representatives respectively. Then, we have that $\mathrm{pr}_n(S) = S_n$.

Proof. It is clear that $\mathrm{pr}_n(S) \subset S_n$. Since each of S_n and S bijectively corresponds to $R/\mathfrak{m}$ by Fact 2.10, the inclusion must be equality. □

Remark 2.12. Let (R, ν) be a complete discrete valuation ring of mixed characteristic $(0, p)$ with perfect residue field. Let S be the set of Teichmüller representatives and let π be a uniformizer. Then, for any $x \in R$, there is a unique infinite sequence $(\lambda_i)_{i \geq 0}$ of elements in S such that $x = \sum_i \lambda_i \pi^i$.

Proof. Fix $x \in R$. By Fact 2.10, we inductively choose λ_i's in S such that $\nu(x - \sum_{i=0}^n \lambda_i \pi^n) > \nu(\pi^n)$ for each $n \geq 0$. Then, we have that $x = \sum_i \lambda_i \pi^i$. It remains to show that such a sequence is unique. Let (λ_i') be a sequence of elements in S such that $x = \sum_i \lambda'_i \pi^i$. Suppose that $\lambda_i \neq \lambda'_i$ for some i. Let i_0 be the smallest index such that $\lambda_{i_0} \neq \lambda'_{i_0}$. Then, we have that

$$\begin{aligned}\mathrm{pr}_1(\lambda_{i_0}) &= \mathrm{pr}_1\left(\frac{x - \sum_{i<i_0} \lambda_i \pi^i}{\pi^{i_0}}\right)\\ &= \mathrm{pr}_1\left(\frac{x - \sum_{i<i_0} \lambda'_i \pi^i}{\pi^{i_0}}\right)\\ &= \mathrm{pr}_1(\lambda'_{i_0}),\end{aligned}$$

which implies that $\lambda_{i_0} = \lambda'_{i_0}$, a contradiction. Thus, $(\lambda_i) = (\lambda'_i)$. □

The following facts are useful to effectively compute N in Question 1.3 (see Theorem 3.7 and Theorem 3.10).

Fact 2.13 *(Krasner's lemma). Let (K, ν) be a henselian valued field and let $a, b \in K^{alg}$. Suppose a is separable over $K(b)$. Suppose that for all embeddings $\sigma(\neq id)$ of $K(a)$ over K, we have*

$$\widetilde{\nu}(b-a) > \widetilde{\nu}\big(\sigma(a) - a\big).$$

Then $K(a) \subset K(b)$.

Proof. See Chapter 2 of [12] or Theorem 4.1.7 of [6]. □

Fact 2.14. *Let $(R, \mathfrak{m}_R) \subset (S, \mathfrak{m}_S)$ be discrete valuation rings. Suppose $S = R[\alpha]$ for some $\alpha \in S$ and S is a finitely generated R-module so that $\mathfrak{m}_R S = \mathfrak{m}_S^e$ for a positive integer e. Suppose the residue fields of R and S are of characteristic $p > 0$. Let $f(x)$ in $R[x]$ be a monic irreducible polynomial of α over R.*

(1) The different $\mathfrak{D}_{S/R}$ of S/R is a principal ideal generated by $f'(\alpha)$.
(2) Let ν_S be the valuation corresponding to S. Let s be the power which satisfies $\mathfrak{m}_S^s = \mathfrak{D}_{S/R}$. Then one has

$$\begin{cases} s = e-1, & \text{if } S \text{ is tamely ramified over } R\text{, that is, } p \nmid e;\\ e \leq s \leq e-1+\nu_S(e), & \text{if } S \text{ is wildly ramified over } R\text{, that is, } p \mid e.\end{cases}$$

Proof. Chapter 3, Section 2 of [13]. □

For model theory of valued fields, we take the language of valued fields with three types of sorts for valuation fields, residue fields, and value groups. Let $\mathcal{L}_K = \{+, -, \cdot; 0, 1; |\}$ be a ring language with a binary relation $|$ for valued fields, where we interpret the binary relation $|$ as $a \mid b$ if $\nu(a) \leq \nu(b)$ for $a, b \in K$, $\mathcal{L}_k = \{+', -', \cdot'; 0', 1'\}$ be the ring language for residue fields, and $\mathcal{L}_\Gamma = \{+^*; 0^*; <\}$ be the ordered group language for valuation groups. The language of valued fields is the language $\mathcal{L}_{val} = \mathcal{L}_K \cup \mathcal{L}_k \cup \mathcal{L}_\Gamma$ equipped with function symbols pr_k and pr_Γ, where pr_k and pr_Γ are interpreted as the canonical surjective maps from the valuation ring to the residue field and from the valued field to the valuation group respectively. Next, we consider an extended language of $\mathcal{L}_{val}$ by adding the ring languages for the n-th residue rings and function symbols pr_n and pr^n_m for $n \geq m$, where pr_n and pr^n_m are interpreted as the canonical epimorphisms from the valuation ring to the n-th residue ring and from the n-th residue ring to the m-th residue ring respectively. For each $n \geq 1$, let $\mathcal{L}_{R_{(n)}} = \{+_n, -_n, \cdot_n; 0_n, 1_n\}$ be the ring language for the n-th residue ring.

For $n = 1$, we identify $\mathcal{L}_{R_{(1)}} = \mathcal{L}_k$. We get an extended language $\mathcal{L}_{val,R} = \mathcal{L}_{val} \cup \bigcup_{n\geq 1} \mathcal{L}_{R_{(n)}}$ for valued fields. Let $(K_1, \nu_1, k_1, \Gamma_1)$ and $(K_2, \nu_2, k_2, \Gamma_2)$ be valued fields, and let $R_{1,(n)}$ and $R_{2,(n)}$ be the n-th residue rings of (K_1, ν_1) and (K_2, ν_2) respectively. We say (K_1, ν_1) and (K_2, ν_2) are elementarily equivalent if they are elementarily equivalent in $\mathcal{L}_K$. If (K_1, ν_1) and (K_2, ν_2) are elementarily equivalent, then

- k_1 and k_2 are elementarily equivalent in $\mathcal{L}_k$;
- Γ_1 and Γ_2 are elementarily equivalent in $\mathcal{L}_\Gamma$; and
- $R_{1,(n)}$ and $R_{2,(n)}$ are elementarily equivalent in $\mathcal{L}_{R_{(n)}}$ for each $n \geq 1$.

Remark 2.15. Let (K_1, ν_1, Γ_1) and (K_2, ν_2, Γ_2) be valued fields. Suppose

- $R_{1,(n)} \equiv R_{2,(n)}$ as rings in the language $\mathcal{L}_{R_{(n)}}$ for each $n \geq 1$;
- $\Gamma_1 \equiv \Gamma_2$ as ordered abelian groups in the language $\mathcal{L}_\Gamma$.

Then there are $\aleph_1$-saturated elementary extensions $(K_1', \nu_1', \Gamma_1')$ and $(K_2', \nu_2', \Gamma_2')$ of K_1 and K_2 such that

- $R'_{1,(n)} \cong R'_{2,(n)}$ for $n \geq 1$;
- $\Gamma_1' \cong \Gamma_2'$,

where $R'_{1,(n)}$ and $R'_{2,(n)}$ are the n-th residue rings of K_1' and K_2' respectively.

Proof. It is easily deduced from the Keisler-Shelah isomorphism theorem in [5]. □

Next, we review coarse valuations. For the coarse valuations, we refer to [11,14].

Remark/Definition 2.16. [14, pp. 25–27] Suppose (K, ν, k, Γ) is finitely ramified. Let π be a uniformizer so that $\nu(\pi)$ is the smallest positive element in Γ. Let Γ° be the convex subgroup of Γ generated by $\nu(\pi)$ and $\dot{\nu} : K \setminus \{0\} \longrightarrow \Gamma/\Gamma^\circ$ be a map sending $x(\neq 0) \in K$ to $\nu(x) + \Gamma^\circ \in \Gamma/\Gamma^\circ$. The map $\dot{\nu}$ is a valuation, called *the coarse valuation*. The residue field K° of $(K, \dot{\nu})$, called *the core field* of (K, ν), forms a valued field equipped with a valuation ν°, whose value group is Γ°. More precisely, the valuation ν° is defined as follows: Let $\mathrm{pr}_{\dot{\nu}} : R_{\dot{\nu}} \longrightarrow K^\circ$ be the canonical epimorphism and let $x \in R_{\dot{\nu}}$. If $x^\circ := \mathrm{pr}_{\dot{\nu}}(x) \in K^\circ \setminus \{0\}$, then $\nu^\circ(x^\circ) := \nu(x)$. And $x^\circ = 0 \in K^\circ$ if and only if $\nu(x) > \gamma$ for all $\gamma \in \Gamma^\circ$.

Remark 2.17.

(1) Let R_ν, $R_{\dot{\nu}}$, and R_{ν° be the valuation rings of (K, ν), $(K, \dot{\nu})$, and (K°, ν°) respectively. Then $(\mathrm{pr}_{\dot{\nu}})^{-1}(R_{\nu^\circ}) = R_\nu$.
(2) Let $R_{(n)}$ and $R^\circ_{(n)}$ be the n-th residue rings of (K, ν) and (K°, ν°) respectively. Then there is a canonical isomorphism $\theta_n : R_{(n)} \longrightarrow R^\circ_{(n)}$ such that $\mathrm{pr}_n^{\nu^\circ} \circ (\mathrm{pr}_{\dot{\nu}}|_{R_\nu}) = \theta_n \circ \mathrm{pr}_n$, where $\mathrm{pr}_n : R_\nu \longrightarrow R_{(n)}$ and $\mathrm{pr}_n^{\nu^\circ} : R_{\nu^\circ} \longrightarrow R^\circ_{(n)}$ are the canonical epimorphisms.
(3) If (K, ν) is henselian, then $(K, \dot{\nu})$ is henselian.
(4) If (K, ν) is $\aleph_1$-saturated, then (K°, ν°) is complete.

Proof. (1) Note that $R_{\dot{\nu}} := \{x \in K|\ \dot{\nu}(x) \geq 0\} = \{x \in K|\ \nu(x) \geq \gamma \text{ for some } \gamma \in \Gamma^\circ\}$. Let $x \in R_{\dot{\nu}}$ be such that $\mathrm{pr}_{\dot{\nu}}(x) =: x^\circ \in R_{\nu^\circ}$, that is, $\nu^\circ(x^\circ)(\in \Gamma^\circ) \geq 0$. If $x^\circ = 0$, $\nu(x) > \gamma$ for all $\gamma \in \Gamma^\circ$ and $x \in R_\nu$. If $x^\circ \neq 0$, then $\nu^\circ(x^\circ) = \nu(x) \geq 0$ in Γ°, and hence $\nu(x) \geq 0$ in Γ. Thus $x \in R_\nu$. Therefore, for $x \in R_{\dot{\nu}}$, $x \in R_\nu$ if and only if $x^\circ \in R_{\nu^\circ}$.

