THE DYNAMIC INSTABILITY OF ADIABATIC BLAST WAVES

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ABSTRACT

Adiabatic blast waves, which have a total energy injected from the center, $E \propto t^q$, and propagate through a preshock medium with a density $\rho_E \propto r^{-\omega}$, are described by a family of similarity solutions. Previous work has shown that adiabatic blast waves with increasing or constant postshock entropy behind the shock front are susceptible to an oscillatory instability, caused by the difference between the nature of the forces on the two sides of the dense shell behind the shock front. This instability sets in if the dense postshock layer is sufficiently thin. In this paper we consider the stability of adiabatic blast waves with a decreasing postshock entropy. Such blast waves, if they are decelerating, always have a region behind the shock front which is subject to convection. Some accelerating blast waves also have such a region, depending on the values of q , ω , and γ , where γ is the adiabatic index. However, since the shock interface stabilizes dynamically induced perturbations, blast waves become convectively unstable only if the convective zone is localized around the origin or a contact discontinuity far from the shock front. On the other hand, the contact discontinuity of accelerating blast waves is subject to a strong Rayleigh-Taylor instability. The frequency spectra of the nonradial, normal modes of adiabatic blast waves have been calculated. We discuss the implications of this work for two astrophysical interests, the instability of shocks propagating through supernova envelopes and the evolution of planetary nebulae.

Subject headings: hydrodynamics — instabilities — interstellar: matter — nebulae: planetary — shock waves — stars: supernovae

1. INTRODUCTION

An understanding of the propagation of shock waves in astrophysical systems can be essential for understanding the history and present state of such systems. A blast shock wave (or a blast wave for short) is generated by an object releasing energy, whether impulsively, as in a supernova explosion, or continuously, as in a stellar or galactic wind. The resulting blast wave can be described by a Sedov-Taylor type similarity solution, provided that the total injected energy E is a power law in time,

$$E = E_0 \left(\frac{t}{t_0} \right)^q, \quad (1.1)$$

and the density of the ambient medium is a power law in radius,

$$\rho_E = \rho_0 \left(\frac{r}{r_0} \right)^{-\omega} \quad (1.2)$$

(Sedov 1946, 1959; Taylor 1950; Ostriker and McKee 1988). The similarity formalism demands that the flow quantities depend only on a single dimensionless parameter involving length and time scales. Such a similarity formalism provides an excellent approximation to realistic systems, when it is applied in the region far from the spatial and temporal boundaries and when it contains an adequate description of physics.

The stability of blast waves has been a long-standing problem to physicists and astrophysicists alike. Erpenbeck (1962) studied the stability of a steady plane-parallel shock dividing space into two homogeneous regions and showed that it is not exponentially unstable against a rippling of the shock front. Subsequently, several workers, including Bertschinger (1986), demonstrated that the shock interface is stable against dynamically induced perturbations because of the stabilizing tangential velocity in the postshock flow, produced by the obliquity of the rippled shock. However, the contact discontinuity between the ejected material and the swept-up gas resembles the classical illustration of a Rayleigh-Taylor instability in accelerating blast waves. In fact, Bernstein and Book (1978) showed that this contact discontinuity in *isentropic* blast waves is indeed unstable.

Ryu and Vishniac (1987, 1988) analyzed the stability of adiabatic blast waves propagating through a uniform medium ($\omega = 0$) and derived the eigenvalues and eigenfunctions for the nonradial, normal modes. Such blast waves are susceptible to an oscillatory instability, owing to the difference between the nature of forces on the two sides of the dense shell behind the shock front. They become subject to growing oscillations if the shell is sufficiently thin. The physical mechanism for this instability was first suggested

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by Vishniac (1983) for an isothermal shell with infinitesimal thickness and later confirmed by Bertschinger (1986). In a subsequent paper, Vishniac and Ryu (1989) gave an analytical derivation of the eigenmodes of isothermal blast waves propagating through a uniform medium under the approximation that the evolutionary time scale of the perturbations is much shorter than the evolutionary time scale of the shock itself. From the above work, it can be seen that adiabatic blast waves with increasing or constant postshock entropy from the shock front, or with

$$\omega \leq \frac{6 - 2q}{2\gamma + (\gamma - 1)q}, \quad (1.3)$$

where γ is the adiabatic index (see § II), are overstable for γ sufficiently close to unity. For a realistic shock this condition is usually equivalent to requiring that the postshock gas cool efficiently. In the absence of radiative cooling, such shocks will not be subject to these growing oscillations.

However, as pointed out by Bandiera (1984), some adiabatic blast waves have a region behind the shock front which is subject to convection and may be convectively unstable. The stability of such blast waves was studied by Chevalier (1976) and Bandiera (1984). Chevalier used the *local* Rayleigh-Taylor criterion to determine the existence of an instability and the dispersion relation for an incompressible fluid under constant gravitational acceleration bounded by rigid boundaries to estimate the growth rate of the perturbations. On the other hand, Bandiera used the *local* Schwarzschild criterion and a dispersion relation based on a *local* stability analysis. Unfortunately, in this case, convection is driven dynamically, not thermally, and such a dynamically driven instability is generally a *global* instability, not a *local* instability. The *global* stability analysis should include the overall dynamics of the blast waves and the correct boundary conditions, in addition to the local structure of the blast waves. Therefore, the *local* satisfaction of the convection criterion does not necessarily guarantee the existence of a *global* convective instability, and estimates of the growth rate of perturbations based on a *local* stability analysis may be misleading. Here we call this instability a convective instability, and reserve the name Rayleigh-Taylor instability to indicate the instability in the contact discontinuity (see § II).

In this paper we consider the stability of adiabatic blast waves with decreasing entropy behind the shock front, or

$$\omega > \frac{6 - 2q}{2\gamma + (\gamma - 1)q}. \quad (1.4)$$

They are subject to convective or Rayleigh-Taylor instability depending on the values of q , ω , and γ . This problem is of more than purely formal interest. Realistic astrophysical shocks often occur in the presence of density gradients and may involve the injection of energy over a time long compared with the evolutionary time scale of the blast wave. Supernova explosions, where the shock propagates outward through the envelope of a massive star, constitute an example of the former. The ejection of planetary nebulae in the later stages of stellar evolution may be an example of both significant density gradients and continuous energy injection. In this case, earlier stellar winds may lead to a significant, but rapidly decreasing, density enhancement around the star. Moreover, the planetary nebula itself is probably driven by a continuous, high-intensity stellar wind. Under these circumstances, instabilities may result in global inhomogeneities in the blast waves at late times. An equally important effect is the tendency of these instabilities to drive mixing in otherwise chemically distinct layers in the postshock gas.

In this paper we have calculated the frequency spectra of the nonradial, normal modes of adiabatic blast waves using a method similar to that used in Ryu and Vishniac (1987, 1988). In § II of this paper we have reviewed the Sedov-Taylor type similarity solutions for adiabatic blast waves. In § III the perturbation equations are derived, and in § IV the boundary conditions at the shock front and at the contact discontinuity or the origin are discussed. In § V the numerical scheme used to solve the perturbation equations is described, and the results are discussed. In § VI the results are applied to the shock propagating through the supernovae envelopes. Finally, in § VII we discuss the application of our results to the evolution of planetary nebulae.

