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Asymptotic analysis for singularly perturbed convection-diffusion equations with a turning point

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Turning points occur in many circumstances in fluid mechanics. When the viscosity is small, very complex phenomena can occur near turning points, which are not yet well understood. A model problem, corresponding to a linear convection-diffusion equation (e.g., suitable linearization of the Navier-Stokes or Bénard convection equations) is considered. Our analysis shows the diversity and complexity of behaviors and boundary or interior layers which already appear for our equations simpler than the Navier-Stokes or Bénard convection equations. Of course the diversity and complexity of these structures will have to be taken into consideration for the study of the nonlinear problems. In our case, at this stage, the full theoretical (asymptotic) analysis is provided. This study is totally new to the best of our knowledge. Numerical treatment and more complex problems will be considered elsewhere. © *2007 American Institute of Physics.* [DOI: 10.1063/1.2347899]

I. INTRODUCTION

Important work has been done in the area of singular perturbations, such as Eckhaus (1972); Lions (1973); O'Malley (1991); (1970); Vishik and Lyusternik (1957), to the point that one may have the impression that the subject has been exhausted. This idea is of course incorrect and many difficult problems still need to be addressed including parabolic boundary layers, corners, turning points, numerical approximation, not mentioning one of the most outstanding problems of fluid mechanics, namely the behavior of viscous fluids at small viscosity, in relation with turbulence. Recent works in these areas include the following: Shih and Kellogg (1987); Jung and Temam (2005a, b) for parabolic boundary layers and Han and Kellogg (1990) and Kellogg and Stynes (2005) for corners; Stynes (2005) is a reference and review article about the difficult problems in numerical approximations. Concerning the convergence of the solutions of the Navier-Stokes equations to those of the Euler equations, see some recent progress in the noncharacteristic case (permeable boundary) in Temam and Wang (2002) and Hamouda and Temam (2006). This is based on the remark implicitly made in Temam and Wang (2002) and explicitly in Temam and Wang (2000) and Hamouda and Temam (2006) that the Prandtl equation for such flows is simple (linear and time independent); see also Xin and Yanagisawa (1999); Grenier and Gues (1998); and Grenier (2004) for the linearized compressible Navier-Stokes equations.

The present article is devoted to turning points. Turning points are a difficult problem in singular perturbation theory for which relatively few results are available. Wasow's (1985) entire

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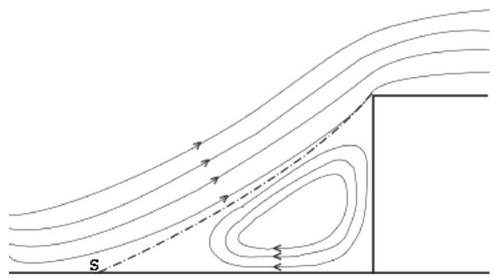

FIG. 1. Separation of a turbulent boundary layer; S: the point of separation [after Fig. 39 from Van Dyke (1998)].

book is devoted to this difficult problem for the linear case; see also Smith (1985) and Desanti (1987a, b) for nonlinear problems. Turning points are an essential feature of turbulent boundary layers and occur at the point(s) where the turbulent boundary layer separates since the tangential velocity vanishes and changes sign at such points [see Batchelor (1988); Lamb (1932) and Fig. 1; see also recent results of topological nature in Ma and Wang (2005)]. Hence among the numerous connections of this article to fluid mechanics one can mention that Eq. (1.1) that we consider can be seen as a suitable linearization of the two-dimensional stationary Navier-Stokes equations for either component of the velocity for a flow as described in Fig. 1 or the linearization of the heat equation for a Bénard convection problem. The linearization procedure is classically used in the study of stability of laminar fluid flows where it leads for instance to the celebrated Orr-Sommerfeld equation (see, e.g., Langer (1957); Reid (1974a, b); Drazin (2002); and Drazin and Reid (2004).

The Orr-Sommerfeld equation appears in the study of the stability of a stationary solution for a flow with velocity U in the direction Ox, and U can be at most a quadratic function of the variable z [see Drazin (2002), p. 156, Sec. 8.5]. To study the stability of a flow of the type depicted in Fig. 1 (which we recall is a common situation for turbulent flows), we would start from a background flow more complex than for the Orr-Sommerfeld equation; either a more complex stationary solution, or a time-dependent one. Hence, from the point of view of fluid mechanics, this article is a very small step in the study of flows more complex than the plane parallel flows considered in the context of the Orr-Sommerfeld equations. And beside its theoretical component, this article gives some qualitative (analytic) indications on the structure of such flows.

Other motivations for studying turning points can be found in the books of Smith (1985) and Wasow (1985), in particular the study of the propagation of light in a nonhomogeneous medium as an application of Maxwell's equations, and some nonlinear differential equations corresponding to simplified models of turbulent boundary layers; see also the double-gyre problem in geophysical fluid mechanics in, e.g., Simonnet *et al.* (2003).

In this article we consider a singularly perturbed problem which has a single turning point, that is

$$L_\epsilon u^\epsilon := -\epsilon^2 u^\epsilon_{xx} - b u^\epsilon_x = f \text{ in } \Omega = (-1,1), \tag{1.1a}$$

$$u^\epsilon(-1) = \alpha, \quad u^\epsilon(1) = \beta, \tag{1.1b}$$

where $0 < \epsilon \ll 1$, $b = b(x)$, $f = f(x)$ are smooth on $[-1,1]$, α, β are constants, and

$$b < 0 \quad \text{for } x < 0, \quad b = 0 \quad \text{for } x = 0, \quad b > 0 \text{ for } x > 0, \tag{1.2a}$$

$$b_x(x) \geq \delta > 0, \quad \delta \text{ constant}, \quad \forall\, x \in [-1,1]. \tag{1.2b}$$

Without loss of generality, we may set $\delta = 1$, and we also note from (1.2) that b has a simple zero at $x = 0$.