(2) Note that each θ_n is induced from $\mathrm{pr}_{\dot{\nu}}|_{R_\nu} : R_\nu \longrightarrow R_{\nu^\circ}$. It is easy to see that each θ_n is surjective. To show that θ_n is injective, it is enough to show that $\nu(x) \geq n$ if and only if $\nu^\circ(x^\circ) \geq n$ for $x \in R_\nu$. It clearly comes from the definition of ν° in (1).

(3)-(4) Section 5 of [11]. □

Remark 2.18. By combining Fact 1.1, Remark 2.15 and Remark 2.17, we reduce the problem on elementary equivalence between finitely ramified henselian valued fields of mixed characteristic to the problem on isometricity between complete discrete valued fields of mixed characteristic whose n-th residue rings are isomorphic for each $n \geq 1$. To our knowledge, this strategy first appeared in [11].

3. Lifting homomorphisms

From now on, if there is no comment, we consider only complete discrete valued fields of mixed characteristic $(0, p)$ with perfect residue fields, and we assume that valuation groups are $\mathbb{Z}$ so that for a valued field (L, R, ν), $\nu(x) = e_\nu(x)$ for $x \in R$. Let (R, ν, k) be a valuation ring. Let π be a uniformizer of R. Let L and K be the fraction fields of R and $W(k)$ respectively.

Definition 3.1. If L is **ramified**, we denote the maximal value

$$\max\left\{\widetilde{\nu}\left(\pi - \sigma(\pi)\right) : \sigma \in \mathrm{Hom}_K\left(L, L^{alg}\right),\ \sigma(\pi) \neq \pi\right\}$$

by $M(R)_\pi$ or $M(L)_\pi$.

Lemma 3.2. *Let $(R_i, \mathfrak{m}_i, \nu_i, k_i)$ be a valuation ring and let π_i be a uniformizer of R_i for $i = 1, 2$. Let S_i be the set of Teichmüller representatives of R_i for $i = 1, 2$.*

(1) For any homomorphism $\phi : R_{1,(n_1)} \longrightarrow R_{2,(n_2)}$, $\phi(S_1 + \mathfrak{m}_1^{n_1})$ is contained in $S_2 + \mathfrak{m}_2^{n_2}$. Similarly, for any homomorphism $g : R_1 \longrightarrow R_2$, $g(S_1)$ is contained in S_2.

(2) For any homomorphism $\phi : R_{1,(n_1)} \longrightarrow R_{2,(n_2)}$, $\phi\left((W(k_1) + \mathfrak{m}_1^{n_1})/\mathfrak{m}_1^{n_1}\right)$ is contained in $(W(k_2) + \mathfrak{m}_2^{n_2})/\mathfrak{m}_2^{n_2}$. Similarly, for any homomorphism $g : R_1 \longrightarrow R_2$, $g(W(k_1))$ is contained in $W(k_2)$.

Proof. (1) This comes from Fact 2.10 and Remark 2.11.

(2) Since $W(k_i)/pW(k_i) \cong R_i/\mathfrak{m}_i \cong k_i$, S_i is contained in $W(k_i)$ by Fact 2.10. Since any element a in $W(k_1)$ can be uniquely written as $a = \sum_{r=0}^{\infty} \lambda_r p^r$ where λ_r is in S_1, we have that $\phi\left((W(k_1) + \mathfrak{m}_1^{n_1})/\mathfrak{m}_1^{n_1}\right) \subset (W(k_2) + \mathfrak{m}_2^{n_2})/\mathfrak{m}_2^{n_2}$ and $g(W(k_1)) \subset W(k_2)$ by Lemma 3.2.(1). □

Lemma 3.3. *Let L_i and K_i be the fraction fields of R_i and $W(k_i)$ respectively for $i = 1, 2$.*

(1) Let α be a uniformizer of R_1. Then $M(R_1)_{\pi_1} = M(R_1)_\alpha$. We write $M(R_1)_{\pi_1} = M(R_1)$.

(2) Suppose $[L_1 : K_1] = [L_2 : K_2] = e$, that is, $\nu_1(p) = \nu_2(p) = e$. Suppose there is an isometry $g : L_1 \longrightarrow L_2$. Then $M(R_1) = M(R_2)$.

Proof. (1) By Remark 2.12, we can write $\alpha = \sum_{r=1}^{\infty} \lambda_r \pi_1^r$ where λ_r is a Teichmüller representative of R_1 for each r and $\lambda_1 \neq 0$. Since $R_1/\mathfrak{m}_1 = k_1$, λ_r is in $W(k_1)$ for each r by Fact 2.10. For any σ in $\mathrm{Hom}_{K_1}(L_1, K_1^{alg})$,

$$\alpha - \sigma(\alpha) = \sum_{r=1}^{\infty} \lambda_r \pi_1^r - \sigma\left(\sum_{r=1}^{\infty} \lambda_r \pi_1^r\right)$$

$$= \sum_{r=1}^{\infty} \lambda_r \Big(\pi_1^r - \sigma(\pi_1^r)\Big)$$

$$= \Big(\pi_1 - \sigma(\pi_1)\Big) \sum_{r=1}^{\infty} \lambda_r \left(\sum_{j=0}^{r-1} \pi_1^{r-1-j} \sigma(\pi_1^j) \right)$$

and $\widetilde{\nu_1}(\alpha - \sigma(\alpha)) = \widetilde{\nu_1}(\pi_1 - \sigma(\pi_1))$ because

$$\widetilde{\nu_1} \left(\sum_{r=1}^{\infty} \lambda_r \left(\sum_{j=0}^{r-1} \pi_1^{r-1-j} \sigma(\pi_1^j) \right) \right) = 0.$$

So, we have $M(R_1)_{\pi_1} = M(R_1)_\alpha$.

(2) By Lemma 3.2.(2), $g(K_1)$ is contained in K_2. Let f_1 be the monic irreducible polynomial of π_1 over $W(k_1)$. Since g is an isometry, we have $\overline{\nu_2}(g(\pi_1)) = \overline{\nu_1}(\pi_1) = 1/e$, and hence, $g(\pi_1)$ is a uniformizer of L_2. Let $\widetilde{g} : L_1^{alg} \longrightarrow L_2^{alg}$ be an extended homomorphism of g. If we write $f_1 = x^e + \cdots + a_1 x + a_0$, we have that

$$g(f_1) = x^e + \cdots + g(a_1)x + g(a_0)$$

is the monic irreducible polynomial of $g(\pi_1)$ over K_2 since $g(K_1)$ is contained in K_2. Then by Lemma 3.3.(1) and Fact 2.7, we get

$$\begin{aligned} M(R_2) &= \max\{\widetilde{\nu_2}\,(g(\pi_1) - \eta) : g(f_1)(\eta) = 0,\ \eta \neq g(\pi_1)\} \\ &= \max\{\widetilde{\nu_2}\,(g(\pi_1) - \widetilde{g}(\pi_1')) : f_1(\pi_1') = 0,\ \pi_1' \neq \pi_1\} \\ &= \max\{\widetilde{\nu_1}\,(\pi_1 - \pi_1') : f_1(\pi_1') = 0,\ \pi_1' \neq \pi_1\} \\ &= M(R_1), \end{aligned}$$

which finishes the proof. □

Now we introduce the notion of lifting maps.

Definition 3.4. Let R_1 and R_2 be complete discrete valuation rings of characteristic 0 with perfect residue fields k_1 and k_2 of characteristic p respectively. Let $\mathfrak{m}_i$ be the maximal ideal of R_i for $i = 1, 2$. Let L_i and K_i be the fraction fields of R_i and $W(k_i)$ for $i = 1, 2$ respectively. For any homomorphism $\phi : R_{1,(n_1)} \longrightarrow R_{2,(n_2)}$, we say that a homomorphism $g : R_1 \longrightarrow R_2$ is a (n_1, n_2)-*lifting* of ϕ if g satisfies the following:

- For any x in R_1, there exists a representative β_x of $\phi(x + \mathfrak{m}_1^{n_1})$ which satisfies

$$\widetilde{\nu_2}\big(g(x) - \beta_x\big) > M(R_1)$$

- $\phi_{red,1} \circ \mathrm{pr}_{1,1} = \mathrm{pr}_{2,1} \circ g$ where $\phi_{red,1} : k_1 \longrightarrow k_2$ denotes the natural reduction map of ϕ and $\mathrm{pr}_{i,1} : R_i \longrightarrow k_i$ is the canonical epimorphism for $i = 1, 2$.

When such g is unique, we denote g by $\mathrm{L}_{n_1,n_2}(\phi)$. When $\mathrm{L}_{n_1,n_2}(\phi)$ exists for all $\phi : R_{1,(n_1)} \longrightarrow R_{2,(n_2)}$, we write $\mathrm{L}_{n_1,n_2} : \mathrm{Hom}(R_{1,(n_1)}, R_{2,(n_2)}) \longrightarrow \mathrm{Hom}(R_1, R_2)$. When $n_1 = n_2 = n$, we denote L_{n_1,n_2} by L_n and say that L_n is an n-*lifting*.

The following example explains why we need our 'approximate' lifting map for the ramified case.

Example 3.5. If we take $R_1 = R_2 = \mathbb{Z}_3[\sqrt{3}]$ and $n_1 = n_2 = 2n$, then $R_{1,(2n)} = R_{2,(2n)} \cong (\mathbb{Z}_3/3^n\mathbb{Z}_3)[x]/(x^2 - 3)$. Then $\phi : a + bx \mapsto a + (1 + 3^{n-1})bx = \phi(a + bx)$ defines an isomorphism between $R_{1,(2n)}$ and $R_{2,(2n)}$. But when $n > 1$, there is no homomorphism $g : R_1 \longrightarrow R_2$ which induces ϕ since the Galois conjugates of $\sqrt{3}$ are $\pm\sqrt{3}$. This shows that we can not guarantee that the following diagram is commutative:

$$\begin{array}{ccc} R_1 & \xrightarrow{\mathrm{L}_{n_1,n_2}(\phi)} & R_2 \\ \downarrow & & \downarrow \\ R_{1,(n_1)} & \xrightarrow{\phi} & R_{2,(n_2)} \end{array}$$

We introduce a weaker condition of lifting map, which will turn out to be equivalent to Definition 3.4 (see Proposition 3.6). This weaker notion is useful to show the functoriality of lifting maps (see Proposition 4.3).