II. BLAST WAVES

We treat the postshock fluid as a one-component ideal fluid with an adiabatic index γ . The flow is governed by the usual conservation equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (2.1)$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p = 0, \quad (2.2)$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0. \quad (2.3)$$

The energy conservation equation can be replaced by the entropy conservation equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \ln \frac{p}{\rho^\gamma} = 0, \quad (2.4)$$

except at the shock discontinuity. The shock is assumed to be very strong, so that the fluid density, velocity, and pressure just behind

the shock front are expressed in terms of the preshock density ρ_E and the shock velocity V_s relative to the preshock medium:

$$\rho_s = \rho_E \frac{\gamma + 1}{\gamma - 1}, \quad (2.5)$$

$$v_s = \frac{2V_s}{\gamma + 1}, \quad (2.6)$$

$$p_s = \frac{2\rho_E V_s^2}{\gamma + 1}, \quad (2.7)$$

by the shock-jump conditions. The flow pattern is entirely determined by two quantities, ρ_E and E , and from them a unique dimensionless parameter of similarity flow can be formed:

$$\begin{aligned} \xi &= r \left(\frac{\rho_E}{Et^2} \right)^{1/5} \\ &= \left(\frac{\rho_0 r_0^{\omega} t_0^q}{E_0} \right)^{1/5} r^{(5-\omega)/5} t^{-(2+q)/5}. \end{aligned} \quad (2.8)$$

The radius and velocity of the shock front change as

$$r_s \propto t^{(2+q)/(5-\omega)}, \quad (2.9)$$

$$V_s \propto t^{-(3-\omega-q)/(5-\omega)}, \quad (2.10)$$

and the fluid density, velocity, and pressure just behind the shock front change as

$$\rho_s \propto t^{-\omega(2+q)/(5-\omega)}, \quad (2.11)$$

$$v_s \propto t^{-(3-\omega-q)/(5-\omega)}, \quad (2.12)$$

$$p_s \propto t^{-(\omega q - 2q + 6)/(5-\omega)}. \quad (2.13)$$

It is convenient to introduce the dimensionless fluid density, velocity, and pressure, and the dimensionless radial coordinate for the flow throughout the region behind the shock front, as defined by

$$\tilde{\rho} \equiv \frac{\rho}{\rho_s}, \quad (2.14)$$

$$\tilde{v} \equiv \frac{v}{v_s}, \quad (2.15)$$

$$\tilde{p} \equiv \frac{p}{p_s}, \quad (2.16)$$

$$\tilde{r} \equiv \frac{r}{r_s}. \quad (2.17)$$

Then the conservation equations (2.1), (2.2), and (2.4) become

$$\left(\frac{2}{\gamma + 1} \tilde{v} - \tilde{r} \right) \frac{1}{\tilde{\rho}} \frac{d\tilde{\rho}}{d\tilde{r}} + \frac{2}{\gamma + 1} \frac{d\tilde{v}}{d\tilde{r}} + \frac{4}{\gamma + 1} \frac{\tilde{v}}{\tilde{r}} - \omega = 0, \quad (2.18)$$

$$\left(\frac{2}{\gamma + 1} \tilde{v} - \tilde{r} \right) \frac{d\tilde{v}}{d\tilde{r}} + \frac{\gamma - 1}{\gamma + 1} \frac{1}{\tilde{\rho}} \frac{d\tilde{\rho}}{d\tilde{r}} - \frac{3 - \omega - q}{2 + q} \tilde{v} = 0, \quad (2.19)$$

$$-\gamma \left(\frac{2}{\gamma + 1} \tilde{v} - \tilde{r} \right) \frac{1}{\tilde{\rho}} \frac{d\tilde{\rho}}{d\tilde{r}} + \left(\frac{2}{\gamma + 1} \tilde{v} - \tilde{r} \right) \frac{1}{\tilde{p}} \frac{d\tilde{p}}{d\tilde{r}} - \left(\frac{6 - 2q + \omega q}{2 + q} - \gamma \omega \right) = 0. \quad (2.20)$$

Blast waves with $\omega < 5$ admit a similarity solution where the radius and velocity of the shock front remain finite, even though the total mass and energy contained in a blast wave are infinite if $\omega \geq 3$.

For $q = 0$ the above equations can be integrated analytically. The integration is given in Sedov (1959), and also in Bandiera (1984) in a compact form. Blast waves with $q = 0$ are *filled*, i.e., the postshock flow extends to the origin, provided that $\omega < (7 - \gamma)/(\gamma + 1)$. In this case, near the origin, the density becomes

$$\tilde{\rho} \propto \tilde{r}^{(3-\gamma\omega)/(\gamma-1)}, \quad (2.21)$$

and the pressure approaches a constant value with

$$\frac{d\tilde{p}}{d\tilde{r}} \propto \tilde{r}^{(2+\gamma-\gamma\omega)/(\gamma-1)}. \quad (2.22)$$

On the other hand, blast waves with $q = 0$ are *hollow*, i.e., the postshock flow ends at a contact discontinuity, provided that $\omega > (7 - \gamma)/(\gamma + 1)$. In this case, near the contact discontinuity, the density and pressure become

$$\tilde{\rho} \propto (\tilde{r} - \tilde{r}_c)^{[6 - (\gamma + 1)\omega]/(3\gamma - 6 + \omega)}, \quad (2.23)$$

$$\tilde{p} \propto (\tilde{r} - \tilde{r}_c)^{(3\gamma - \gamma\omega)/(3\gamma - 6 + \omega)}. \quad (2.24)$$

Here $\tilde{r}_c$ is the position of the contact discontinuity. The special case with $\omega = (7 - \gamma)/(\gamma + 1)$ is sometimes referred as the Primakoff blast wave, and the postshock fluid quantities are expressed by a simple power law in radius:

$$\tilde{\rho} \propto \tilde{r}, \quad (2.25)$$

$$\tilde{p} \propto \tilde{r}^3. \quad (2.26)$$

The blastwaves with $q = 0$ can be classified into seven characteristic types depending on the values of ω and γ , and the ranges of ω for each type are given in Bandiera (1984). The plots of the postshock flow for each characteristic type of the blast waves with $\gamma = 4/3$ are shown in Figure 2 of Chevalier (1976).