We shall consider the Sobolev spaces $H^m(\Omega)$, m integer, equipped with the semi-norm, $|u|_{H^m}=(\Sigma_{|\alpha|=m}\int_\Omega|D^\alpha u|^2 \mathrm{d}x\,\mathrm{d}y)^{1/2}$, and the norm, $\|u\|_{H^m}=(\Sigma_{j=0}^m|u|_{H^j}^2)^{1/2}$. We define the corresponding inner product in the space $H^m(\Omega)$: $((u,v))_{H^m(\Omega)}=\Sigma_{|\alpha|\leq m}(D^\alpha u,D^\alpha v)$, where $(u,v)=\int_\Omega uv\,\mathrm{d}x\,\mathrm{d}y$. In particular we use the Sobolev space $H_0^1(\Omega)$, which is the closure in the space $H^1(\Omega)$ of C^∞ functions compactly supported in Ω.

In the text κ (κ_j, κ_{jm} depending on j or m, etc.) and c denote generic constants which are independent of ϵ and that may be different at different occurrences.

We then consider the weak formulation of (1.1) as follows: *To find* $u=v+\alpha(1-x)/2+\beta(1+x)/2$ *with* $v\in H_0^1(\Omega)$ *such that*

$$a_\epsilon(v,w)=F(w),\qquad \forall\, w\in H_0^1(\Omega), \tag{1.3a}$$

where

$$a_\epsilon(v,w)=\epsilon^2((v,w))-(bv_x,w), \tag{1.3b}$$

$$F(w)=(\tilde{f},w),\quad \tilde{f}=f+\frac{\beta-\alpha}{2}b; \tag{1.3c}$$

thanks to the Poincaré inequality, the space $H_0^1(\Omega)$ is equipped with the inner product $((\cdot,\cdot))$, and the norm $\|\cdot\|$:

$$((v,w))=\int_{-1}^1 v_x w_x \mathrm{d}x,\quad \|v\|=|v|_{H^1}=((v,v))^{1/2}. \tag{1.4}$$

It is then easy to verify the coercivity of a_ϵ, because for all $v\in H_0^1(\Omega)$,

$$a_\epsilon(v,v)=\epsilon^2\|v\|^2+\int_{-1}^1\left(\frac{b_x}{2}\right)v^2\mathrm{d}x\geq\epsilon^2\|v\|^2. \tag{1.5}$$

We also easily verify the continuity of the bilinear form a_ϵ on $H_0^1(\Omega)\times H_0^1(\Omega)$ and the continuity of the linear form F on $H_0^1(\Omega)$ and thus, by the Lax-Milgram theorem, there exists a unique function $v\in H_0^1(\Omega)$ satisfying Eq. (1.3), hence $u=u^\epsilon$ satisfying (1.1).

Lemma 1.1: The following regularity results and a priori estimates of the solutions $u=u^\epsilon$ *of Eq. (1.1) with* $\alpha=\beta=0$ *hold: if* $f\in H^{m-2}(\Omega)$, $m\geq 2$, *then* $u\in H^m(\Omega)$ *and*

$$|u|_{L^2(\Omega)}\leq\kappa|f|_{L^2(\Omega)},\quad \|u\|_{H^1(\Omega)}\leq\kappa\epsilon^{-1}|f|_{L^2(\Omega)}, \tag{1.6a}$$

$$\|u\|_{H^m(\Omega)}\leq\kappa_m\epsilon^{-2m+1}|f|_{L^2(\Omega)}+\kappa_m\sum_{l=2}^{m-1}\epsilon^{-2(l-1)}|f|_{H^{m-l}(\Omega)}. \tag{1.6b}$$

Proof: Multiplying Eq. (1.1) by u and then integrating over $(-1,1)$ we find that

$$\epsilon^2|u|_{H^1}^2+\tfrac{1}{2}|u|_{L^2}^2\leq(\text{by }(1.5))\leq a_\epsilon(u,u)=(f,u)\leq|f|_{L^2}^2+\tfrac{1}{4}|u|_{L^2}^2; \tag{1.7}$$

the H^2-estimate is easily derived from (1.1a). Differentiating (1.1a) we inductively find the higher estimates H^m, $m\geq 3$. □

We notice that the characteristics are $x'(t)=-b(x(t))$ and hence $x'>0$ for $x\in(-1,0)$, $x'<0$ for $x\in(0,1)$. We thus observe that the characteristics converge to the point $x=0$. If we are away from $x=0$, the solution u^ϵ behaves like u^0 (when $\epsilon=0$). Various complicated behaviors may occur near $x=0$, in particular if certain compatibility conditions between b and f are not satisfied [see Desanti (1987a, b); Kevorkian and Cole (1996); O'Malley (1991, 1970); Smith (1985); and Wasow (1985)]. For example, if $b=x$, $f=1$, then $-xu_x^0=1$ (thus $u^0=-\ln(|x|)$ for $|x|>0$). Here we observe the logarithmic singularities at $x=0$. Notice that u^0 only belongs to $L^2(\Omega)$. These singularities arise due to the inconsistency between b and f (note that if $|u_x^0(0)|$ were bounded, $-b(0)u_x^0(0)=0\neq 1=f(0)$). These issues are addressed in Sec. III.

In Sec. II we start with the case $f=0$, α, β arbitrary. Since $f=0$, we easily see that u^0 is constant where $x\neq 0$. Because of the boundary conditions, $u^0=\alpha$ in $x<0$ and $u^0=\beta$ in $x>0$. The discrepancies (differences) between α and β lead to the interior layers θ^j below. Section II thus deals with the homogeneous problems with inhomogeneous boundary conditions.