Proposition 3.6. *For a homomorphism $\phi : R_{1,(n_1)} \longrightarrow R_{2,(n_2)}$, suppose that a homomorphism $g : R_1 \longrightarrow R_2$ satisfies the following:*

- *There exists a representative β of $\phi(\pi_1 + \mathfrak{m}_1^{n_1})$ which satisfies*

$$\widetilde{\nu_2}\left(g(\pi_1) - \beta\right) > \max_{\sigma} \left\{\widetilde{\nu_2}\left(\sigma\left(g(\pi_1)\right) - \beta\right) : \sigma\left(g(\pi_1)\right) \neq g(\pi_1)\right\}$$

  where σ runs through all of $\mathrm{Hom}_{K_2}(L_2, L_2^{alg})$.
- *$\phi_{red,1} \circ \mathrm{pr}_{1,1} = \mathrm{pr}_{2,1} \circ g$ where $\phi_{red,1} : k_1 \longrightarrow k_2$ is the natural reduction map of ϕ.*

(1) We have that

$$\max_{\sigma} \left\{\widetilde{\nu_2}\left(\sigma\left(g(\pi_1)\right) - \beta\right) : \sigma\left(g(\pi_1)\right) \neq g(\pi_1)\right\} = M(R_1).$$

(2) For any x in R_1, there exists a representative β_x of $\phi(x + \mathfrak{m}_1^{n_1})$ which satisfies

$$\widetilde{\nu_2}\left(g(x) - \beta_x\right) > M(R_1)$$

so that g is a (n_1, n_2)-lifting of ϕ.

Proof. (1) For $\sigma \in \mathrm{Hom}_{K_2}(L_2, L_2^{alg})$ with $\sigma(g(\pi_1)) \neq g(\pi_1)$, we have

$$\begin{aligned} \widetilde{\nu_2}\left(\sigma\left(g(\pi_1)\right) - g(\pi_1)\right) &= \widetilde{\nu_2}\left(\sigma\left(g(\pi_1)\right) - \beta + \beta - g(\pi_1)\right) \\ &= \min\left\{\widetilde{\nu_2}\left(\sigma\left(g(\pi_1)\right) - \beta\right),\ \widetilde{\nu_2}\left(g(\pi_1) - \beta\right)\right\} \\ &= \widetilde{\nu_2}\left(\sigma\left(g(\pi_1)\right) - \beta\right) \end{aligned}$$

where the second equality follows from the ultrametric inequality and the assumption $\widetilde{\nu_2}(g(\pi_1) - \beta) > \widetilde{\nu_2}(\sigma(g(\pi_1)) - \beta)$.

This shows

$$\begin{aligned} M(R_1) &= \max_{\sigma'} \left\{\widetilde{\nu_1}\left(\pi_1 - \sigma'(\pi_1)\right) : \sigma'(\pi_1) \neq \pi_1\right\} \\ &= \max_{\sigma} \left\{\widetilde{\nu_2}\left(g(\pi_1) - \sigma\left(g(\pi_1)\right)\right) : \sigma\left(g(\pi_1)\right) \neq g(\pi_1)\right\} \\ &= \max_{\sigma} \left\{\widetilde{\nu_2}\left(\sigma\left(g(\pi_1)\right) - \beta\right) : \sigma\left(g(\pi_1)\right) \neq g(\pi_1)\right\} \end{aligned}$$

where σ' runs through all of $\mathrm{Hom}_{K_1}(L_1, L_1^{alg})$. The second equality follows from Lemma 3.3.(2) because $[K_2(g(\pi_1)) : K_2]$ is equal to $[L_1 : K_1]$ and $g(\pi_1)$ is a uniformizer of $K_2(g(\pi_1))$ by Fact 2.6.

(2) For any x in R_1, we can write $x = \sum_{r=0}^{\infty} \lambda_r \pi_1^r$ where λ_r is in S_1 for each r. Then

$$\phi(x + \mathfrak{m}_1^{n_1}) = \phi\left(\left(\sum_{r=0}^{\infty} \lambda_r \pi_1^r\right) + \mathfrak{m}_1^{n_1}\right) = \left(\sum_{r=0}^{\infty} \tau_r \beta^r\right) + \mathfrak{m}_2^{n_2}$$

where τ_r is a representative of $\phi(\lambda_r + \mathfrak{m}_1^{n_1})$ contained in S_2 which is guaranteed by Lemma 3.2.(1). In particular $\sum_{r=0}^{\infty} \tau_r \beta^r$ is a representative of $\phi(x+\mathfrak{m}_1^{n_1})$, say β_x. By Lemma 3.2.(1) again, we have $g(\lambda_r) = \tau_r$, and hence,

$$g(x) = g\left(\sum_{r=0}^{\infty} \lambda_r \pi_1^r\right) = \sum_{r=0}^{\infty} \tau_r g(\pi_1)^r.$$

We obtain

$$\begin{aligned}\widetilde{\nu_2}(g(x) - \beta_x) &= \widetilde{\nu_2}\left(\sum_{r=0}^{\infty} \tau_r g(\pi_1)^r - \sum_{r=0}^{\infty} \tau_r \beta^r\right) \\ &= \widetilde{\nu_2}\left(\Big(g(\pi_1) - \beta\Big) \sum_{r=1}^{\infty} \tau_r \left(\sum_{j=0}^{r-1} g(\pi_1)^{r-1-j} \beta^j\right)\right) \\ &> M(R_1)\end{aligned}$$

because

$$\begin{aligned}\widetilde{\nu_2}\left(g(\pi_1) - \beta\right) &> \max_{\sigma} \left\{\widetilde{\nu_2}\left(\sigma\left(g(\pi_1)\right) - \beta\right) : \sigma\left(g(\pi_1)\right) \neq g(\pi_1)\right\} \\ &= M(R_1).\end{aligned}$$

So g is a (n_1, n_2)-lifting of ϕ. □

The following theorem shows that there is a unique lifting if we enlarge the lengths of residue rings.

Theorem 3.7. *Suppose $n_2 > M(R_1)\nu_1(p)\nu_2(p)$ and $\mathrm{Hom}(R_{1,(n_1)}, R_{2,(n_2)})$ is not empty. Then there exists a unique (n_1, n_2)-lifting $\mathrm{L}_{n_1,n_2} : \mathrm{Hom}(R_{1,(n_1)}, R_{2,(n_2)}) \longrightarrow \mathrm{Hom}(R_1, R_2)$. Also, $\mathrm{L}_{n_1,n_2}(\phi)$ is an isomorphism when $\phi : R_{1,(n_1)} \longrightarrow R_{2,(n_2)}$ is an isomorphism.*

Proof. Let ϕ be a homomorphism from $R_{1,(n_1)}$ to $R_{2,(n_2)}$. By Lemma 3.2.(2), let

$$\phi_{res} : \frac{W(k_1) + \mathfrak{m}_1^{n_1}}{\mathfrak{m}_1^{n_1}} \longrightarrow \frac{W(k_2) + \mathfrak{m}_2^{n_2}}{\mathfrak{m}_2^{n_2}}$$

be the restriction map of ϕ. For an element $a = \sum_{r=0}^{\infty} \lambda_r p^r$ in $W(k_1)$, we define $g_{res} : W(k_1) \longrightarrow W(k_2)$ by the rule

$$g_{res} : W(k_1) \longrightarrow W(k_2),\ a \mapsto g_{res}(a) = \sum_{r=0}^{\infty} \tau_r p^r$$

where τ_r is a unique representative of $\phi_{res}(\lambda_r + \mathfrak{m}_1^{n_1})$ which is contained in S_2, the set of Teichmüller representatives of R_2. Then, by the proof of Fact 1.5.(2) (cf. the proof of [15, Proposition 10]), g_{res} is a homomorphism and g_{res} induces ϕ_{res}. By Fact 2.9, $L_1 = K_1(\alpha)$ is totally ramified of degree $\nu_1(p)$ over K_1, that is, $[L_1 : K_1] = \nu_1(p)$, where $\alpha = \pi_1$ is a uniformizer of R_1. Let f be the monic irreducible polynomial of α over K_1. The ring homomorphism g_{res} induces a field homomorphism from K_1 into K_2. We still denote the fraction field homomorphism by g_{res} if there is no confusion. Then $g_{res} : K_1 \longrightarrow K_2$ is an isometry by Fact 2.6. Let $\widetilde{g_{res}} : K_1^{alg} \longrightarrow K_2^{alg}$ be an extended field homomorphism of g_{res}, which is also an isometry by Fact 2.7. Write

$$\begin{aligned} f =& x^{\nu_1(p)} + \cdots + a_1 x + a_0 \\ =& (x - \alpha_1) \cdots (x - \alpha_{\nu_1(p)}) \end{aligned}$$

where $\alpha = \alpha_1$, and put

$$\begin{aligned} g_{res}(f) =& x^{\nu_1(p)} + \cdots + g_{res}(a_1)x + g_{res}(a_0) \\ =& \left(x - \widetilde{g_{res}}(\alpha_1)\right) \cdots \left(x - \widetilde{g_{res}}(\alpha_{\nu_1(p)})\right). \end{aligned}$$

We have that $[K_2(\widetilde{g_{res}}(\alpha)) : K_2] \leq [K_1(\alpha) : K_1] = \nu_1(p)$ and that $\widetilde{\nu_2}(\widetilde{g_{res}}(\alpha)) = \widetilde{\nu_1}(\alpha) = 1/\nu_1(p)$ because $\widetilde{g_{res}}$ is an isometry. Therefore $g_{res}(f)$ is the monic irreducible polynomial of $\widetilde{g_{res}}(\alpha)$ over K_2. Let β be any representative of $\phi(\alpha + \mathfrak{m}_1^{n_1})$. Since g_{res} induces ϕ_{res}, we can write

$$\begin{aligned} 0 + \mathfrak{m}_2^{n_2} &= \phi(f(\alpha) + \mathfrak{m}_1^{n_1}) \\ &= \phi(\alpha + \mathfrak{m}_1^{n_1})^{\nu_1(p)} + \cdots + \phi(a_1 + \mathfrak{m}_1^{n_1})\phi(\alpha + \mathfrak{m}_1^{n_1}) + \phi(a_0 + \mathfrak{m}_1^{n_1}) \\ &= g_{res}(f)(\beta) + \mathfrak{m}_2^{n_2}. \end{aligned}$$

This shows that $g_{res}(f)(\beta)$ is in $\mathfrak{m}_2^{n_2}$ and

$$\nu_2\left(g_{res}(f)(\beta)\right) \geq n_2 > M(R_1)\nu_1(p)\nu_2(p).$$

We claim that there exists an index i_0 satisfying $\widetilde{\nu_2}(\beta - \widetilde{g_{res}}(\alpha_{i_0})) > M(R_1)$. If $\widetilde{\nu_2}(\beta - \widetilde{g_{res}}(\alpha_i)) \leq M(R_1)$ for all i, then

$$\widetilde{\nu_2}\left(g_{res}(f)(\beta)\right) = \widetilde{\nu_2}\left(\prod_i \left(\beta - \widetilde{g_{res}}(\alpha_i)\right)\right) \leq M(R_1)\nu_1(p).$$