For $q > 0$ equations (2.18), (2.19), and (2.20) cannot be integrated analytically, but the behavior of the postshock flow can be induced from the equations. All the blast waves with $q > 0$ are *hollow*. Near the contact discontinuity,

$$\begin{aligned} \frac{\tilde{r} - \tilde{r}_c}{\tilde{\rho}} \frac{d\tilde{\rho}}{d\tilde{r}} &\propto \frac{6 - 2q + \omega q}{2 + q} - \gamma\omega & \text{for } \omega \leq 3, \\ \frac{\tilde{r} - \tilde{r}_c}{\tilde{\rho}} \frac{d\tilde{\rho}}{d\tilde{r}} &< 0 & \text{for } \omega > 3. \end{aligned} \quad (2.27)$$

The density converges to zero or a constant if

$$\omega \leq \frac{6 - 2q}{2\gamma + (\gamma - 1)q}, \quad (2.28)$$

and diverges otherwise. Also, near the contact discontinuity,

$$\frac{1}{\tilde{\rho}} \frac{d\tilde{p}}{d\tilde{r}} \propto 3 - \omega - q, \quad (2.29)$$

and the pressure approaches a constant value if $\omega < 3$ and diverges otherwise. On the other hand, near the shock front,

$$\frac{d\tilde{p}}{d\tilde{r}} \propto 6(3 - q) - 4 \frac{\gamma - 1}{\gamma + 1} (2 + q) - [2(2 + \gamma) + (\gamma - 1)q]\omega, \quad (2.30)$$

$$\frac{d\tilde{p}}{d\tilde{r}} \propto 2(2\gamma - 1)(3 - q) - 4\gamma \frac{\gamma - 1}{\gamma + 1} (2 + q) - [2\gamma - (\gamma - 1)q]\omega. \quad (2.31)$$

(The above relations also hold for the cases with $q = 0$.) The blast waves with $q > 0$ can be classified into five characteristic types depending on the values of q , ω , and γ , and the ranges of ω for each type are the following:

Figure 1a:

$$\omega < \frac{6 - 2q}{2\gamma + (\gamma - 1)q}, \quad (2.32)$$

Figure 1b:

$$\frac{6 - 2q}{2\gamma + (\gamma - 1)q} < \omega < \frac{6(3 - q)}{2(2 + \gamma) + (\gamma - 1)q} - \frac{\gamma - 1}{\gamma + 1} \frac{4(2 + q)}{2(2 + \gamma) + (\gamma - 1)q}, \quad (2.33)$$

Figure 1c:

$$\frac{6(3 - q)}{2(2 + \gamma) + (\gamma - 1)q} - \frac{\gamma - 1}{\gamma + 1} \frac{4(2 + q)}{2(2 + \gamma) + (\gamma - 1)q} < \omega < \frac{2(2\gamma - 1)(3 - q)}{2\gamma - (\gamma - 1)q} - \frac{\gamma - 1}{\gamma + 1} \frac{4\gamma(2 + q)}{2\gamma - (\gamma - 1)q}, \quad (2.34)$$

Figure 1d:

$$\frac{2(2\gamma - 1)(3 - q)}{2\gamma - (\gamma - 1)q} - \frac{\gamma - 1}{\gamma + 1} \frac{4\gamma(2 + q)}{2\gamma - (\gamma - 1)q} < \omega < 3 - q, \quad (2.35)$$

Figure 1e:

$$3 - q < \omega < 5. \quad (2.36)$$

Figure 1 shows the plots of the postshock flow for each characteristic type of blast wave with $\gamma = 5/3$ and $q = 1$. Each plot is meant to be illustrative of a range of values of ω which yields similar results.

FIG. 1a

FIG. 1b

FIG. 1c

FIG. 1d

FIG. 1e

FIG. 1.—Characteristic types of the postshock flow of blast waves with $\gamma = 5/3$ and $q = 1$. The ranges of ω for which each particular type applies are given in the text.

It is well known that a region is subject to a convective motion if the entropy increases in the direction of gravity (see, for example, Landau and Lifshitz 1959). Denoting by $\mathcal{S}$ the entropy per unit mass, which is defined as

$$\mathcal{S} \equiv \ln \left(\frac{\tilde{p}}{\tilde{\rho}^\gamma} \right), \quad (2.37)$$

it can be seen from equation (2.20) that the entropy increases or stays constant from the shock front toward the origin or the contact

discontinuity provided that

$$\omega \leq \frac{6 - 2q}{2\gamma + (\gamma - 1)q}, \quad (2.38)$$

and decreases otherwise. On the other hand, even though we are neglecting gravity, the presence of a pressure gradient implies that the flow is accelerated and gives rise to an effective gravity

$$g_{\text{eff}} \equiv -\frac{1}{\bar{\rho}} \frac{d\bar{p}}{d\bar{r}}. \quad (2.39)$$

The postshock region around the contact discontinuity is decelerated (or accelerated in the direction of increasing entropy) provided that

$$\omega < 3 - q \quad (2.40)$$

(see eq. [2.29]), and the postshock region behind the shock front is decelerated provided that

$$\omega < \frac{2(2\gamma - 1)(3 - q)}{2\gamma - (\gamma - 1)q} - \frac{\gamma - 1}{\gamma + 1} \frac{4\gamma(2 + q)}{2\gamma - (\gamma - 1)q} \quad (2.41)$$

(see eq. [2.31]). Hence, all or a portion of the postshock region is subject to a convective motion if

$$\frac{6 - 2q}{2\gamma + (\gamma - 1)q} < \omega < \max \left[(3 - q), \frac{2(2\gamma - 1)(3 - q)}{2\gamma - (\gamma - 1)q} - \frac{\gamma - 1}{\gamma + 1} \frac{4\gamma(2 + q)}{2\gamma - (\gamma - 1)q} \right]. \quad (2.42)$$

In such region, the *local* strength of the convective motion may be estimated by the *local* growth rate given by Bandiera (1984),

$$\sigma = \left(g_{\text{eff}} \frac{d\mathcal{S}}{d\bar{r}} \right)^{1/2}. \quad (2.43)$$

From equations (2.18), (2.19), and (2.20), it can be seen that σ^2 approaches a constant near the origin in filled blast waves, or if

$$q = 0 \quad \text{and} \quad \omega \leq \frac{7 - \gamma}{\gamma + 1} \quad (2.44)$$

and $\sigma^2 \propto 1/(r - r_c)$ near the contact discontinuity in the hollow blast waves, or if

$$q > 0 \quad \text{or} \quad q = 0 \quad \text{and} \quad \omega > \frac{7 - \gamma}{\gamma + 1}. \quad (2.45)$$

Hence, if convection occurs in the postshock flow, it is expected to be highly localized in hollow blast waves but not in filled blast waves. However, the question whether the blast waves *locally* subject to a convective motion becomes dynamically unstable is a different problem, as discussed in § I. Since the shock interface stabilizes dynamically induced perturbations, we expect that only those blast waves with a highly localized convective zone far from the shock interface, or with

$$\frac{7 - \gamma}{\gamma + 1} < \omega < 3 \quad \text{and} \quad q = 0 \quad (2.46)$$

or

$$\frac{6 - 2q}{2\gamma + (\gamma - 1)q} < \omega \leq q - 3 \quad \text{and} \quad q > 0,$$

are convectively unstable.

The accelerating contact discontinuity with

$$\omega > 3 - q \quad (2.47)$$

and the decelerating shock front with

$$\omega < \frac{2(2\gamma - 1)(3 - q)}{2\gamma - (\gamma - 1)q} - \frac{\gamma - 1}{\gamma + 1} \frac{4\gamma(2 + q)}{2\gamma - (\gamma - 1)q} \quad (2.48)$$

have configurations subject to the Rayleigh-Taylor instability. However, again, since the shock front tends to suppress the instability, we expect that only those blast waves with the accelerating contact discontinuity are subject to the Rayleigh-Taylor instability, i.e., only those blast-waves where the instability is localized near the contact discontinuity and far from the shock front.

In the following sections, we will provide a proof of these arguments and find the relevant frequency spectra by performing a linear stability analysis.

III. PERTURBATION EQUATIONS

In this paper we will follow the notation used in Ryu and Vishniac (1987, 1988), except that ω and q have been added to the list of parameters.