In Secs. III and IV we discuss the inhomogeneous problems with homogeneous boundary conditions. In Sec. III, assuming enough compatibility conditions between b and f at $x=0$ [see (3.1)], we will observe discrepancies between $u_l^0(0^-)$ and $u_r^0(0^+)$ [more generally, between the outer solutions $u_l^j(0^-)$ and $u_r^j(0^+)$, $j\geq 0$, on the left and right of zero, see (2.1)] which result in the interior layers θ_r^j, θ_l^j, and ζ^j below.

We will also consider the case f arbitrary, not necessarily satisfying the compatibility conditions, by decomposing f into $\hat{f}$ and B_k where $\hat{f}=f-\Sigma_k\gamma_k B_k$, the B_k's are defined in (4.1), and the γ_k's are chosen so that $\hat{f}$ satisfies the compatibility conditions which appeared in Sec. III. We thus need only to investigate the case $f=B_k$ which we do in Sec. IV. We will observe the interior layers $\widetilde{\theta}_r^j$, $\widetilde{\theta}_l^j$, and $\widetilde{\zeta}^j$ in the following and they display very sharp transitions due to the singularities of the outer solutions u^j at $x=0$, e.g., $u^0=-\ln(|x|)$.

II. ASYMPTOTIC ANALYSIS I: $f=0$, α, β ARBITRARY

A. Outer expansions

We start with the formal expansions $u^\epsilon\sim\Sigma_{j=0}^\infty\epsilon^j u_l^j$ in $x<0$ and $u^\epsilon\sim\Sigma_{j=0}^\infty\epsilon^j u_r^j$ in $x>0$. Substituting these expansions in Eq. (1.1a) we find that, by identification at each power of ϵ,

$$O(1):-bu_{lx}^0=f \text{ in } [-1,0),\quad -bu_{rx}^0=f \text{ in } (0,1], \tag{2.1a}$$

$$O(\epsilon):-bu_{lx}^1=0 \text{ in } [-1,0),\quad -bu_{rx}^1=0 \text{ in } (0,1], \tag{2.1b}$$

$$O(\epsilon^j):-bu_{lx}^j=u_{lxx}^{j-2} \text{ in } [-1,0)\quad -bu_{rx}^j=u_{rxx}^{j-2}\quad \text{in } (0,1]\quad \text{for } j\geq 2. \tag{2.1c}$$

In this section since we consider the problem (1.1) with $f=0$, the outer solutions are very simple, namely $u_l^0=\alpha$, $u_r^0=\beta$ and $u_l^j=u_r^j=0$ for $j\geq 1$. Here we imposed the boundary conditions: $u_l^0(-1)=\alpha$, $u_r^0(1)=\beta$ and $u_l^j(-1)=u_r^j(1)=0$, $j\geq 1$, which will be justified in the following (by Theorems 2.1, 3.1, 4.1, and 4.2).

B. Interior layers θ^j

To resolve the discrepancies between u_l^0 and u_r^0 at $x=0$ (namely α and β if these numbers are different), we introduce the so-called ordinary interior layers which are defined by the inner expansions $u^\epsilon\sim\Sigma_{j=0}^\infty\epsilon^j\theta^j$ with a stretched variable $\bar{x}=x/\epsilon$, $\theta^j=\theta^j(\bar{x})$, $\bar{x}\in(-\infty,\infty)$ as follows. Using the formal Taylor expansion for $b=b(x)$ at $x=0$ we obtain the asymptotic expansion for b:

$$b(x)=\sum_{j=1}^\infty b_j x^j=\sum_{j=1}^\infty b_j\epsilon^j\bar{x}^j; \tag{2.2}$$

note that $b_0=b(0)=0$ and $b_1=b_x(0)\geq 1$ from (1.2). Substituting (2.2) and the inner expansions $(\Sigma_{j=0}^\infty\epsilon^j\theta^j)$ for b and u^ϵ, respectively, in Eq. (1.1a), we then obtain (with $b_0=0$) the following formal expansion:

$$\sum_{j=0}^\infty\left\{-\epsilon^j\theta_{\bar{x}\bar{x}}^j-\epsilon^j\left[\sum_{k=0}^j b_{j-k+1}\bar{x}^{j-k+1}\theta_{\bar{x}}^k\right]\right\}=0. \tag{2.3}$$

By identification at each power of ϵ, we find

$$O(1):- \theta^0_{\bar{x}\bar{x}} - b_1\bar{x}\theta^0_{\bar{x}} = 0, \tag{2.4a}$$

$$O(\epsilon):- \theta^1_{\bar{x}\bar{x}} - b_1\bar{x}\theta^1_{\bar{x}} = b_2\bar{x}^2\theta^0_{\bar{x}} \tag{2.4b}$$

$$O(\epsilon^j):- \theta^j_{\bar{x}\bar{x}} - b_1\bar{x}\theta^j_{\bar{x}} = \sum_{k=0}^{j-1} b_{j-k+1}\bar{x}^{j-k+1}\theta^k_{\bar{x}}. \tag{2.4c}$$

We impose the boundary conditions:[1] $\theta^0(x=-1)=\alpha$, $\theta^0(x=1)=\beta$, and $\theta^j(x=-1)=\theta^j(x=1)=0$, $j \geqslant 1$. But for the purpose of the analysis to follow it is convenient to consider the approximate form of θ^j, namely $\bar{\theta}^j$ satisfying Eq. (2.4) on all of $\mathbb{R}$ (for the variable $\bar{x}$) with the following boundary conditions:

$$\bar{\theta}^0 \to \alpha \quad \text{as } \bar{x} \to -\infty, \quad \bar{\theta}^0 \to \beta \quad \text{as } \bar{x} \to \infty, \tag{2.5a}$$

$$\bar{\theta}^j \to 0 \quad \text{as } \bar{x} \to \pm\infty, \quad j \geqslant 1. \tag{2.5b}$$