This shows

$$n_2 \leq \nu_2\left(g_{res}(f)(\beta)\right) = \nu_2(p)\widetilde{\nu_2}\left(g_{res}(f)(\beta)\right) \leq M(R_1)\nu_1(p)\nu_2(p),$$

which is impossible. Thus there is an index i_0 satisfying

$$\widetilde{\nu_2}\left(\beta - \widetilde{g_{res}}(\alpha_{i_0})\right) > M(R_1) = \max\left\{\widetilde{\nu_2}\left(\widetilde{g_{res}}(\alpha_1) - \widetilde{g_{res}}(\alpha_j)\right) : j = 2, ..., \nu_1(p)\right\}$$

where the equality follows from the fact that $\widetilde{g_{res}}$ is an isometry. Hence, by Fact 2.13, $K_2(\widetilde{g_{res}}(\alpha_{i_0})) \subset K_2(\beta) \subset L_2$. We define an extended homomorphism $g : L_1 \longrightarrow L_2$ of $g_{res} : K_1 \longrightarrow K_2$ by the rule $\pi_1 \mapsto g(\pi_1) = \widetilde{g_{res}}(\alpha_{i_0})$. Then, g induces the restricted homomorphism from R_1 to R_2 which is still denoted by g. Also, g is a (n_1, n_2)-lifting of ϕ because g_{res} induces ϕ_{res} and

$$M(R_1) = \max_{\sigma}\left\{\widetilde{\nu_2}\left(\sigma\left(g(\pi_1)\right) - \beta\right) : \sigma\left(g(\pi_1)\right) \neq g(\pi_1)\right\}$$

by Lemma 3.6.

Suppose that $g_1 : R_1 \longrightarrow R_2$ is an (n_1, n_2)-lifting of ϕ other than g. We note that the restriction $g|_{S_1}$ of g to S_1 is equal to $g_1|_{S_1}$ by Fact 1.5. From Remark 2.12 and $g|_{S_1} = g_1|_{S_1}$, it follows that $g_1|_{W(k_1)} = g|_{W(k_1)}$. Since $R_1 = W(k_1)[\pi_1]$, $g = g_1$ if $g(\pi_1) = g_1(\pi_1)$. So, $g(\pi_1) \neq g_1(\pi_1)$, and by Proposition 3.6,

$$\widetilde{\nu_2}\left(g_1(\pi_1) - \beta\right) > \max_{\sigma} \left\{\widetilde{\nu_2}\left(\sigma\left(g_1(\pi_1)\right) - \beta\right) : \sigma\left(g_1\left(\pi_1\right)\right) \neq g_1(\pi_1)\right\}.$$

Since $g_1|_{W(k_1)} = g|_{W(k_1)}$, $g(\pi_1)$ and $g_1(\pi_1)$ have the same minimal polynomial over $W(k_2)$ and

$$\left\{\sigma\left(g_1(\pi_1)\right) : \sigma \in \mathrm{Hom}_{K_2}(L_2, L_2^{alg})\right\} = \left\{\sigma\left(g(\pi_1)\right) : \sigma \in \mathrm{Hom}_{K_2}(L_2, L_2^{alg})\right\}.$$

In particular $g_1(\pi_1) = \sigma(g(\pi_1))$ for some $\sigma \in \mathrm{Hom}_{K_2}(L_2, L_2^{alg})$. Since $g_1(\pi_1) \neq g(\pi_1)$, we have the inequalities $\widetilde{\nu_2}(g_1(\pi_1) - \beta) > \widetilde{\nu_2}(g(\pi_1) - \beta)$ and $\widetilde{\nu_2}(g_1(\pi_1) - \beta) < \widetilde{\nu_2}(g(\pi_1) - \beta)$ simultaneously by the first bullet point of Proposition 3.6. This gives a contradiction, and hence, we obtain the uniqueness of the lifting.

When ϕ is an isomorphism, so are ϕ_{res} and g_{res}. We obtain $[L_2 : K_2] = [L_1 : K_1]$ from the assumption that $n_2 > M(R_1)\nu_1(p)\nu_2(p)$, and hence, $\mathrm{L}_{n_1,n_2}(\phi)$ is also an isomorphism. □

We note that the proof of Theorem 3.7 works for any representative β of $\phi(\pi_1 + \mathfrak{m}_1^{n_1})$.

Example 3.8. Let $R_1 = \mathbb{Z}_3[\sqrt{3}]$ and $R_2 = \mathbb{Z}_3[\sqrt{-3}]$. There is no homomorphism between R_1 and R_2 by Kummer theory. But there is an isomorphism

$$\phi : R_{1,(2)} = \frac{\mathbb{Z}_3[\sqrt{3}]}{3\mathbb{Z}_3[\sqrt{3}]} \longrightarrow R_{2,(2)} = \frac{\mathbb{Z}_3[\sqrt{-3}]}{3\mathbb{Z}_3[\sqrt{-3}]}$$

given by the rule $a + b\sqrt{3} \mapsto a + b\sqrt{-3}$. Since $\nu_1(3) = \nu_2(3) = 2$ and $M(R_1) = \widetilde{\nu_1}(\sqrt{3} - (-\sqrt{3})) = 1/2$, we obtain $M(R_1)\nu_1(3)\nu_2(3) = 2$. Hence the lower bound for n_2 in Theorem 3.7 is the best possible in this case. This phenomenon will be generalized in Proposition 4.5.

We give a generalized version of Fact 1.5.(1) for the ramified case. We first give a useful upper bound for $M(R)$.

Lemma 3.9. *Let (R, ν, k) be a valuation ring and let π be a uniformizer of R. Let L and K be fraction fields of R and $W(k)$ respectively. Then,*

$$M(R) \leq \frac{1 + \nu(\nu(p))}{\nu(p)}.$$

Proof. Let f be the monic irreducible polynomial of π over K, which is of degree $e := \nu(p)$. Let $\pi_1(:= \pi), \ldots, \pi_e$ be the distinct zeros of f. We have $\widetilde{\nu}(\pi) = 1/e$ and hence $\widetilde{\nu}(\pi_i - \pi_j) \geq 1/e$ for all i and j. Furthermore, by definition of $M(R)$, we have that for some $2 \leq i_0 \leq e$,

- $M(R) \geq \widetilde{\nu}(\pi) = \frac{1}{e}$; and
- $M(R) = \widetilde{\nu}\big(\pi_1 - \pi_{i_0}\big)$.

Consider the differentiation

$$f' = \sum_{i=1}^{e} \frac{f}{(x - \pi_i)}.$$

There are two cases.

- Tame case: Suppose L/K is tamely ramified. Hence, $\nu(\nu(p)) = \nu(e) = 0$. It follows from Fact 2.14 that

$$\frac{e-1}{e} = \widetilde{\nu}\big(f'(\pi_1)\big) = \widetilde{\nu}\left(\prod_{j\neq 1}(\pi_1 - \pi_j)\right) = \sum_{j\neq 1}\widetilde{\nu}(\pi_1 - \pi_j).$$

  Since $\widetilde{\nu}(\pi_i - \pi_j) \geq 1/e$, $\widetilde{\nu}(\pi_1 - \pi_j) = 1/e = M(R)$ for $j \neq 1$. Hence, we have that

$$M(R) = \frac{1}{e} = \frac{1+\nu(e)}{e}.$$

- Wild case: Suppose L/K is wildly ramified. Noting that $\widetilde{\nu}(\pi_i - \pi_j) \geq 1/e$, we have that

$$\begin{aligned} M(R) &\leq \widetilde{\nu}(\pi_1 - \pi_{i_0}) + \sum_{2\leq i\neq i_0\leq e}\left(\widetilde{\nu}(\pi_1 - \pi_i) - \frac{1}{e}\right) \\ &= \widetilde{\nu}\left(\prod_{i\neq 1}(\pi_1 - \pi_i)\right) - \frac{(e-2)}{e} \;=\; \widetilde{\nu}(f'(\pi)) - \frac{e-2}{e} \\ &\leq \frac{e-1+\nu(e)}{e} - \frac{e-2}{e} \;=\; \frac{1+\nu(e)}{e} \end{aligned}$$

  by Fact 2.14 again.

Therefore we get the desired result. □

Theorem 3.10. *Let $\overline{R}$ be a principal Artinian local ring of length n with perfect residue field k of characteristic p and maximal ideal $\overline{\mathfrak{m}}$, that is, $\overline{\mathfrak{m}}^n = 0$ and $\overline{\mathfrak{m}}^{n-1} \neq 0$. Suppose that $\overline{R}$ has no finite subfield as a subring. For any positive integer a, if a generates a nonzero ideal $\overline{\mathfrak{m}}^k$, we denote k by $\nu(a)$. Suppose*

$$\nu(p)\overline{R} \neq 0 \text{ and } n > \nu(p) + \nu(p)\nu\big(\nu(p)\big).$$

Then there exists a complete discrete valuation ring of characteristic 0 which has $\overline{R}$ as its n-th residue ring. Also such a ring is unique up to isomorphism.

Proof. Any principal Artinian local ring is a homomorphic image of a discrete valuation ring. This can be proved by Cohen structure theorem for complete local rings (cf. [10]) or, more directly, by the property of CPU-rings (cf. [9]). Since the completion of a discrete valuation ring R has the same n-th residue ring as that of R, we may assume that there are complete discrete valuation rings R_1 and R_2 such that $R_{1,(n)}$ and $R_{2,(n)}$ are isomorphic to $\overline{R}$. We note that R_i is of characteristic 0 for $i = 1, 2$ because $\overline{R}$ has no finite subfield as a subring. Let L_i and K_i be the fraction fields of R_i and $W(k_i)$ for $i = 1, 2$ respectively. Then by Fact 2.9, $L_1 = K_1(\alpha)$ where $\alpha = \pi_1$ is a uniformizer of R_1. By Lemma 3.9, we have that

$$M(R_1)\nu_1(p)\nu_2(p) \leq \nu_2(p)(1 + \nu_1(\nu_1(p))) = \nu(p)(1 + \nu(\nu(p))).$$

Note that $\nu(\nu(p))$ and $\nu(p)$ are well-defined since $\nu(p)\overline{R} \neq 0$ and $\overline{R}$ has no finite subfield. The desired result follows from Theorem 3.7. □

Note that the notation $\nu(p)$ in Theorem 3.10 is compatible with the previously defined valuation. Suppose that a discrete valuation ring R with valuation ν and maximal ideal $\mathfrak{m}$ has $\overline{R}$ as its residue ring. Then $\nu(p)$ is equal to a power of the maximal ideal generated by p, that is, $pR = \mathfrak{m}^{\nu(p)}$ as we noted in the proof of Theorem 3.10.