Let $\delta\tilde{\rho}$, $\delta\tilde{v}$, and $\delta\tilde{p}$ be the normalized, perturbed variables defined in the Eulerian coordinates as

$$\delta\tilde{\rho}(r, \theta, \phi, t) \equiv \frac{\rho(r, \theta, \phi, t) - \rho_0(r, t)}{\rho_s}, \quad (3.1)$$

$$\delta\tilde{v}(r, \theta, \phi, t) \equiv \frac{\mathbf{v}(r, \theta, \phi, t) - v_0(r, t)\hat{\mathbf{r}}}{v_s}, \quad (3.2)$$

$$\delta\tilde{p}(r, \theta, \phi, t) \equiv \frac{p(r, \theta, \phi, t) - p_0(r, t)}{p_s}, \quad (3.3)$$

where ρ_0 , v_0 , and p_0 are the unperturbed fluid quantities considered in the previous section. Then, by linearizing the hydrodynamic conservation equations (2.1), (2.2), and (2.3) about the perturbed variables, we obtain the perturbation equations

$$\frac{\gamma+1}{2} \frac{5-\omega}{2+q} t \frac{\partial \delta\tilde{\rho}}{\partial t} - \frac{\gamma+1}{2} \omega \delta\tilde{\rho} + \tilde{v} \frac{\partial \delta\tilde{\rho}}{\partial \tilde{r}} + \frac{\partial \tilde{v}}{\partial \tilde{r}} \delta\tilde{\rho} + 2 \frac{\tilde{v}}{\tilde{r}} \delta\tilde{\rho} + \tilde{\rho} \tilde{\nabla} \cdot \delta\tilde{\mathbf{v}} + \frac{\partial \tilde{\rho}}{\partial \tilde{r}} \delta\tilde{v}_r = 0, \quad (3.4)$$

$$\frac{\gamma+1}{2} \frac{5-\omega}{2+q} t \frac{\partial \delta\tilde{v}}{\partial t} - \frac{\gamma+1}{2} \frac{3-\omega-q}{2+q} \delta\tilde{v} + \tilde{v} \frac{\partial \delta\tilde{v}}{\partial \tilde{r}} + \frac{\partial \tilde{v}}{\partial \tilde{r}} \delta\tilde{v}_r + \frac{\tilde{v}}{\tilde{r}} \delta\tilde{v}_T + \frac{\gamma-1}{2} \frac{1}{\tilde{\rho}} \tilde{\nabla} \delta\tilde{p} - \frac{\gamma-1}{2} \frac{1}{\tilde{\rho}^2} \frac{\partial \tilde{\rho}}{\partial \tilde{r}} \delta\tilde{p} \tilde{\mathbf{r}} = 0, \quad (3.5)$$

$$\frac{\gamma+1}{2} \frac{5-\omega}{2+q} t \frac{\partial}{\partial t} \left(\frac{\delta\tilde{p}}{\tilde{\rho}} - \gamma \frac{\delta\tilde{\rho}}{\tilde{\rho}} \right) + \tilde{v} \frac{\partial}{\partial \tilde{r}} \left(\frac{\delta\tilde{p}}{\tilde{\rho}} - \gamma \frac{\delta\tilde{\rho}}{\tilde{\rho}} \right) + \left(\frac{1}{\tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial \tilde{r}} - \gamma \frac{1}{\tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial \tilde{r}} \right) \delta\tilde{v}_r = 0, \quad (3.6)$$

where $\tilde{\nabla}$ is the normalized spatial derivative and $\delta\tilde{v}_r$ and $\delta\tilde{v}_T$ are the radial and tangential components of the normalized perturbed velocity.

Since the unperturbed flow is self-similar, we have seen in the previous section that the normalized, unperturbed fluid quantities $\tilde{\rho}$, $\tilde{v}$, and $\tilde{p}$ depend only on a single, normalized, dimensionless radius $\tilde{r}$. Hence, we assume that the normalized, perturbed variables can be written in a self-similar form consisting of a term which depends only on the normalized radius, the usual expansion in spherical harmonics, and a power law in time:

$$\delta\tilde{\rho}(r, \theta, \phi, t) = \sum_{l,m} \delta\tilde{\rho}_{lm}(\tilde{r}) Y_{lm}(\theta, \phi) t^s, \quad (3.7)$$

$$\delta\tilde{v}(r, \theta, \phi, t) = \sum_{l,m} [\delta\tilde{v}_{rlm}(\tilde{r}) Y_{lm}(\theta, \phi) t^s \hat{\mathbf{r}} + \delta\tilde{v}_{Tlm}(\tilde{r}) \tilde{\nabla}_T Y_{lm}(\theta, \phi) t^s], \quad (3.8)$$

$$\delta\tilde{p}(r, \theta, \phi, t) = \sum_{l,m} \delta\tilde{p}_{lm}(\tilde{r}) Y_{lm}(\theta, \phi) t^s, \quad (3.9)$$

where the operator $\tilde{\nabla}_T$ is defined by

$$\tilde{\nabla}_T \equiv \frac{1}{\tilde{r}} \frac{\partial}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{\tilde{r} \sin \theta} \frac{\partial}{\partial \phi} \hat{\boldsymbol{\phi}}. \quad (3.10)$$

Here s is the dimensionless perturbation growth rate. It depends on l and may be either real or a complex number. The first-order perturbation equations, which are the partial differential equations, are reduced to a set of ordinary differential equations:

$$\left(\tilde{v} - \frac{\gamma+1}{2} \tilde{r} \right) \frac{d\delta\tilde{\rho}}{d\tilde{r}} + \tilde{\rho} \frac{d\delta\tilde{v}_r}{d\tilde{r}} + \left(\frac{d\tilde{v}}{d\tilde{r}} + 2 \frac{\tilde{v}}{\tilde{r}} - \frac{\gamma+1}{2} \omega + \frac{\gamma+1}{2} \frac{5-\omega}{2+q} s \right) \delta\tilde{\rho} + \left(\frac{d\tilde{\rho}}{d\tilde{r}} + 2 \frac{\tilde{\rho}}{\tilde{r}} \right) \delta\tilde{v}_r - l(l+1) \frac{\tilde{\rho}}{\tilde{r}} \delta\tilde{v}_T = 0, \quad (3.11)$$

$$\left(\tilde{v} - \frac{\gamma+1}{2} \tilde{r} \right) \tilde{\rho} \frac{d\delta\tilde{v}_r}{d\tilde{r}} + \frac{\gamma-1}{2} \frac{d\delta\tilde{p}}{d\tilde{r}} - \frac{\gamma-1}{2} \frac{1}{\tilde{\rho}} \frac{d\tilde{\rho}}{d\tilde{r}} \delta\tilde{p} + \left(\frac{d\tilde{v}}{d\tilde{r}} - \frac{\gamma+1}{2} \frac{3-\omega-q}{2+q} + \frac{\gamma+1}{2} \frac{5-\omega}{2+q} s \right) \tilde{\rho} \delta\tilde{v}_r = 0, \quad (3.12)$$