We show in the following that θ^j and $\bar{\theta}^j$ differ by an *exponentially small term* (denoted e.s.t). The reason for considering the $\bar{\theta}^j$ is that we are able to obtain the explicit solutions for $\bar{\theta}^j$: in particular,

$$\bar{\theta}^0 = c_0^{-1}\left[\alpha\int_{\bar{x}}^{\infty} \exp\left(-\frac{b_1 s^2}{2}\right)\mathrm{d}s + \beta\int_{-\infty}^{\bar{x}} \exp\left(-\frac{b_1 s^2}{2}\right)\mathrm{d}s\right], \tag{2.6a}$$

$$\bar{\theta}^1 = (\alpha-\beta)b_2 3^{-1}c_0^{-1}\int_{-\infty}^{\bar{x}} s^3 \exp\left(-\frac{b_1 s^2}{2}\right)\mathrm{d}s, \tag{2.6b}$$

where $c_0=\int_{-\infty}^{\infty}\exp(-b_1s^2/2)\mathrm{d}s=\sqrt{2\pi/b_1}$; see Figs. 2(a) and 2(d).

We claim that

$$\bar{\theta}^j_{\bar{x}} = P_{3j}(\bar{x})\exp\left(-\frac{b_1\bar{x}^2}{2}\right), \quad \forall j \geqslant 0, \tag{2.7}$$

where $P_s(\bar{x})$ denotes a polynomial in $\bar{x}$ of degree s with coefficients independent of ϵ but its expression may be different at different occurrences. Indeed, (2.7) for $j=0$ follows from (2.6a); then we assume that (2.7) is valid for $0 \leqslant j \leqslant n$. For $j=n+1$, the claim (2.7) follows observing that from (2.4c), θ^j being replaced by $\bar{\theta}^j$, we can write

$$-\left\{\bar{\theta}^{n+1}_{\bar{x}} \exp\left(\frac{b_1\bar{x}^2}{2}\right)\right\}_{\bar{x}} = \left\{\sum_{k=0}^{n} b_{n-k+2}\bar{x}^{n-k+2}\bar{\theta}^k_{\bar{x}}\right\}\exp\left(\frac{b_1\bar{x}^2}{2}\right) = \sum_{k=0}^{n} b_{n-k+2}\bar{x}^{n-k+2}P_{3k}(\bar{x}) = P_{3n+2}(\bar{x}), \tag{2.8}$$

and hence with a suitable constant C_{n+1},

[1] These boundary conditions would be different if α, β would contain lower order terms, e.g., $\theta^j \to \beta_j$ as $\bar{x}\to\infty$ with say $\beta=\beta_0+\epsilon\beta_1+\epsilon^2\beta_2+\cdots$.

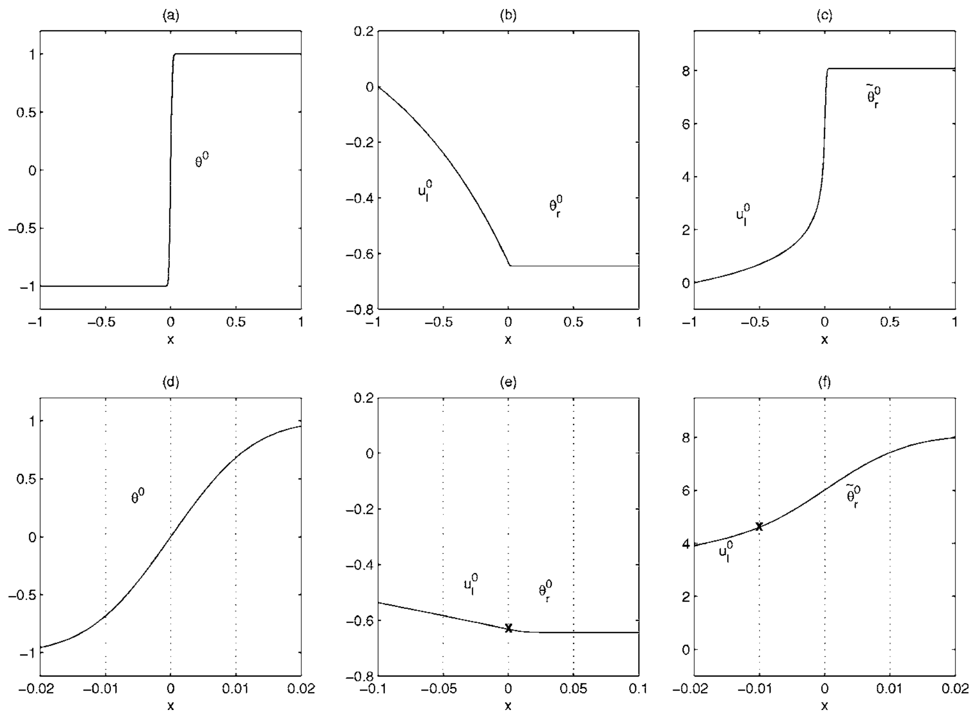

FIG. 2. The interior layers, θ^0, θ_r^0, $\tilde{\theta}_r^0$, for $b(x)=x$, $\epsilon=0.01$; (a) $f=0$, $\alpha=-1$, $\beta=1$, $\theta^0=\bar{\theta}^0+e.s.t.$ with $\bar{\theta}^0=\text{erf}(70.710\,678\,10*x)$; (b) $f=xe^x$, $\alpha=\beta=0$, $u_l^0=-e^x+e^{-1}$, $\theta_r^0=-0.012\,533\,141\,37*\text{erf}(70.710\,678\,10*x)-0.632\,120\,558\,8$, u_l^0 and θ_r^0 are matched at $x=0$ with C^1-smoothness; (c) $f=1$, $\alpha=\beta=0$, $u_l^0=-\ln(-x)$, $\tilde{\theta}_r^0=2.066\,365\,677*\text{erf}(70.710\,678\,10*x)+6.015\,856\,320$, u_l^0 and $\tilde{\theta}_r^0$ are matched at $x=-\epsilon=-0.01$ with C^1-smoothness; (d), (e), (f) are, respectively, zooming of (a), (b), (c) near $x=0$, see the matching points $x=0$, -0.01 marked "x," respectively, in (e), (f).