4. Functoriality

The main purpose of this section is to give a generalized version of Fact 1.6 for the ramified case. For a prime number p and a positive integer e, let $\mathcal{C}_{p,e}$ be a category consisting of the following data:

- $\mathrm{Ob}(\mathcal{C}_{p,e})$ is the family of complete discrete valuation rings of mixed characteristic having perfect residue fields of characteristic p and the ramification index e; and
- $\mathrm{Mor}_{\mathcal{C}_{p,e}}(R_1, R_2) := \mathrm{Hom}(R_1, R_2)$ for R_1 and R_2 in $\mathrm{Ob}(\mathcal{C}_{p,e})$.

Let $\mathcal{R}^n_{p,e}$ be a category consisting of the following data:

- For $n \le e$, $\mathrm{Ob}(\mathcal{R}^n_{p,e})$ is the family of principal Artinian local rings $\overline{R}$ of length n with perfect residue fields of characteristic p, and for $n > e$, $\mathrm{Ob}(\mathcal{R}^n_{p,e})$ is the family of principal Artinian local rings $\overline{R}$ of length n with perfect residue fields of characteristic p such that $p \in \overline{\mathfrak{m}}^e \setminus \overline{\mathfrak{m}}^{e+1}$ where $\overline{\mathfrak{m}}$ is the maximal ideal of $\overline{R}$; and
- $\mathrm{Mor}_{\mathcal{R}^n_{p,e}}(\overline{R_1}, \overline{R_2}) := \mathrm{Hom}(\overline{R_1}, \overline{R_2})$ for $\overline{R_1}$ and $\overline{R_2}$ in $\mathrm{Ob}(\mathcal{R}^n_{p,e,})$,

Note that for $e_1, e_2 \ge 1$ and for $n \le e_1, e_2$, two categories $\mathcal{R}^n_{p,e_1}$, $\mathcal{R}^n_{p,e_2}$ are the same. For each $m > n$, let $\mathrm{Pr}_n : \mathcal{C}_{p,e} \to \mathcal{R}^n_{p,e}$ and $\mathrm{Pr}^m_n : \mathcal{R}^m_{p,e} \to \mathcal{R}^n_{p,e}$ be the canonical functors respectively.

Definition 4.1. Fix a prime number p and a positive integer e.

(1) We say that the category $\mathcal{C}_{p,e}$ is *n-liftable* if there is a functor $\mathrm{L} : \mathcal{R}^n_{p,e} \longrightarrow \mathcal{C}_{p,e}$ which satisfies the following:
   - $(\mathrm{Pr}_n \circ \mathrm{L})(\overline{R}) \cong \overline{R}$ for each $\overline{R}$ in $\mathrm{Ob}(\mathcal{R}_{p,e})$.
   - $\mathrm{Pr}_1 \circ \mathrm{L}$ is equivalent to Pr^n_1.
   - $\mathrm{L} \circ \mathrm{Pr}_n$ is equivalent to $\mathrm{Id}_{\mathcal{C}_{p,e}}$, the identity functor.

   We say that L is a *n-th lifting functor* of $\mathcal{C}_{p,e}$.
(2) The *lifting number* for $\mathcal{C}_{p,e}$ is the smallest positive integer n such that $\mathcal{C}_{p,e}$ is n-liftable. If there is no such n, we define the lifting number for $\mathcal{C}_{p,e}$ to be ∞.

We note that the condition $(\mathrm{Pr}_n \circ \mathrm{L})(\overline{R}) \cong \overline{R}$ in the first bullet point in Definition 4.1.(1) is weaker than the condition that $\mathrm{Pr}_n \circ \mathrm{L}$ is equivalent to $\mathrm{Id}_{\mathcal{R}^n_{p,e}}$. By Example 3.5, $\mathrm{Pr}_n \circ \mathrm{L}$ is not equivalent to $\mathrm{Id}_{\mathcal{R}^n_{p,e}}$ in general.

Remark 4.2.

(1) Suppose that there is a n-th lifting functor $\mathrm{L} : \mathcal{R}^n_{p,e} \to \mathcal{C}_{p,e}$. For any $\overline{R}$ in $\mathrm{Ob}(\mathcal{R}_{p,e})$, $\mathrm{L}(\overline{R})$ is the unique (up to isomorphism) object in $\mathrm{Ob}(\mathcal{C}_{p,e})$ which has $\overline{R}$ as its n-th residue ring. Indeed, suppose that R in $\mathrm{Ob}(\mathcal{C}_{p,e})$ has $\overline{R}$ as its n-th residue ring. Since $\mathrm{L} \circ \mathrm{Pr}_n$ is equivalent to the identity functor $\mathrm{Id}_{\mathcal{C}_{p,e}}$, $R = \mathrm{Id}_{\mathcal{C}_{p,e}}(R)$ is isomorphic to $(\mathrm{L} \circ \mathrm{Pr}_n)(R) = \mathrm{L}(\overline{R})$.
(2) The lifting number for $\mathcal{C}_p$ is 1 by Fact 1.6. We will see that the lifting number for $\mathcal{C}_{p,e}$ is always larger than e whenever $e > 1$ in Corollary 4.11.

(3) For $n \geq e$, a functor $\mathrm{L}_{n+1} := \mathrm{L}_n \circ \mathrm{Pr}_n^{n+1}$ is a $(n+1)$-th lifting functor of $\mathcal{C}_{p,e}$ for any n-th lifting functor $\mathrm{L}_n : \mathcal{R}_{p,e}^n \to \mathcal{C}_{p,e}$. The proof is as follows: For $\overline{R}$ in $\mathrm{Ob}(\mathcal{R}_{p,e}^{n+1})$, there exists a ring R in $\mathrm{Ob}(\mathcal{C}_{p,e})$ which satisfies $\mathrm{Pr}_{n+1}(R) = \overline{R}$ as noted in the proof of Theorem 3.10. Since there is a unique object in $\mathrm{Ob}(\mathcal{C}_{p,e})$ which has $\mathrm{Pr}_n(R)$ as its n-th residue ring by Remark 4.2.(1), we have that

$$(\mathrm{Pr}_{n+1} \circ \mathrm{L}_{n+1})\left(\overline{R}\right) = \mathrm{Pr}_{n+1} \circ (\mathrm{L}_n \circ \mathrm{Pr}_n^{n+1})\left(\overline{R}\right) = \mathrm{Pr}_{n+1}(R) = \overline{R}.$$

Also, $\mathrm{Pr}_1 \circ \mathrm{L}_{n+1} = (\mathrm{Pr}_1 \circ \mathrm{L}_n) \circ \mathrm{Pr}_n^{n+1}$ is equivalent to $\mathrm{Pr}_1^n \circ \mathrm{Pr}_n^{n+1} = \mathrm{Pr}_1^{n+1}$ and

$$\mathrm{L}_{n+1} \circ \mathrm{Pr}_{n+1} = (\mathrm{L}_n \circ \mathrm{Pr}_n^{n+1}) \circ \mathrm{Pr}_{n+1} = \mathrm{L}_n \circ \mathrm{Pr}_n$$

is equivalent to $\mathrm{Id}_{\mathcal{C}_{p,e}}$.

Proposition 4.3. *For $1 \leq i \leq 3$, let $(R_i, \mathfrak{m}_i, \nu_i)$ be a complete discrete valuation ring of mixed characteristic $(0, p)$ with a perfect residue field and let π_i be a uniformizer of R_i. For $\phi^{1,2} : R_{1,(n_1)} \longrightarrow R_{2,(n_2)}$ and $\phi^{2,3} : R_{2,(n_2)} \longrightarrow R_{3,(n_3)}$, suppose that there are liftings $g^{1,2} : R_1 \longrightarrow R_2$ and $g^{2,3} : R_2 \longrightarrow R_3$ of $\phi^{1,2}$ and $\phi^{2,3}$ respectively.*

If $\nu_1(p) = \nu_2(p)$, then $g = g^{2,3} \circ g^{1,2}$ is a lifting of $\phi^{2,3} \circ \phi^{1,2}$. Moreover g is the unique lifting of $\phi^{2,3} \circ \phi^{1,2}$ when $n_3 > M(R_2)\nu_2(p)\nu_3(p)$.

Proof. By Fact 2.6, the liftings $g^{1,2}$ and $g^{2,3}$ are isometries. Also, since both $\widetilde{\nu_2}$ and $\widetilde{\nu_3}$ are normalized, we have $\widetilde{\nu_3}(g^{2,3}(x)) = \widetilde{\nu_2}(x)$ for any $x \in R_2$. By Lemma 3.3, $M(R_1) = M(R_2)$, say M. Since $g^{1,2}$ is a lifting of $\phi^{1,2}$, there is a representative β_1 of $\phi^{1,2}(\pi_1 + \mathfrak{m}_1^{n_1})$ such that $\widetilde{\nu_2}(g^{1,2}(\pi_1) - \beta_1) > M$. We note that β_1 is a uniformizer of R_2. Since $g^{2,3}$ is a lifting of $\phi^{2,3}$, there is a representative β_2 of

$$(\phi^{2,3} \circ \phi^{1,2})(\pi_1 + \mathfrak{m}_1^{n_1}) = \phi^{2,3}(\beta_1 + \mathfrak{m}_2^{n_2})$$

such that $\widetilde{\nu_3}(g^{2,3}(\beta_1) - \beta_2) > M$.

If we write $g^{1,2}(\pi_1) = \beta_1 + x_M$ where $\widetilde{\nu_2}(x_M) > M$, then

$$g(\pi_1) = g^{2,3}(g^{1,2}(\pi_1)) = g^{2,3}(\beta_1 + x_M).$$

Since $\widetilde{\nu_3}(g^{2,3}(\beta_1) - \beta_2) > M$ and $\widetilde{\nu_3}(g^{2,3}(x_M)) = \widetilde{\nu_2}(x_M) > M$,

$$\widetilde{\nu_3}\big(g(\pi_1) - \beta_2\big) = \widetilde{\nu_3}\big(g^{2,3}(\beta_1) - \beta_2 + g^{2,3}(x_M)\big) > M.$$

The equality $(\phi^{2,3} \circ \phi^{1,2})_{red,1} \circ \mathrm{pr}_{1,1} = \mathrm{pr}_{3,1} \circ g$ follows directly from $g = g^{2,3} \circ g^{1,2}$. By Proposition 3.6, g is a lifting of $\phi^{2,3} \circ \phi^{1,2}$.