$$\left(\tilde{v} - \frac{\gamma+1}{2} \tilde{r} \right) \tilde{\rho} \frac{d\delta\tilde{v}_T}{d\tilde{r}} + \left(\frac{\tilde{v}}{\tilde{r}} - \frac{\gamma+1}{2} \frac{3-\omega-q}{2+q} + \frac{\gamma+1}{2} \frac{5-\omega}{2+q} s \right) \tilde{\rho} \delta\tilde{v}_T + \frac{\gamma-1}{2} \frac{1}{\tilde{r}} \delta\tilde{p} = 0, \quad (3.13)$$

$$\begin{aligned} & -\gamma \left(\tilde{v} - \frac{\gamma-1}{2} \tilde{r} \right) \frac{d\delta\tilde{\rho}}{d\tilde{r}} + \left(\tilde{v} - \frac{\gamma+1}{2} \tilde{r} \right) \frac{\tilde{\rho}}{\tilde{\rho}} \frac{d\delta\tilde{\rho}}{d\tilde{r}} + \left[-\gamma \frac{1}{\tilde{\rho}} \frac{d\tilde{\rho}}{d\tilde{r}} + \frac{1}{\tilde{\rho}} \frac{d\tilde{\rho}}{d\tilde{r}} \right] \tilde{\rho} \delta\tilde{v}_r \\ & + \gamma \left[\left(\tilde{v} - \frac{\gamma+1}{2} \tilde{r} \right) \frac{1}{\tilde{\rho}} \frac{d\tilde{\rho}}{d\tilde{r}} - \frac{\gamma+1}{2} \frac{5-\omega}{2+q} s \right] \delta\tilde{p} - \left[\left(\tilde{v} - \frac{\gamma+1}{2} \tilde{r} \right) \frac{1}{\tilde{\rho}} \frac{d\tilde{\rho}}{d\tilde{r}} - \frac{\gamma+1}{2} \frac{5-\omega}{2+q} s \right] \frac{\tilde{\rho}}{\tilde{\rho}} \delta\tilde{p} = 0. \end{aligned} \quad (3.14)$$

Here, we have dropped the subscript lm from $\delta\tilde{\rho}_{lm}$, $\delta\tilde{v}_{rlm}$, $\delta\tilde{v}_{Tlm}$, and $\delta\tilde{p}_{lm}$ for clarity. Although we do not distinguish notationally between, say, $\delta\tilde{\rho}(t, \theta, \phi, t)$ and $\delta\tilde{\rho}(\tilde{r})$, it is clear that the latter form is applied in the above equations. In deriving equations

(3.11)–(3.14), we have used the relations

$$\frac{\partial \delta \tilde{p}}{\partial t} = -\frac{2+q}{5-\omega} \frac{\tilde{r}}{t} \frac{\partial \delta \tilde{p}}{\partial \tilde{r}} + \frac{s}{t} \delta \tilde{p}, \quad (3.15)$$

$$\frac{\partial \delta \tilde{v}}{\partial t} = -\frac{2+q}{5-\omega} \frac{\tilde{r}}{t} \frac{\partial \delta \tilde{v}}{\partial \tilde{r}} + \frac{s}{t} \delta \tilde{v}, \quad (3.16)$$

$$\frac{\partial \delta \tilde{p}}{\partial t} = -\frac{2+q}{5-\omega} \frac{\tilde{r}}{t} \frac{\partial \delta \tilde{p}}{\partial \tilde{r}} + \frac{s}{t} \delta \tilde{p}. \quad (3.17)$$

IV. BOUNDARY CONDITIONS

a) Outer Boundary Conditions

The outer boundary conditions at the shock front for the perturbation equations (3.11)–(3.14) are derived from the requirement of the mass, momentum, and energy flux conservations across the shock front:

$$[\rho u_r] = 0, \quad (4.1)$$

$$[p + \rho u_r^2] = 0, \quad (4.2)$$

$$[u_T] = 0, \quad (4.3)$$

$$\left[\frac{1}{2} u^2 + \frac{\gamma}{\gamma-1} \frac{p}{\rho} \right] = 0. \quad (4.4)$$

Here u_r is the fluid velocity relative to and perpendicular to the shock front, and u_T is the fluid velocity parallel to the shock front. The square brackets denote the difference in the enclosed quantities across the shock front.

Let η_s be the perturbation in the position of the shock front, r_s , and let $\tilde{\eta}_s$ be the normalized perturbation defined as

$$\tilde{\eta}_s(\theta, \phi, t) \equiv \frac{\eta_s(\theta, \phi, t)}{r_s}. \quad (4.5)$$

Following equations (3.7)–(3.10), we assume that $\tilde{\eta}_s$ can be written as

$$\tilde{\eta}_s(\theta, \phi, t) = \sum_{l,m} \tilde{\eta}_{slm} Y_{lm}(\theta, \phi) t^s. \quad (4.6)$$

Then equations (4.1)–(4.4) give the normalized boundary conditions at the shock front for the perturbation equations (3.11)–(3.14):

$$\delta \tilde{p} = -\tilde{\eta}_s \frac{d\tilde{p}}{d\tilde{r}} - \omega \tilde{\eta}_s, \quad (4.7)$$

$$\delta \tilde{v}_r = -\tilde{\eta}_s \frac{d\tilde{v}}{d\tilde{r}} + \left(\frac{5-\omega}{2+q} s + 1 \right) \tilde{\eta}_s, \quad (4.8)$$

$$\delta \tilde{p} = -\tilde{\eta}_s \frac{d\tilde{p}}{d\tilde{r}} - \omega \tilde{\eta}_s + 2 \left(\frac{5-\omega}{2+q} s + 1 \right) \tilde{\eta}_s, \quad (4.9)$$

$$\delta \tilde{v}_T = -\tilde{\eta}_s, \quad (4.10)$$

where the subscript lm has been dropped for clarity. We have assumed that the ambient medium is unperturbed in deriving the above conditions. We will set $\tilde{\eta}_s = 1$ for convenience.

b) Inner Boundary Conditions

The derivation of the inner boundary conditions is subtler than that of the outer boundary conditions. The physical arguments for the inner boundary conditions are different for *filled* and *hollow* blast waves. The inner boundary conditions at the origin for filled blast waves, i.e., for

$$q = 0 \quad \text{and} \quad \omega \leq \frac{7-\gamma}{\gamma+1}, \quad (4.11)$$

were considered in Ryu and Vishniac (1987). From the requirement that the fluid should not undergo divergent perturbations in the Lagrangian sense at the origin, or the fractional density change of the displaced volume elements should go to zero at the origin, the boundary condition

$$\delta p = 0 \quad (4.12)$$

is obtained. The same boundary condition is also obtained from the requirement that the energy perturbation should vanish at the origin.