$$\bar{\theta}_{\bar{x}}^{n+1}=(P_{3(n+1)}(\bar{x})+C_{n+1})\exp\left(-\frac{b_1\bar{x}^2}{2}\right). \tag{2.9}$$

We then have to show that the coefficients in the polynomial $P_{3(n+1)}(\bar{x})+C_{n+1}$ corresponding to $\bar{\theta}_{\bar{x}}^{n+1}$ are independent of ϵ. We first notice that by the induction assumption, the coefficients of $P_{3n+2}(\bar{x})$ in (2.8) are independent of ϵ, and so are those of $P_{3(n+1)}(\bar{x})$ in (2.9). It thus suffices to show that C_{n+1} is independent of ϵ. Indeed, we find that for a constant D_{n+1}, $n\geqslant 0$,

$$\bar{\theta}^{n+1}=\int_{-\infty}^{\bar{x}}\bar{\theta}_{\bar{x}}^{n+1}(s)\mathrm{d}s+D_{n+1}=\int_{-\infty}^{\bar{x}}P_{3(n+1)}(s)\exp\left(-\frac{b_1s^2}{2}\right)\mathrm{d}s+C_{n+1}\int_{-\infty}^{\bar{x}}\exp\left(-\frac{b_1s^2}{2}\right)\mathrm{d}s+D_{n+1}. \tag{2.10}$$

By the boundary conditions (2.5b), we first notice that $D_{n+1}=0$ and

$$C_{n+1}c_0=-\int_{-\infty}^{\infty}P_{3(n+1)}(s)\exp\left(-\frac{b_1s^2}{2}\right)\mathrm{d}s, \tag{2.11}$$

which is independent of ϵ because so are the coefficients of $P_{3(n+1)}(s)$ in (2.10) and (2.11).

We now show that the $\theta^j-\bar{\theta}^j$, $j\geqslant 0$, are *exponentially small terms*, more precisely,

$$\|\theta^j-\bar{\theta}^j\|_{H^m(\Omega)}\leqslant \kappa_{jm}e^{-c/\epsilon},\quad \forall\, j,\quad m\geqslant 0. \tag{2.12}$$

To see this, from (2.7) we first notice that $\bar{\theta}^j(\bar{x})=\int_{-\infty}^{\bar{x}}P_{3j}(s)e^{-b_1s^2/2}\mathrm{d}s+\bar{\theta}^j(-\infty)$, or $-\int_{\bar{x}}^{\infty}P_{3j}(s)e^{-b_1s^2/2}\mathrm{d}s+\bar{\theta}^j(\infty)$, where $\bar{\theta}^j(\pm\infty)$ is given in (2.5). Since $\bar{\theta}^j(-\infty)=\theta^j(x=-1)$, $\bar{\theta}^j(\infty)=\theta^j(x=1)$, we find that

$$\begin{aligned}|(\theta^j-\bar{\theta}^j)(x=-1)| &= |\bar{\theta}^j(-\infty)-\bar{\theta}^j(-1/\epsilon)|\leqslant \int_{-\infty}^{-1/\epsilon}|P_{3j}(s)|e^{-b_1s^2/2}\mathrm{d}s\\ &\leqslant (\text{by (2.19a), (2.19b), see below})\leqslant \kappa_j e^{-c/\epsilon},\end{aligned} \tag{2.13}$$

and similarly $|(\theta^j-\bar{\theta}^j)(x=1)|\leqslant \kappa_j e^{-c/\epsilon}$. Setting then $\delta^j=\delta^j(x)=\theta^j-\bar{\theta}^j-[(\theta^j-\bar{\theta}^j)(x=-1)](1-x)/2-[(\theta^j-\bar{\theta}^j)(x=1)](1+x)/2$, it suffices to show that $\|\delta^j\|_{H^m(\Omega)}\leqslant \kappa_{jm}e^{-c/\epsilon}$. To show this we write with Eq. (2.4) for θ^j and $\bar{\theta}^j$:

$$\begin{gathered}-\epsilon^2\delta^j_{xx}-b_1x\delta^j_x=\sum_{k=0}^{j-1}\epsilon^{-(j-k)}b_{j-k+1}x^{j-k+1}\delta^k_x+\tilde{\delta}^j,\\ \delta^j(-1)=\delta^j(1)=0,\end{gathered} \tag{2.14}$$

where $\tilde{\delta}^j=-b_1x/2\{(\theta^j-\bar{\theta}^j)(x=-1)-(\theta^j-\bar{\theta}^j)(x=1)\}$, $\forall j$, $m\geqslant 0$; note that $|\tilde{\delta}^j|_{H^m(\Omega)}\leqslant \kappa_{jm}e^{-c/\epsilon}$. By Lemma 1.1, we first find $\|\delta^0\|_{H^m(\Omega)}\leqslant \kappa_m P(\epsilon^{-1})e^{-c/\epsilon}$, $P(\epsilon^{-1})$ a polynomial in ϵ^{-1}, and again recursively we also find that $\|\delta^j\|_{H^m(\Omega)}\leqslant \kappa_{jm}P(\epsilon^{-1})e^{-c/\epsilon}$, $\forall j\geqslant 1$. This implies that $\|\delta^j\|_{H^m(\Omega)}\leqslant \kappa_{jm}e^{-c/(2\epsilon)}$, $\forall j\geqslant 0$.

The following pointwise and norm estimates can then be derived.