When $n_3 > M(R_2)\nu_2(p)\nu_3(p) = M(R_1)\nu_1(p)\nu_3(p)$, g is the unique lifting of $\phi^{2,3} \circ \phi^{1,2}$ by Theorem 3.7. □

Theorem 4.4. *The lifting number for $\mathcal{C}_{p,e}$ is finite. More precisely, $\mathcal{C}_{p,e}$ is $(e + e\nu(e) + 1)$-liftable. Here $\nu(e)$ denotes the exponent n such that e generates an ideal $\mathfrak{m}^n$ of R in $\mathrm{Ob}(\mathcal{C}_{p,e})$ where $\mathfrak{m}$ denotes the maximal ideal of R. The value $\nu(e)$ depends only on the prime number p and the ramification index e, in particular $\nu(e)$ is independent of the choice of R in $\mathrm{Ob}(\mathcal{C}_{p,e})$.*

Proof. Suppose n is bigger than $e + e\nu(e)$. For any $\overline{R}, \overline{R_1}$ and $\overline{R_2}$ in $\mathrm{Ob}(\mathcal{R}_{p,e}^n)$, by Theorem 3.10, we define $\mathrm{L}_n(\overline{R})$ to be a unique ring R in $\mathrm{Ob}(\mathcal{C}_{p,e})$ which satisfies $\mathrm{Pr}_n(R) = \overline{R}$. By Lemma 3.9, $e + e\nu(e) \geq M(R)e^2$. By Theorem 3.7, for any $\phi : \overline{R_1} \longrightarrow \overline{R_2}$, there exists a unique n-th lifting map $\mathrm{L}(\phi) : \mathrm{L}(\overline{R_1}) \longrightarrow \mathrm{L}(\overline{R_2})$, and hence we obtain a lifting functor $\mathrm{L}_n : \mathcal{R}_{p,e}^n \longrightarrow \mathcal{C}_{p,e}$ by Proposition 4.3. □

Example 3.8 can be generalized as follows.

Proposition 4.5. *Let $R_1/W(k)$ and $R_2/W(k)$ be totally ramified extensions of degree e. Then $R_{1,(e)}$ is isomorphic to $R_{2,(e)}$ as $W(k)$-algebras.*

Proof. Let π_i be a uniformizer of R_i and let ν_i be the valuation corresponding to R_i for $i = 1, 2$. By the theory of totally ramified extensions (see Chapter 2 of [12] for example), the monic irreducible polynomial f_i of π_i over $W(k)$ is an Eisenstein polynomial for $i = 1, 2$. If we write $f_i = x^e + a_{i,e-1}x^{e-1} + \cdots + a_{i,1}x + a_{i,0}$, then $\nu_i(p) = \nu_i(a_{i,0}) = e$ and $\nu_i(a_{i,j}) \geq e$ for $i = 1, 2$ and $j = 1, 2, \ldots, e-1$. This shows

$$\begin{aligned} R_{i,(e)} = \frac{W(k)[\pi_i]}{(\pi_i)^e} &\cong \frac{W(k)[x]}{(p, f_i)} \\ &= \frac{k[x]}{(x^e + \cdots + a_{i,1}x + a_{i,0})} \\ &= \frac{k[x]}{(x^e)}, \end{aligned}$$

and hence, $R_{1,(e)}$ is isomorphic to $R_{2,(e)}$ as $W(k)$-algebras. □

For the tame case, we can calculate the lifting number. We denote a primitive n-th root of unity by ζ_n.

Lemma 4.6. *Let k be a perfect field of characteristic p and let K be the fraction field of $W(k)$. Let e be a positive integer prime to p. Suppose that there is a prime divisor l of e such that ζ_{l^n} is in $k^\times$ and $\zeta_{l^{n+1}}$ is not in $k^\times$ for some $n > 0$. Then there are two totally ramified extensions L_1 and L_2 of degree e over K which are not isomorphic over $\mathbb{Q}$.*

Proof. We have ζ_{l^n} is in $W(k)^\times$ by Hensel's lemma, and $\zeta_{l^{n+1}}$ is not in $W(k)^\times$. Then $L_1 = K(\sqrt[e]{p})$ and $L_2 = K(\sqrt[e]{p\zeta_{l^n}})$ are totally ramified extensions of degree e over K. Suppose that there is an isomorphism $\sigma : L_2 \longrightarrow L_1$. Since Galois conjugates of $\sqrt[e]{p}$ and ζ_{el^n} over $\mathbb{Q}$ are of the form $\sqrt[e]{p}\zeta_e^i$ and $\zeta_{el^n}^j$ respectively for some i and j with $(j, e) = 1$,

$$\sigma\left(\sqrt[e]{p\zeta_{l^n}}\right) = \sigma\left(\sqrt[e]{p}\zeta_{el^n}\right) = \sqrt[e]{p}\zeta_{el^n}^k$$

for some k prime to l. In particular, L_1 contains both $\sqrt[e]{p}$ and $\sqrt[e]{p}\zeta_{el^n}^k$, and hence, $\zeta_{l^{n+1}}$ is in L_1. This is a contradiction because L_1/K is totally ramified. □

Corollary 4.7. *Suppose that p does not divide e and $e > 1$. Then $e+1$ is the lifting number for $\mathcal{C}_{p,e}$.*

Proof. Since $\nu(p) = 0$, $e + e\nu(e) + 1 = e + 1$. By Theorem 4.4, $\mathcal{C}_{p,e}$ is $(e+1)$-liftable. Let $\mathbb{F}_p$ be the prime field of p elements. Let K be the fraction field of the Witt ring $W(k)$ of $k = \mathbb{F}_p(\zeta_e)$. By Lemma 4.6, there are two totally ramified extensions L_1 and L_2 of degree e over K such that there is no isomorphism between L_1 and L_2. If $\mathcal{C}_{p,e}$ is e-liftable, L_1 and L_2 are isomorphic over K by Proposition 4.5 and it is a contradiction. □

Remark 4.8. Proposition 4.5 and Corollary 4.7 show the difference between the unramified case and the tamely ramified case. We can regard the unramified valued fields of mixed characteristic as the tamely ramified valued fields having the ramification index $e = 1$. If we apply Corollary 4.7 to $\mathcal{C}_p$, the lifting number for $\mathcal{C}_p$ should be $1+1 = 2$. However the argument in the proof of Corollary 4.7 does not work for $\mathcal{C}_p$. For an unramified complete discrete valued field K, there is a unique totally ramified extension of degree 1 over K, that is, K itself. Hence the fact that the lifting number for $\mathcal{C}_p$ is 1 does not contradict Corollary 4.7.

For the wild case, we have the following example. Let $R_1 = \mathbb{Z}_2[\sqrt{2}]$ and $R_2 = \mathbb{Z}_2[\sqrt{10}]$. There is no homomorphism between R_1 and R_2 by Kummer theory. But there is an isomorphism between $R_{1,(6)}$ and $R_{2,(6)}$ because

$$\begin{aligned} R_{1,(6)} = \frac{\mathbb{Z}_2[\sqrt{2}]}{(\sqrt{2}^6)} &\cong \frac{\mathbb{Z}_2[x]}{(x^2-2,8)} \\ = \frac{\mathbb{Z}_2[x]}{(x^2-10,8)} &\cong \frac{\mathbb{Z}_2[\sqrt{10}]}{(\sqrt{2}^6)} = R_{2,(6)}. \end{aligned}$$

Note that the last equality holds because $(\sqrt{10})^6\mathbb{Z}_2[\sqrt{10}] = (\sqrt{2})^6\mathbb{Z}_2[\sqrt{10}]$. This shows that the lifting number for $\mathcal{C}_{2,2}$ is $2 + 2\nu(2) + 1 = 7$ by Theorem 4.4. In general, we have a lower bound $e + 1$ of the lifting number for the wild case. To prove this, we need the following lemma.

Lemma 4.9. *Let k be a perfect field of characteristic p and let K be the fraction field of the Witt ring $W(k)$ of k. Let e be a positive integer divisible by p. Then there are two totally ramified extensions L_1 and L_2 of degree e over K which are not isomorphic over $\mathbb{Q}$.*

Proof. We write $e = sp^r$ for some positive integers s and r where s is prime to p. Let $\mathbb{Q}_\infty/\mathbb{Q}$ be the cyclotomic $\mathbb{Z}_p$-extension, in particular $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \cong \mathbb{Z}_p$. Let M_r be a unique subfield of $\mathbb{Q}_\infty$ such that $[M_r : \mathbb{Q}] = p^r$. By the theory of cyclotomic fields (cf. [13, Chapter 1]), the Galois extension $M_r/\mathbb{Q}$ is totally ramified at the place above p. Let α be a uniformizer of M_r corresponding to the place above p. Since $M_r/\mathbb{Q}$ is a Galois extension, $M_r = \mathbb{Q}(\alpha) = \mathbb{Q}(\sigma(\alpha))$ for any embedding σ. We fix an embedding $\mathbb{Q}^{alg} \subset K^{alg}$.

Let $L_1 = K(p^{1/e}) = K(p^{1/s}, p^{1/p^r})$ and $L_2 = K(p^{1/s}, \alpha)$. Then L_1 and L_2 are totally ramified extensions of degree e over K. If there is an isomorphism $\sigma : L_2 \longrightarrow L_1$, L_1 contains both $\sigma(\alpha)$ and p^{1/p^r}. Since $\mathbb{Q}(\alpha) = \mathbb{Q}(\sigma(\alpha))$, $K(\sigma(\alpha)) = K(\alpha)$ is contained in L_1. We note that $[K(p^{1/p^r}, \alpha) : K(p^{1/p^r})]$ divides $[K(\alpha) : K] = p^r$ because $K(\alpha)/K$ is a Galois extension. Since

$$s = \left[L_1 : K\left(p^{1/p^r}\right)\right] = \left[L_1 : K\left(p^{1/p^r}, \alpha\right)\right]\left[K\left(p^{1/p^r}, \alpha\right) : K(p^{1/p^r})\right],$$

$[K(p^{1/p^r}, \alpha) : K(p^{1/p^r})]$ divides s. Hence we obtain $[K(p^{1/p^r}, \alpha) : K(p^{1/p^r})] = \gcd(s, p^r) = 1$. This shows $K(p^{1/p^r}) = K(\alpha)$ because $[K(p^{1/p^r}) : K] = [K(\alpha) : K]$. This is a contradiction, and hence, L_1 and L_2 are not isomorphic. □

Proposition 4.10. *Let p be a prime number and let e be a positive integer divisible by p. Then the lifting number for $\mathcal{C}_{p,e}$ is bigger than e.*

Proof. By Lemma 4.9, there are two totally ramified extensions L_1 and L_2 of degree e over $\mathbb{Q}_p$ such that there is no isomorphism over $\mathbb{Q}_p$ between L_1 and L_2. If $\mathcal{C}_{p,e}$ is e-liftable, L_1 and L_2 are isomorphic over $\mathbb{Q}_p$ by Proposition 4.5 and it is a contradiction. Hence, the lifting number for $\mathcal{C}_{p,e}$ is bigger than e. □

Corollary 4.11. *The lifting number for $\mathcal{C}_{p,e}$ is bigger than e whenever $e > 1$.*

Although we have the lower bound $e + 1$ and the upper bound $e + e\nu(e) + 1$ of the lifting number for $\mathcal{C}_{p,e}$, we have no clue to calculate the lifting number explicitly for the wild case.