The inner boundary conditions at the contact discontinuity for *hollow* blast waves, i.e., for

$$q > 0 \quad \text{or} \quad q = 0 \quad \text{and} \quad \omega > \frac{7 - \gamma}{\gamma + 1}, \quad (4.13)$$

were considered in Ryu and Vishniac (1988). They are derived from the requirement that there are no mass, momentum, and energy fluxes across the contact discontinuity, or

$$u_r = 0 \quad (4.14)$$

at the contact discontinuity and

$$[P] = 0 \quad (4.15)$$

across the contact discontinuity. Here u_r is the fluid velocity relative to and perpendicular to the contact discontinuity. Let η_c be the perturbation in the position of the contact discontinuity, r_c , and $\tilde{\eta}_c$ be the normalized perturbation defined as

$$\tilde{\eta}_c(\theta, \phi, t) \equiv \frac{\eta_c(\theta, \phi, t)}{r_s}. \quad (4.16)$$

Then the above two equations give

$$\delta \tilde{v}_r + \tilde{\eta}_c \frac{d\tilde{v}}{d\tilde{r}} - \frac{\gamma + 1}{2} \left(\frac{5 - \omega}{2 + q} s + 1 \right) \tilde{\eta}_c = 0, \quad (4.17)$$

$$\delta \tilde{p} + \tilde{\eta}_c \frac{d\tilde{p}}{d\tilde{r}} = 0. \quad (4.18)$$

In deriving the above conditions, it has been assumed that there is no perturbation inside the contact discontinuity, where a vacuum forms if $q = 0$ and $\omega > (7 - \gamma)/(\gamma + 1)$ or the sound speed is so large as to smooth out immediately all the perturbations if $q > 0$. In the case that $d\tilde{p}/d\tilde{r} = 0$ near the contact discontinuity, or for

$$\begin{aligned} q = 0, \quad \omega &< \frac{6}{\gamma + 1} \\ q > 0, \quad \omega &< \frac{6 - 2q}{2\gamma + (\gamma - 1)q}, \end{aligned} \quad (4.19)$$

the second condition becomes

$$\delta \tilde{p} = 0, \quad (4.20)$$

which is completely sufficient as the inner boundary condition. In this case the other one may be used to compute $\tilde{\eta}_c$. However, in the case that $d\tilde{p}/d\tilde{r} \neq 0$ near the contact discontinuity, both conditions must be invoked to obtain a solution.

V. NUMERICAL SOLUTIONS

The first-order perturbation equations (3.11)–(3.14) with the outer boundary conditions (4.7)–(4.11) and the inner boundary conditions (4.12) or (4.17) and (4.18) constitute eigenvalue equations for the eigenvalue s , which we have solved using the following iteration method. First, the derivatives of the normalized, perturbed variables with respect to the dimensionless radius have been calculated from equations (3.11)–(3.14):

$$\frac{d\delta\tilde{\mathbf{Q}}}{d\tilde{r}} = \tilde{\mathcal{A}} \cdot \delta\tilde{\mathbf{Q}}, \quad (5.1)$$

where $\delta\tilde{\mathbf{Q}}$ is a vector with

$$\delta\tilde{\mathbf{Q}} \equiv \begin{pmatrix} \tilde{p} \\ \delta\tilde{v}_r \\ \delta\tilde{p} \\ \delta\tilde{v}_T \end{pmatrix} \quad (5.2)$$

and $\tilde{\mathcal{A}}$ is a 4×4 matrix whose elements are functions of $\tilde{p}$, $\tilde{v}$, $\tilde{p}$, $d\tilde{p}/d\tilde{r}$, $d\tilde{v}/d\tilde{r}$, $d\tilde{p}/d\tilde{r}$, and $\tilde{r}$. Second, for given values of q , ω , γ , and l , the eigenvalue s has been estimated. Using the outer boundary conditions at the shock front, the above equation has been integrated numerically up to the origin or to the contact discontinuity. The integration has been carried out using the Runge-Kutta-Verner fifth-order and sixth-order method. In order to improve the accuracy around the origin or the contact discontinuity, where some quantities diverge, (i) $\ln \tilde{p}$, $\ln \tilde{v}$, or $\ln \tilde{r}$ instead of $\tilde{r}$ has been used as the independent variable and (ii) the unperturbed fluid quantities near the origin or the contact discontinuity have been replaced by the leading power-law solutions discussed in § II when possible. Finally, the integrated perturbed quantities at the origin or at the contact discontinuity have been checked against the inner

boundary conditions. If they have not satisfied the inner boundary conditions, s has been modified to a better value $s + \Delta s$ by the formula

$$\Delta s = -\delta\tilde{p}\left(\frac{\partial\delta\tilde{p}}{\partial s}\right)^{-1} \quad (5.3)$$

or

$$\Delta s = -\left[\delta\tilde{v}_r - \frac{d\tilde{v}}{d\tilde{r}}\left(\frac{d\tilde{p}}{d\tilde{r}}\right)^{-1}\delta\tilde{p} + \frac{\gamma+1}{2}\left(\frac{5-\omega}{2+q}s+1\right)\left(\frac{d\tilde{p}}{d\tilde{r}}\right)^{-1}\delta\tilde{p}\right] \\ \times \left[\frac{\partial\delta\tilde{v}_r}{\partial s} - \frac{d\tilde{v}}{d\tilde{r}}\left(\frac{d\tilde{p}}{d\tilde{r}}\right)^{-1}\frac{\partial\delta\tilde{p}}{\partial s} + \frac{\gamma+1}{2}\left(\frac{5-\omega}{2+q}s+1\right)\left(\frac{d\tilde{p}}{d\tilde{r}}\right)^{-1}\frac{\partial\delta\tilde{p}}{\partial s} + \frac{\gamma+1}{2}\frac{5-\omega}{2+q}\left(\frac{d\tilde{p}}{d\tilde{r}}\right)^{-1}\delta\tilde{p}\right]^{-1}. \quad (5.4)$$

Here $\partial\delta\tilde{p}/\partial s$ and $\partial\delta\tilde{v}_r/\partial s$ have been calculated from

$$\frac{d}{d\tilde{r}}\frac{\partial\delta\tilde{Q}}{\partial s} = \frac{\partial\tilde{\mathcal{Q}}}{\partial s} \cdot \delta\tilde{Q} + \tilde{\mathcal{Q}} \cdot \frac{\partial\delta\tilde{Q}}{\partial s}. \quad (5.5)$$

The iteration has been continued until $|\Delta s| \leq 10^{-4}$, and the resulting s has been taken as the eigenvalue for given q , ω , γ , and l . Typically, s converges in 3 or 4 iterations.

In the stability analysis, the blast waves with $q = 0$ can be classified into four characteristic types, as discussed in § II (refer, for instance, to eqs. [2.38], [2.40], [2.44], and [2.45]). The ranges of ω for which each type applies are given below:

$$\text{Type } a: \quad \omega \leq \frac{3}{\gamma} \quad \text{and} \quad q = 0, \quad (5.6)$$

$$\text{Type } b: \quad \frac{3}{\gamma} < \omega \leq \frac{7-\gamma}{\gamma+1} \quad \text{and} \quad q = 0, \quad (5.7)$$

$$\text{Type } c: \quad \frac{7-\gamma}{\gamma+1} < \omega \leq 3 \quad \text{and} \quad q = 0, \quad (5.8)$$

$$\text{Type } d: \quad \omega > 3 \quad \text{and} \quad q = 0. \quad (5.9)$$

The plots of the resulting frequency spectra for each characteristic type of blast wave with $\gamma = 4/3$ are shown in Figure 2. Each plot is meant to be illustrative of a range in ω which yields similar results.