Lemma 2.1: There exist positive constants κ_{jm} and c such that the following pointwise estimate holds:[2]

$$\left|\frac{\mathrm{d}^m\theta^j}{\mathrm{d}x^m}\right|\leqslant \kappa_{jm}\begin{cases}1 & \text{for } j=0 \text{ and } m=0\\ \epsilon^{-m}\exp\left(-c\dfrac{|x|}{\epsilon}\right) & \text{for } j\geqslant 1 \text{ or } m\geqslant 1.\end{cases} \tag{2.15}$$

Furthermore, for $\sigma\in[0,1)$,

$$|\theta^j|_{H^m((-1,-\sigma)\cup(\sigma,1))}\leqslant \kappa_{jm}\begin{cases}1 & \text{for } j=0 \text{ and } m=0\\ \epsilon^{-m+1/2}\exp\left(-c\dfrac{\sigma}{\epsilon}\right) & \text{for } j\geqslant 1 \text{ or } m\geqslant 1,\end{cases} \tag{2.16}$$

and, for $m\geqslant 0$,

$$|\theta^j|_{H^m(-1,1)}\leqslant \kappa_{jm}(1+\epsilon^{-m+1/2}). \tag{2.17}$$

Proof: Differentiating (2.7) in $\bar{x}$, we find that for $m\geqslant 1$,

$$\frac{\mathrm{d}^m\bar{\theta}^j}{\mathrm{d}\bar{x}^m}=P_{3j+m-1}(\bar{x})\exp\left(-\frac{b_1\bar{x}^2}{2}\right). \tag{2.18}$$

We thus derive (2.15) for $\bar{\theta}^j$, $m\geqslant 1$, $j\geqslant 0$ from the fact that for every $c>0$,

$$\exp\left(-\frac{b_1\bar{x}^2}{2}\right)\leqslant \kappa(c)\exp(-2c|\bar{x}|), \tag{2.19a}$$

[2] $c>0$ is arbitrary but $k_{jm}=k_{jm}(c)$. In what follows, throughout this paper, the c is understood in this manner.

$$|P_{3j+m-1}(\bar{x})| \leq \kappa_{jm}(c)\exp(c|\bar{x}|), \tag{2.19b}$$

so that, for any $c > 0$,

$$\left| \frac{\mathrm{d}^m \bar{\theta}^j}{\mathrm{d}\bar{x}^m}(\bar{x}) \right| \leq \kappa_{jm}(c)\exp(-c|\bar{x}|), \quad \forall\, \bar{x} \in (-\infty,\infty). \tag{2.19c}$$

For $m=0$, $j=0$, (2.15) for $\bar{\theta}^0$ is obvious from (2.6a). For $m=0$, $j\geq 1$, using the estimates (2.19) (2.15) for $\bar{\theta}^j$ directly follows from (2.10) and (2.11). Then thanks to (2.12), (2.15) follows.

The norm estimates (2.16) and (2.17) are directly deduced from (2.15). □

Remark 2.1: It follows from Lemma 2.1 that the interior layers θ^j, $j\geq 1$, and their derivatives are *exponentially small terms (e.s.t)* in the regions $|x| > \kappa\epsilon^{\gamma}$, $0<\gamma<1$, κ fixed.

C. Asymptotic errors

Let

$$w_{\epsilon n} = u^{\epsilon} - \theta_{\epsilon n}, \tag{2.20a}$$

where

$$\theta_{\epsilon n} = \sum_{j=0}^{n} \epsilon^j \theta^j. \tag{2.20b}$$

Multiplying (2.4c) by ϵ^j and summing over $j=0,\ldots,n$, we find that

$$L_{\epsilon}\theta_{\epsilon n} = -R_1^n, \tag{2.21a}$$

where

$$R_1^n = \sum_{j=0}^{n} \epsilon^j \theta_x^j R^{j,n}(b), \tag{2.21b}$$

with

$$R^{j,n}(b) = b(x) - \sum_{k=1}^{n+1-j} b_k x^k; \tag{2.21c}$$

here we used the fact that, by permuting the summations:

$$\sum_{j=0}^{n}\left\{ \epsilon^j \sum_{k=0}^{j} b_{j-k+1}\bar{x}^{j-k+1}\theta_{\bar{x}}^k \right\} = \sum_{j=0}^{n}\sum_{k=0}^{j} b_{j-k+1}x^{j-k+1}\epsilon^k\theta_x^k = \sum_{j=0}^{n}\epsilon^j\theta_x^j\left\{ \sum_{k=1}^{n+1-j} b_k x^k \right\}. \tag{2.21d}$$

We now estimate the L^2- norm of R_1^n as follows. We first notice that, by Taylor expansion,

$$|R^{j,n}(b)| = \left| b(x) - \sum_{k=1}^{n+1-j} b_k x^k \right| \leq \kappa_n |x|^{n+2-j} \leq \kappa_n \epsilon^{n+2-j}|\bar{x}|^{n+2-j}. \tag{2.22}$$

From (2.15) and (2.21b) we thus find that

$$|R_1^n| \leq \kappa_n \epsilon^{n+2}\sum_{j=0}^{n}|\bar{x}|^{n+2-j}|\theta_x^j| \leq \kappa_n\epsilon^{n+1}\exp\left(-\frac{c|x|}{2\epsilon}\right) \tag{2.23a}$$

and hence

$$|R_1^n|_{L^2} \leq \kappa_n \epsilon^{n+3/2}. \tag{2.23b}$$

Subtracting (2.21a) from (1.1a) with $f=0$, we find that

$$L_\epsilon w_{\epsilon n} = R_1^n + e.s.t. \text{ in } \Omega, \tag{2.24a}$$

$$w_{\epsilon n}(-1) = w_{\epsilon n}(1) = 0. \tag{2.24b}$$

Applying Lemma 1.1 to Eq. (2.24) with $u=w_{\epsilon n}$ and using (2.23b) we thus obtain the following theorem.