Question 4.12. What is the lifting number for the wild case?

5. Ax-Kochen-Ershov principle for finitely ramified valued fields

Our main goal in this section is to strengthen Basarab's result on relative completeness for finitely ramified henselian valued fields of mixed characteristic with perfect residue fields. **In this section, we drop the restriction that a valuation group is $\mathbb{Z}$ so that a valuation group can be an arbitrary ordered abelian group.** Recall that for a valued field (K, R, ν, Γ), $e_\nu(x)$ is the number of the positive elements of Γ less than or equal to $\nu(x)$ for $x \in R$.

Remark 5.1. Let (K_1, ν_1) and (K_2, ν_2) be finitely ramified valued fields of mixed characteristic $(0, p)$. Suppose $R_{1,n} \equiv R_{2,n}$ for some $n > \min\{e_{\nu_1}(p), e_{\nu_2}(p)\}$, where $R_{1,(n)}$ and $R_{2,(n)}$ are the n-th residue rings of K_1 and K_2 respectively. Then, $e_{\nu_1}(p) = e_{\nu_2}(p)$.

Proof. Without loss of generality, we may assume that $e_1 := e_{\nu_1}(p) \leq e_2 := e_{\nu_2}(p)$. By the Keisler-Shelah isomorphism theorem, we may assume that $R_{1,(n)}$ and $R_{2,(n)}$ are isomorphic. Since $n > e_1$, we have that $pR_{1,(n)} = \bar{\mathfrak{m}}_1^{e_1} \neq 0$ where $\bar{\mathfrak{m}}_1$ is the maximal ideals of $R_{1,(n)}$. Since $R_{1,(n)}$ and $R_{2,(n)}$ are isomorphic, $0 \neq pR_{2,(n)} = \bar{\mathfrak{m}}_2^{e_2}$, where $\bar{\mathfrak{m}}_2$ is the maximal ideal of $R_{2,(n)}$, and $e_1 = e_2$. □

Theorem 5.2. *Let (K_1, ν_1, Γ_1) and (K_2, ν_2, Γ_2) be finitely ramified henselian valued fields of mixed characteristic $(0, p)$ with perfect residue fields. Let $n_0 > e_{\nu_2}(p)(1 + e_{\nu_1}(e_{\nu_1}(p)))$. Then, the following are equivalent:*

(1) $K_1 \equiv K_2$;
(2) $\Gamma_1 \equiv \Gamma_2$ and $R_{1,(n)} \equiv R_{2,(n)}$ for each $n \geq 1$; and
(3) $\Gamma_1 \equiv \Gamma_2$ and $R_{1,(n_0)} \equiv R_{2,(n_0)}$.

Proof. It is easy to check (1) ⇒ (2) ⇒ (3). We show (3) ⇒ (1). Suppose $R_{1,(n_0)} \equiv R_{2,(n_0)}$ and $\Gamma_1 \equiv \Gamma_2$. By Remark 2.15, we may assume that $R_{1,(n_0)} \cong R_{2,(n_0)}$ and $\Gamma_1 \cong \Gamma_2$, and that (K_1, ν_1, Γ_1) and (K_2, ν_2, Γ_2) are $\aleph_1$-saturated. Consider the coarse valuations $\dot{\nu}_1$ and $\dot{\nu}_2$ of ν_1 and ν_2 respectively and the valued fields $(K_1, \dot{\nu}_1, \Gamma_1/\Gamma_1^\circ)$ and $(K_2, \dot{\nu}_2, \Gamma_2/\Gamma_2^\circ)$, where Γ_i° is the convex subgroup of Γ_i generated by the minimum positive element in Γ_i for $i = 1, 2$. Since (K_1, ν_1) and (K_2, ν_2) are $\aleph_1$-saturated, by Remark 2.17.(4), the core fields (K_1°, ν_1°) and (K_2°, ν_2°) are complete discrete valued fields, where ν_1° and ν_2° are the valuations induced from ν_1 and ν_2 respectively. Since the n_0-th residue rings of (K_1, ν_1) and (K_2, ν_2) are isomorphic, by Remark 2.17.(2), the n_0-th residue rings of (K_1°, ν_1°) and (K_2°, ν_2°) are isomorphic.

By Theorem 3.7, K_1° and K_2° are isomorphic. Since $\Gamma_1 \cong \Gamma_2$, $\Gamma_1/\Gamma_1^\circ \cong \Gamma_2/\Gamma_2^\circ$. Furthermore, $(K_1, \dot{\nu}_1, (K_1^\circ, \nu_1^\circ)) \equiv (K_2, \dot{\nu}_2, (K_2^\circ, \nu_2^\circ))$ because Fact 1.1 holds after adding structure on residue fields. To get that $(K_1, \nu_1) \equiv (K_2, \nu_2)$, it is enough to show that $(K_1, R_{\nu_1}) \equiv (K_2, R_{\nu_2})$ in the ring language with a unary predicate. By Remark 2.17.(1), the valuation rings R_{ν_1} and R_{ν_2} are definable by the same formula in $(K_1, \dot{\nu}_1, (K_1^\circ, \nu_1^\circ))$ and $(K_2, \dot{\nu}_2, (K_2^\circ, \nu_2^\circ))$ so that $(K_1, R_{\nu_1}) \equiv (K_2, R_{\nu_2})$. □

We give several corollaries of Theorem 5.2. First, we improve the result in [3] on a decidability of finitely ramified henselian valued fields in the case of perfect residue field.

Corollary 5.3. *Let (K, ν, Γ) be a finitely ramified henselian valued field of mixed characteristic with a perfect residue field. Let $n_0 > e_\nu(p)(1 + e_\nu(e_\nu(p)))$. Let $\mathrm{Th}(K, \nu)$ be the theory of (K, ν), $\mathrm{Th}(\Gamma)$ the theory of Γ, and $\mathrm{Th}(R_{(n)})$ the theory of $R_{(n)}$. The following are equivalent:*

(1) $\mathrm{Th}(K, \nu)$ is decidable.
(2) $\mathrm{Th}(\Gamma)$ is decidable, and $\mathrm{Th}(R_{(n)})$ is decidable for each $n \geq 1$.
(3) $\mathrm{Th}(\Gamma)$ is decidable, and $\mathrm{Th}(R_{(n_0)})$ is decidable.

Note that the lower bound of n_0 depends only on e and p.

Proof. (1) $\Leftrightarrow$ (2) This was already given by Basarab in [3].

(1) $\Leftrightarrow$ (3) Let $e(:= e_\nu(p))$ be the ramification index of (K, ν). Consider the following theory $T_{p,e}$ consisting of the following statements, which can be expressed by the first order logic;

- (K, ν) is a henselian valued field of characteristic zero;
- Γ is an abelian ordered group having the minimum positive element;
- k is a perfect field of characteristic $p > 0$;
- (K, ν) has the ramification index e.

By Theorem 5.2, the theory $T_{p,e} \cup \mathrm{Th}(\Gamma) \cup \mathrm{Th}(R_{(n_0)})$ is complete. Thus $\mathrm{Th}(K, \nu)$ is decidable if and only if $\mathrm{Th}(\Gamma)$ and $\mathrm{Th}(R_{(n_0)})$ are decidable. $\square$

Next we recall the following definition introduced in [4]:

Definition 5.4. [4] Let T be the theory of a finitely ramified henselian valued field (K, ν, Γ) of mixed characteristic. Let $\lambda(T) \in \mathbb{N} \cup \{\infty\}$ be defined as the smallest positive integer n (if such a number exists) such that for every finitely ramified henselian valued field (K', ν', Γ') of mixed characteristic having the same ramification index of (K, ν, Γ), the following are equivalent:

(1) $(K', \nu', \Gamma') \models T$.
(2) $\Gamma \equiv \Gamma'$ and the n-th residue rings of (K, ν) and (K', ν') are elementarily equivalent.

Otherwise, $\lambda(T) = \infty$.

Basarab in [4] showed that $\lambda(T)$ is finite if T is the theory of a local field of mixed characteristic. In general, for the perfect residue field case, we prove that Basarab's invariant $\lambda(T)$ is always finite and smaller than or equal to the lifting number.

Corollary 5.5. *Let (K, ν) be a finitely ramified henselian valued field of mixed characteristic $(0, p)$ having finite ramification index $e = e_\nu(p)$ with a perfect residue field. Let T be the theory of (K, ν). Then*

(1) $\lambda(T)$ is smaller than or equal to the lifting number for $\mathcal{C}_{p,e}$.
(2) $\lambda(T) \leq e_\nu(p)(1 + e_\nu(e_\nu(p)) + 1$.

Next, we compute explicitly $\lambda(T)$ for the theories T of some tamely ramified valued fields. We say that an abelian group G is e-divisible when the multiplication by e map, $e : G \longrightarrow G$ is surjective. We denote the unit group of a ring R by $R^\times$.

Lemma 5.6. *Let $(K, W(k), \mathfrak{m}, k)$ be an unramified complete discrete valued field of mixed characteristic $(0, p)$ with a perfect residue field. Suppose that $k^\times$ is e-divisible for a positive integer e prime to p.*

(1) If ζ_e is contained in $W(k)$, then there exists a unique totally ramified extension L of degree e over K.
(2) If ζ_e is not contained in $W(k)$, then there exists a unique totally ramified extension L of degree e over K up to K-isomorphism.

Proof. Let S be the set of Teichmüller representatives of $W(k)$. By Hensel's lemma, $1 + \mathfrak{m}$ is e-divisible, and so is $W(k)^\times = S \setminus \{0\} \times (1 + \mathfrak{m})$ because $k^\times \cong S \setminus \{0\}$ is e-divisible.