The blast waves with $q > 0$ can be classified into three characteristic types, and the ranges of ω for which each type applies are given below:

$$\text{Type } a: \quad \omega \leq \frac{6-2q}{2\gamma + (\gamma-1)q}, \quad (5.10)$$

$$\text{Type } b: \quad \frac{6-2q}{2\gamma + (\gamma-1)q} < \omega \leq q-3, \quad (5.11)$$

$$\text{Type } c: \quad \omega > q-3. \quad (5.12)$$

The plots of the resulting frequency spectra for each characteristic type of blast wave with $\gamma = 5/3$ and $q = 1$ are shown in Figure 3. Figure 3b actually corresponds to the shock wave structure given in Figure 1c, but the frequency spectra corresponding to perturbations of Figures 1b and 1d are only slightly different. Consequently, we have omitted the corresponding plots and allowed the remaining figure to stand for all three cases.

The frequency spectra of the filled decelerating blast waves in Figures 2a and 3a are the same frequency spectra as those in Ryu and Vishniac (1987, 1988) and have been reproduced here for completeness. Such blast waves are unstable to growing oscillatory modes only when the thickness of the shell behind the shock front is sufficiently thin, or for γ sufficiently close to unity.

Blast waves with ω in the range given in formulae (5.7) are apparently subject to a convective motion but have a convective region extending from the origin up to the shock front. The corresponding frequency spectra, such as Figure 2b, indicate that such blast waves are actually stable against convection because the shock interface stabilizes the dynamically induced perturbations. On the other hand, blast waves with ω in the ranges given in formulae (5.8) and (5.11) have a convective region highly localized around the contact discontinuity. The corresponding frequency spectra, such as Figures 2c and 3b, indicate that such blast waves are unstable against the convective instability. In the blast waves subject to convection, the sound wave modes at long wavelengths bifurcate progressively into growing modes and decaying modes. The first mode to split, which we call the *primary mode*, is also the mode which becomes unstable first. The modes which split at higher l , which we call the *higher harmonic modes*, become unstable at progressively shorter wavelengths. The primary mode is always the most unstable mode. The bifurcating wavelength, k_b , at which a given mode splits (s becomes real), and the critical wavenumber, k_c , at which a given mode becomes unstable (s becomes positive), depend on the pressure and density distributions in the postshock shell. The detailed dependence of k_b and k_c on the internal structure will not be given here but will be investigated in the subsequent paper (Ryu 1990). In order for the convective instability to

FIG. 2a

FIG. 2b

FIG. 2c

FIG. 2d

FIG. 2.—Characteristic types of the frequency spectra of blast waves with $\gamma = 4/3$ and $q = 0$. (a) Blast wave with $\omega = 0$ which is subject to an oscillatory instability but stable against the dynamically induced perturbations. (b) Blast wave with $\omega = 2.3$ which is subject to a convective motion but stable against the dynamically induced perturbations. (c) Blast wave with $\omega = 2.7$ which is subject to a convective motion and unstable against the dynamically induced perturbations. (d) Blast wave with $\omega = 3.2$ which is subject to a Rayleigh-Taylor instability and unstable against the dynamically induced perturbations. The solid lines show the real eigenvalues s , and the dashed lines show the real parts of the complex eigenvalues s . In (b) and (c), at the points where two solid lines meet, a dashed line starts and extends up to $l = 1$. Some dashed lines are omitted for clarity.

be effective, the pressure gradient should be large (see Fig. 3b). However, the region initially subject to convection may become less convective as the instability grows, since the effect of convection will be to reduce the gradient of the entropy. Clearly, the detailed nonlinear evolution of the convective instability in shocks should be investigated by high-resolution, three-dimensional numerical simulations.

The accelerating contact discontinuity in blast waves with $\omega > 3 - q$ is unstable against the Rayleigh-Taylor mode over *all* wavelengths. Figures 2d and 3c show the frequency spectra of a growing mode, which is the only nonoscillatory mode we can identify. Of course, it is expected that there are an infinite number of oscillatory modes corresponding to sound waves propagating throughout the postshock region. However, blast waves with $\omega > 3 - q$ are generally stable against such modes over all wavelengths. We have actually identified several such modes, but the real part of s of such modes is always negative. In the subsequent paper (Ryu 1990), we will show that, for perturbations with wavelengths smaller than the thickness of the shell, the growth rate of the Rayleigh-Taylor instability is given by

$$\Gamma \sim \sqrt{gk}, \quad (5.13)$$

where g is the effective gravity exerted on the shell.

Figure 4 shows the postshock flow of the Primakoff blast wave with $\gamma = 5/3$, $q = 0$, and $\omega = 2$ and its frequency spectra. This is particularly interesting case, since this corresponds to a supernova explosion in a preexisting supersonic stellar wind. Bernstein and Book (1980) considered the stability of the Primakoff blast wave and showed that it is stable against perturbations on all wavelengths. However, Gaffet (1984a, b) argued that Bernstein and Book had neglected the entropy perturbations and concluded

FIG. 3a

FIG. 3b

FIG. 3c

FIG. 3.—Characteristic types of the frequency spectra of blast waves with $\gamma = 5/3$ and $q = 1$. (a) Blast wave with $\omega = 0$ which is subject to an oscillatory instability but stable against the dynamically induced perturbations. (b) Blast wave with $\omega = 1.4$ which is subject to a convective motion and unstable against the dynamically induced perturbations. (c) Blast wave with $\omega = 2.5$ which is subject to a Rayleigh-Taylor instability and unstable against the dynamically induced perturbations. The solid lines show the real eigenvalues s , and the dashed lines show the real parts of the complex eigenvalues s .

that the Primakoff blast wave can be proved to be stable only against radial modes. Our analysis shows that the Primakoff blast wave is stable also against nonradial, normal modes, even though it is formally subject to a convective motion.

Figure 5 shows the distributions of the perturbed fluid quantities $\delta\bar{p}$, $\delta\bar{v}_r$, $\delta\bar{p}$, and $\delta\bar{v}_T$ of blast waves which are unstable against the convective instability or the Rayleigh-Taylor instability. The perturbation wavelength corresponds to

$$\frac{\lambda}{r_s} = \frac{2\pi}{l} = 0.628, \quad (5.14)$$

which is larger than the thickness of the shells. In both cases, the perturbations are highly localized around the contact discontinuity.

VI. MIXING IN SUPERNOVA ENVELOPES

The advent of SN 1987A in the Large Magellanic Cloud provided the first direct observational evidence that nonspherical hydrodynamic processes (hydrodynamics instabilities) may play an important role in the early stage of a supernova explosion. Large-scale mixing of the ejecta was suggested from the following observations: (1) Both X-rays and γ -rays were discovered earlier than expected (see, e.g., Sunyaev *et al.* 1987; Dotani *et al.* 1987; Matz *et al.* 1988). (2) After a rapid initial decline, the bolometric light curve increased very slowly and then reached a plateau-like maximum (see, e.g., Catchpole *et al.* 1987). (3) Broad lines with FWHM $v \simeq 2000\text{--}3000 \text{ km s}^{-1}$ of Fe II, Ni II, Ar II, and Co II were observed in late 1987 and early 1988 (see, e.g., Erickson *et al.* 1988; Rank *et al.* 1988; Witteborn *et al.* 1989). Several possible mechanisms for mixing were suggested (see Arnett, Fryxell, and Müller 1989 for more discussion and references), and among these some hydrodynamic instability has been considered the most probable solution.