Theorem 2.1: *Let u^ϵ be the solution of (1.1) with $f=0$. Then for m, $n \geq 0$, there exists a constant $\kappa_n > 0$ independent of ϵ such that*

$$\|u^\epsilon - \theta_{\epsilon n}\|_{H^m(\Omega)} \leq \kappa_n \begin{cases} \epsilon^{n+3/2} & \text{for } m=0 \\ \epsilon^{-2m+n+5/2} & \text{for } m=1,2, \end{cases} \tag{2.25}$$

where $\theta_{\epsilon n}$ is as in (2.20b).

Remark 2.2: For $n=0$ we can write the left-hand side of (2.25) as $\|u^\epsilon - u^0 - (\theta^0 - u^0)\|_{H^m(\Omega)}$, where the corrector $\theta^0 - u^0$ vanishes at $x=-1$ and 1, and is discontinuous at $x=0$. Now we notice that from Lemma 2.1

$$|\theta^0 - \beta| \leq \int_{\bar{x}}^{\infty} |\theta^0_{\bar{x}}(s)| \mathrm{d}s \leq \kappa \exp(-c\bar{x}) \quad \text{for } \bar{x} \geq 0, \tag{2.26}$$

and similarly, $|\theta^0 - \alpha| \leq \kappa \exp(c\bar{x})$, for $\bar{x} < 0$. Setting $u^0 = \alpha$ for $x \in [-1,0)$, $u^0 = \beta$ for $x \in (0,1]$, we then find that from Theorem 2.1

$$\begin{aligned} |u^\epsilon - u^0|_{L^2(-1,1)} &\leq |u^\epsilon - u^0 - (\theta^0 - u^0)|_{L^2(-1,1)} + \kappa|\theta^0 - \alpha|_{L^2(-1,0)} + \kappa|\theta^0 - \beta|_{L^2(0,1)} \leq \kappa\epsilon^{3/2} + \kappa\epsilon^{1/2} \\ &\leq \kappa\epsilon^{1/2}. \end{aligned} \tag{2.27}$$

Hence we write: $u^\epsilon = u^0 + O(\epsilon^{1/2})$ in $L^2(\Omega)$.

III. ASYMPTOTIC ANALYSIS II: *f*, *b* COMPATIBLE, $\alpha=\beta=0$

In this section, we consider the problem (1.1) with $\alpha=\beta=0$ and f arbitrary satisfying the compatibility conditions (3.1) to follow. If $f \not\equiv 0$, in particular if $f(0) \neq 0$, since $b(0)=0$, the limit problem $-bu^0_x = f$ has an inconsistency at $x=0$. That is its solution cannot be smooth (C^1). To avoid the inconsistency between b and f, in this section we assume the following compatibility conditions:

$$\frac{\mathrm{d}^i f}{\mathrm{d}x^i}(0) = 0, \quad i = 0,1,\ldots,N; \tag{3.1}$$

the integer $N \geq 0$ will be specified later on. If (3.1) does not hold, we will see that the solution u^ϵ of (1.1) possesses logarithmic singularities at $x=0$ as already indicated in Sec. I. The case where the compatibility conditions (3.1) are not satisfied is addressed in Sec. IV.

We first construct the outer expansions u_l^j, u_r^j as in (2.1). Here (also in Sec. IV) we impose the following boundary conditions: for $j \geq 0$,

$$u_l^j(-1) = u_r^j(1) = 0, \tag{3.2}$$

which will be justified in the following (by Theorems 3.1, 4.1, and 4.2). We then notice with (2.1) that $u_l^j = u_r^j = 0$ for all odd $j \geq 1$. Furthermore, we are able to obtain the following explicit expressions:

$$u_l^0 = -\int_{-1}^{x} b(s)^{-1} f(s)\mathrm{d}s, \quad u_r^0 = \int_{x}^{1} b(s)^{-1} f(s)\mathrm{d}s, \tag{3.3a}$$

and for all $j=2k$, $k \geqslant 1$,

$$u_l^{2k} = -\int_{-1}^{x} b(s)^{-1} u_{xx}^{2(k-1)}(s)\mathrm{d}s, \quad u_r^{2k} = \int_{x}^{1} b(s)^{-1} u_{xx}^{2(k-1)}(s)\mathrm{d}s. \tag{3.3b}$$

Thanks to the compatibility conditions (3.1), the values of $u_l^j(0^-)$ and $u_r^j(0^+)$, of $u_{lx}^j(0^-)$ and $u_{rx}^j(0^+)$ or of higher order derivatives, are finite if we take N sufficiently large. For example, to guarantee that $|u_{lx}^2(0^-)|$, $|u_{rx}^2(0^+)| < \infty$, $N=2$ is required. Indeed, $u_{lx}^2(0^-) = -b(0^-)^{-1} u_{xx}^0(0^-)$ and by the L'Hospital's rule, we find that $u_{lxx}^0(0^-)=0$ and $|u_{lxxx}^0(0^-)| < \infty$ are needed (we then find $u_{lx}^2(0^-) = -b_x(0)^{-1} u_{lxxx}^0(0^-)$). Assuming that condition (3.1) with $N=2$ holds, by some elementary calculations, we find

$$u_{lxx}^0(0^-) = -\frac{f_{xx}(0)}{2b_x(0)} = 0, \tag{3.4a}$$

$$|u_{lxxx}^0(0^-)| = \left| \frac{f_{xx}(0) b_{xx}(0)}{2b_x(0)^2} - \frac{f_{xxx}(0)}{3b_x(0)} \right| < \infty, \tag{3.4b}$$

and the arguments are similar for $u_{rx}^2(0^+)$. In the following lemma, we precisely specify N so that $|\mathrm{d}^m u_l^{2k}/\mathrm{d}x^m(0^-)|$, $|\mathrm{d}^m u_r^{2k}/\mathrm{d}x^m(0^+)| \leqslant \kappa_{km}$ and thus, in the following, (3.20) for $m=1$, and (3.36c) for $m=2$ make sense.