For a totally tamely ramified extension L of degree e over K, there is u in $W(k)^\times$ such that $L = K(\sqrt[e]{pu})$ by the theory of tamely ramified extensions (cf. [12, Chapter 2]). Since $W(k)^\times$ is e-divisible, there is v in $W(k)^\times$ such that $v^e = u$. Hence, $\sqrt[e]{pu} = \sqrt[e]{p}v\zeta_e^i$ for some i. This shows that $L = K(\sqrt[e]{pu}) = K(\sqrt[e]{p}\zeta_e^i)$ is isomorphic to $K(\sqrt[e]{p})$ over K because the irreducible polynomial of $\sqrt[e]{p}$ over K is $x^e - p$. Furthermore, $L = K(\sqrt[e]{p})$ when ζ_e is contained in $W(k)$. $\square$

Proposition 5.7. *Let (K, ν, Γ, k) be a finitely tamely ramified henselian valued field of mixed characteristic $(0, p)$ with a perfect residue field. Let $e \geq 2$ be the ramification index of (K, ν). Let T be the theory of (K, ν).*

(1) If $k^\times$ is e-divisible, then $\lambda(T) = 1$.
(2) If there is a prime divisor l of e such that $\zeta_{l^n} \in k^\times$ and $\zeta_{l^{n+1}} \notin k^\times$ for some n, then $\lambda(T) = e + 1$.

Proof. (1) Suppose $k^\times$ is e-divisible. Let (K', ν', Γ', k') be a henselian valued field of mixed characteristic having ramification index e. Suppose $k \equiv k'$ and $\Gamma \equiv \Gamma'$. By Remark 2.15, we may assume that $k \cong k'$, $\Gamma \cong \Gamma'$, and both K and K' are $\aleph_1$-saturated. Consider the core fields $(K^\circ, \nu^\circ, k^\circ)$ and $((K')^\circ, (\nu')^\circ, (k')^\circ)$ of (K, ν) and (K', ν') respectively. Since $k^\times$ is e-divisible, so is $(k^\circ)^\times$. Then by Lemma 5.6, $(K^\circ, \nu^\circ) \cong ((K')^\circ, (\nu')^\circ)$. By the proof of Theorem 5.2, we have $(K, \nu) \equiv (K', \nu')$. Thus $\lambda(T) = 1$.

(2) Suppose there is a prime divisor l of e and a natural number n such that $\zeta_{l^n} \in k^\times$ and $\zeta_{l^{n+1}} \notin k^\times$. Let $T_{p,e}$ be the theory introduced in the proof of Corollary 5.3. Set $T_0 = T_{p,e} \cup \mathrm{Th}(R_e)$. Consider the following theories:

- $T_1 = T_0 \cup \{\exists x(x^e - p = 0)\}$;
- $T_2 = T_0 \cup \big\{\exists xy\big((x^e - py = 0) \wedge \Phi_{l^n}(y) = 0\big)\big\}$,

where $\Phi_{l^n}(X) \in \mathbb{Z}[X]$ is the l^n-th cyclotomic polynomial. By the proof of Lemma 4.6, we have

- $T_1 \cup T_2$ is inconsistent;
- T_1 and T_2 are consistent.

So, there are at least two different complete theories containing T_0, and we have $\lambda(T) \geq e + 1$. By Corollary 5.5, we conclude that $\lambda(T) = e + 1$. $\square$

For some wild cases, we have a lower bound for $\lambda(T)$.

Proposition 5.8. *Let p be a prime number and let e be a positive integer divisible by p. Let (K, ν, Γ, k) be a finitely ramified henselian valued field of mixed characteristic $(0, p)$ with a perfect residue field having the ramification index $e \geq 2$. Then $\lambda(T) \geq e + 1$ for the theory T of (K, ν).*

Proof. The proof is similar to the proof of Proposition 5.7. Let $T_{p,e}$ and T_0 be the theory introduced in the proof of Proposition 5.7. We write $e = sp^r$ for positive integers s and r where s is prime to p. Let $\alpha \in \mathbb{Q}^{alg}$ be as in the proof of Lemma 4.9. In particular, α is a uniformizer of M_r corresponding to the place above p where $M_r = \mathbb{Q}(\alpha)$ is the r-th subfield of the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty$ of degree p^r over $\mathbb{Q}$. Let $f(X)$ be the minimal polynomial of α over $\mathbb{Q}$. Consider the following theories:

- $T_1 = T_0 \cup \{\exists x(x^e - p = 0)\}$;
- $T_2 = T_0 \cup \{\exists x(x^s - p = 0),\ \exists x(f(x) = 0)\}$.

By the proof of Lemma 4.9, we have

- $T_1 \cup T_2$ is not consistent;
- T_1 and T_2 are consistent.

So, there are at least two different complete theories containing T_0, we have $\lambda(T) \geq e+1$. $\square$

We list some special cases of Proposition 5.7 and Proposition 5.8 (see Corollary 5.10). For a positive integer s, we say that s^∞ divides $[k : \mathbb{F}_p]$ if there is a subfield k_n of k such that $[k_n : \mathbb{F}_p]$ is finite and s^n divides $[k_n : \mathbb{F}_p]$ for each $n \geq 1$. For $m \geq 1$, let μ_m be the group generated by ζ_m and let $\mu_{m^\infty} = \bigcup_{n\geq 1} \mu_{m^n}$.

Remark 5.9. Let k be an algebraic extension of $\mathbb{F}_p$. Let $e > 1$ be coprime to p, and let s be the order of the group $\mu_e \cap k^\times$. Suppose s^∞ divides $[k : \mathbb{F}_p]$. Then, $k^\times$ is e-divisible.

Proof. Note that $(k^{alg})^\times \cong \oplus \mu_{q^\infty}$ where q runs through all primes not equal to p. To show that $k^\times$ is e-divisible, it is enough to show that $k^\times$ is r-divisible for each prime factor r of e.

Case $r \nmid s$. $k^\times$ is contained in $\oplus_{q\neq p,r} \mu_{q^\infty}$. Since μ_{q^n} is r-divisible for each $q \neq r$, $k^\times$ is r-divisible.

Case $r \mid s$. Note that r^∞ divides $[k : \mathbb{F}_p]$ because s^∞ divides $[k : \mathbb{F}_p]$. It is enough to show that $\mu_{r^\infty} \subset k^\times$. Clearly, we have that $\zeta_r \in k$. By Kummer theory, for any positive integer n, we have $[\mathbb{F}_p(\zeta_{r^{n+1}}) : \mathbb{F}_p] = r^{d_n}[\mathbb{F}_p(\zeta_r) : \mathbb{F}_p]$ for some $d_n \leq n$. Since r^∞ divides $[k : \mathbb{F}_p]$, there is a subfield $k_{r,n}$ of k with $[k_{r,n} : \mathbb{F}_p] = r^n$ so that $[k_{r,n}(\zeta_r) : \mathbb{F}_p] = r^n[\mathbb{F}_p(\zeta_r) : \mathbb{F}_p]$. So, $\mathbb{F}_p(\zeta_{r^{n+1}}) \subset k_{r,n}(\zeta_r) \subset k$. Therefore, we conclude that $\mu_{r^\infty} \subset k$. $\square$

Corollary 5.10. *Let (K, ν, Γ, k) be a finitely ramified henselian valued field of mixed characteristic $(0, p)$ with a perfect residue field. Let e be the ramification index of K and let s be the order of the group $\mu_e \cap k^\times$ where μ_e is the group generated by ζ_e. For the theory T of (K, ν):*

Case $p \nmid e$.

- $\lambda(T) = 1$ *when* $k = k^{alg}$*;*
- $\lambda(T) = 1$ *when K is a subfield of $\mathbb{C}_p$ and s^∞ divides $[k : \mathbb{F}_p]$;*
- $\lambda(T) = e + 1$ *when K is a subfield of $\mathbb{C}_p$ and s^∞ does not divide $[k : \mathbb{F}_p]$.*

Case $p|e$.

- $\lambda(T) \geq e + 1$ *when K is a subfield of $\mathbb{C}_p$.*

Proposition 5.7.(1) shows that Basarab's invariant $\lambda(T)$ can be strictly smaller than the bound in Corollary 5.5 for the tame case. In the following example, the same thing can happen for the wild case.

Example 5.11. Let $(K, R, \nu) = (\mathbb{Q}_3(\sqrt[3]{3}), \mathbb{Z}_3[\sqrt[3]{3}], \nu)$, $f(x) = x^3 - 3$ and $\alpha_1 = \sqrt[3]{3}$, $\alpha_2 = \sqrt[3]{3}\zeta_3$, and $\alpha_3 = \sqrt[3]{3}\zeta_3^2$. Since $f(x) = (x-\sqrt[3]{3})(x-\sqrt[3]{3}\zeta_3)(x-\sqrt[3]{3}\zeta_3^2) = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$ and $[\mathbb{Q}_3(\sqrt[3]{3}, \zeta_3) : \mathbb{Q}_3(\sqrt[3]{3})] = 2$,

$$\frac{x^3 - 3}{x - \sqrt[3]{3}} = \left(x - \sqrt[3]{3}\zeta_3\right)\left(x - \sqrt[3]{3}\zeta_3^2\right) = (x - \alpha_2)(x - \alpha_3)$$

is irreducible over $\mathbb{Q}_3(\sqrt[3]{3})$, that is, α_2 and α_3 are conjugate each other over $\mathbb{Q}_3(\sqrt[3]{3})$. It follows that $\widetilde{\nu}(\alpha_1 - \alpha_2) = \widetilde{\nu}(\alpha_1 - \alpha_3)$. By Fact 2.14,

$$\widetilde{\nu}(f'(\alpha_1)) = \widetilde{\nu}((\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)) = 2\widetilde{\nu}(\alpha_1 - \alpha_2) \leq \frac{\nu(3) - 1 + \nu\big(\nu(3)\big)}{\nu(3)}.$$

Hence we have the following bound

$$\begin{aligned} M(R) &= \max\left\{\widetilde{\nu}\big(\alpha_1-\alpha_j\big) : j \neq 1\right\} \\ &= \widetilde{\nu}(\alpha_1-\alpha_2) \; = \; \widetilde{\nu}(\alpha_1-\alpha_3) \; = \; \frac{\widetilde{\nu}(f'(\alpha_1))}{2} \\ &\leq \frac{\nu(3)-1+\nu\big(\nu(3)\big)}{2\nu(3)} \; = \; \frac{3-1+\nu(3)}{6} \\ &= \frac{5}{6}. \end{aligned}$$

So we have

$$M(R)\nu(3)^2 \leq \frac{5}{6}3^2 = \frac{15}{2} \leq 8 < \nu(3)+\nu(3)\nu(\nu(3)) = 3+3\nu(3) = 12.$$

Thus, Theorem 3.7 shows that Basarab's invariant $\lambda(T)$ for K is smaller than or equal to 8, which is strictly smaller than $\nu(3)(1+\nu(\nu(3)))+1 = 12$.

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