Nagasawa, Nakamura, and Miyama (1988) performed numerical simulations of supernova explosion in $n = 3$ and $n = 1.5$ polytropic stars using a three-dimensional smoothed particle hydrodynamic (SPH) code and found the formation of clumpy structures produced by a compressible Rayleigh-Taylor instability. However, Müller *et al.* (1989) redid the multidimensional calculations of Nagasawa, Nakamura, and Miyama using Eulerian hydrodynamic codes and concluded that the initial perturbations in polytropic models do not grow significantly in the shock propagation time scale. In addition, in recent papers, Nagasawa

FIG. 4a

FIG. 4b

FIG. 4.—(a) Primakoff blast wave with $\gamma = 5/3$, $q = 0$, and $\omega = 2$ and (b) its frequency spectra. The Primakoff blast wave is subject to a convective motion but stable against the dynamically induced perturbations. The solid lines show the real eigenvalues s , and the dashed lines show the real parts of the complex eigenvalues s . In (b), at the points where two solid lines meet, a dashed line starts and extends up to $l = 1$. Some dashed lines are omitted for clarity.

(1990) and Yamada, Nakamura, and Oohara (1990) showed that the original calculations of Nagasawa, Nakamura, and Miyama suffered from numerical instabilities caused by using too few particles.

On the other, Ebisuzaki, Shigeyama, and Nomoto (1989) and Benz and Thielemann (1990) considered the stability of a blast wave propagating through a realistic progenitor star and applied a *local* stability analysis based on the Rayleigh-Taylor criterion of an *incompressible* fluid (Ebisuzaki, Shigeyama, and Nomoto) and the Schwarzschild criterion of a compressible fluid (Benz and Thielemann). Both groups concluded that the two composition interfaces between the hydrogen-rich and helium zones and the helium and metal zones are *extremely* unstable. However, as mentioned in § II, an analysis of the dynamic instability of blast waves should include the overall dynamics as well as the correct boundary conditions. Therefore, estimates of the growth rate of the perturbations based on local criteria may be misleading.

In more accurate numerical simulations using a hydrodynamic code based on the piecewise parabolic method (PPM), Arnett, Fryxell, and Müller (1989) considered a supernova explosion in a realistic progenitor star with $M = 15 M_{\odot}$ proposed by Arnett (1987). They claimed that the metal/He interface is strongly unstable against the Rayleigh-Taylor instability and heavier metals penetrate into the He zone, but the He/H interface stays relatively stable. However, in more recent and extensive calculations by Fryxell, Müller, and Arnett (1990) using the same progenitor star as before, they found instabilities in both interfaces. They explained that, owing to insufficient resolution, they could not clearly distinguish the instability in the He/H interfaces in the original calculations. Hachisu *et al.* (1990) performed similar calculations using a different hydrodynamic code and a different presupernova model and got similar results.

Even though a simple model based on power-law density distributions may not be directly applicable to the realistic model of

FIG. 5a

FIG. 5b

FIG. 5.—Perturbations of the postshock flow. (a) Blast wave with $\gamma = 4/3$, $q = 0$, and $\omega = 2.7$, which is convectively unstable, for $l = 10$ and $s = 0.239$. (b) Blast wave with $\gamma = 4/3$, $q = 0$, and $\omega = 3.2$, which is Rayleigh-Taylor unstable, for $l = 10$ and $s = 0.713$.

supernova envelopes, the results in § V may still be used to interpret the instability of blast waves propagating through the supernova envelopes. Here we wish to make two general points: First, a very steep gradient in density appears in both metal/He and He/H interfaces after the shock pass through them (see, e.g., Figs. 1 and 2 in Ebisuzaki, Shigeyama, and Nomoto 1989). Such steep density gradients may be fairly approximated as a contact discontinuity, even though the flow is still continuous, inasmuch as we are interested in the instability with modes whose wavelength is much longer than the width of the interface. As a result, the composite interfaces are expected to be subject to the Rayleigh-Taylor instability and strongly unstable over all wavelengths, if $dp/dr > 0$ or pressure decreases in the direction of increasing density around the interfaces. However, different models for the supernova progenitor may result in composition interfaces with $dp/dr < 0$, which are stable against the Rayleigh-Taylor instability. The stability of the composition interfaces may be tested by

$$\Sigma = \exp \left\{ \int_0^t \operatorname{Re} \left[\left(\frac{1}{\rho} \frac{dp}{dr} k \right)^{1/2} \right] dt' \right\}. \quad (6.1)$$

The growth time scale of the perturbations is comparable to the dynamical time scale of the contact discontinuity for perturbation wavelengths comparable to the thickness of the unstable zone (at least in the linear regime). Consequently, the long-wavelength perturbations will grow about as fast as the supernova explosion evolves. According to the linear stability analysis, perturbations with smaller wavelengths grow faster. However, perturbations with wavelengths smaller than the thickness of the unstable zone are expected to be strongly localized around the composition interfaces. This should prevent them from dominating the long-wavelength perturbations. We conclude that the modes that most effectively promote mixing will be the long-wavelength modes, i.e., those with wavelengths comparable to the local pressure scale height. Second, the regions between the two composition interfaces and the shock front are expected to be only *weakly*, or *not*, unstable against the convective motions. This is because the small pressure gradient in both regions makes the convective instability ineffective compared with the Rayleigh-Taylor instability of the composition interfaces, and also because convective motions in the region between the He/H interface and the shock front will become stabilized if they extend to the shock front.

VII. PLANETARY NEBULAE

In the case of a stellar wind driving a shock into its surroundings, it is necessary to take $q \geq 1$. For planetary nebulae the standard model is that the bright shell of ionized material represents a shock being driven into gas which originates from an earlier, and gentler, stellar wind emitted by the same star. When the shock emerges from this neutral material, the shock wave will expand rapidly into the low-density interstellar medium and the nebula will eventually become less conspicuous. At earlier times the shock will encounter a density gradient that depends on the history of mass loss from the central star. In general, we have $\omega > 2$, the value we would have for a steady mass loss at a constant velocity.

In an earlier paper Breitschwerdt and Kahn (1990) pointed out that under certain reasonable assumptions concerning the evolution of a planetary nebula and its central star, the nebula will be subject to the Rayleigh-Taylor instability. They pointed out that this may explain the small-scale features seen in most planetary nebulae. This result is consistent with the work we have presented here, but we wish to make a more general point as well. Taking $q \geq 1$ and $\omega > 2$, we see immediately from equation (2.48) that we are always in the regime where the contact discontinuity is accelerating and the dense shell is subject to Rayleigh-Taylor instabilities. In particular, we see that this result does not depend on the value of γ in the dense shell (or, equivalently, on the importance of radiative cooling). In other words, regardless of the exact model one uses for the evolution of planetary nebulae, the implication of this paper is that small-scale instabilities and mixing will necessarily be part of their evolution. Whether this results in large-scale features depends on whether or not there are other stabilizing influences at work. For example, a sufficiently intense ionizing flux may stabilize large-amplitude ripples in the inner boundary by evaporating lagging dense regions, an effect which is beyond the scope of this paper.

Note added in manuscript.—During the completion of this paper, we received a preprint from J. Goodman. He considered the stability of the blast waves with $q = 0$ and $(7 - \gamma)/(\gamma + 1) < \omega < 3$. We found that his results agree with ours for that particular case.

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