Lemma 3.1: Let $m \geqslant 1$ and $k \geqslant 0$. Assume that the compatibility conditions (3.1) hold with $N=m+2k-1$. Then there exists a positive constant κ_{km} such that

$$\left| \frac{\mathrm{d}^m u_l^{2k}}{\mathrm{d}x^m}(0^-) \right|, \quad \left| \frac{\mathrm{d}^m u_r^{2k}}{\mathrm{d}x^m}(0^+) \right| \leqslant \kappa_{km}. \tag{3.5}$$

Proof: Set $u_{lxx}^{-2} = f$ for convenience. We then claim that for $m \geqslant 1$, $x \in [-1,0)$,

$$\left| \frac{\mathrm{d}^m u_l^{2k}}{\mathrm{d}x^m}(x) \right| \leqslant \kappa_m \sum_{r=0}^{m-1} |b|^{r-m} \left| \frac{\mathrm{d}^{r+2} u_l^{2(k-1)}}{\mathrm{d}x^{r+2}}(x) \right|. \tag{3.6}$$

Indeed, we prove (3.6) using an induction argument on m. For $m=1$, we easily derive (3.6) from (2.1c). Assume that (3.6) holds for $m \leqslant s$. For $m=s+1$, differentiating (2.1c) s times in x we find that

$$-b \frac{\mathrm{d}^{s+1} u_l^{2k}}{\mathrm{d}x^{s+1}} = \sum_{r=1}^{s} \binom{s}{r} \frac{\mathrm{d}^r b}{\mathrm{d}x^r} \frac{\mathrm{d}^{s-r+1} u_l^{2k}}{\mathrm{d}x^{s-r+1}} + \frac{\mathrm{d}^{s+2} u_l^{2(k-1)}}{\mathrm{d}x^{s+2}}. \tag{3.7}$$

The claim (3.6) follows observing that

$$\begin{aligned} \left| \frac{\mathrm{d}^{s+1} u_l^{2k}}{\mathrm{d}x^{s+1}} \right| &\leqslant \kappa_s |b|^{-1} \left\{ \sum_{r=1}^{s} \left| \frac{\mathrm{d}^{s-r+1} u_l^{2k}}{\mathrm{d}x^{s-r+1}} \right| + \left| \frac{\mathrm{d}^{s+2} u_l^{2(k-1)}}{\mathrm{d}x^{s+2}} \right| \right\} \\ &\leqslant \kappa_s |b|^{-1} \left\{ \sum_{r=1}^{s} \left(\sum_{l=0}^{s-r} |b|^{l-(s-r+1)} \left| \frac{\mathrm{d}^{l+2} u_l^{2(k-1)}}{\mathrm{d}x^{l+2}} \right| \right) + \left| \frac{\mathrm{d}^{s+2} u_l^{2(k-1)}}{\mathrm{d}x^{s+2}} \right| \right\} \\ &\leqslant \kappa_s |b|^{-1} \left\{ \sum_{l=0}^{s} |b|^{l-s} \left| \frac{\mathrm{d}^{l+2} u_l^{2(k-1)}}{\mathrm{d}x^{l+2}} \right| \right\}. \end{aligned} \tag{3.8}$$

We next claim that

$$\frac{d^n u_l^{2k}}{dx^n}(0^-) = 0 \tag{3.9a}$$

provided that

$$\frac{d^i f}{dx^i}(0) = 0, \quad 0 \leqslant i \leqslant n + 2k. \tag{3.9b}$$

To prove this claim, thanks to the L'Hospital rule, from (3.6) with $m=n$ we easily find that if

$$\frac{d^i u_l^{2(k-1)}}{dx^i}(0^-) = 0 \quad \text{for } 2 \leqslant i \leqslant n+2, \tag{3.10}$$

then (3.9a) follows. Similarly, (3.10) follows if

$$\frac{d^i u_l^{2(k-2)}}{dx^i}(0^-) = 0 \quad \text{for } 2 \leqslant i \leqslant n+4. \tag{3.11}$$

We then recursively find that (3.10) (and thus (3.9a)) follows if

$$\frac{d^{i-2} f}{dx^{i-2}}(0^-) = \frac{d^i u_l^{-2}}{dx^i}(0^-) = 0 \quad \text{for } 2 \leqslant i \leqslant n + 2(k+1); \tag{3.12}$$

this is exactly (3.9b).

From (3.6) we now derive the following recursive relation: to guarantee that

$$\left| \frac{d^m u_l^{2k}}{dx^m}(0^-) \right| \leqslant \kappa_{km}, \tag{3.13a}$$

we require that

$$\frac{d^{r+s+2} u_l^{2(k-1)}}{dx^{r+s+2}}(0^-) = 0 \quad \text{for } 0 \leqslant s \leqslant m - r - 1 \tag{3.13b}$$

and

$$\left| \frac{d^{m+2} u_l^{2(k-1)}}{dx^{m+2}}(0^-) \right| \leqslant \kappa_{k-1,m+2}. \tag{3.13c}$$

Due to (3.9), the compatibility conditions (3.1) with $0 \leqslant i \leqslant m+2k-1$ imply (3.13b). The lemma follows from the recursive relation (3.13) and the fact that $u_{lxx}^{-2} = f$, f smooth. The estimates for u_r^{2k} can be similarly deduced. □

Remark 3.1: We easily find that (3.5) in Lemma 3.1 can be replaced by

$$\left| \frac{d^m u_l^j}{dx^m}(0^-) \right|, \quad \left| \frac{d^m u_r^j}{dx^m}(0^+) \right| \leqslant \kappa_{jm} \quad \text{for all } 0 \leqslant j \leqslant 2k+1; \tag{3.14}$$

note that $u_l^j = u_r^j = 0$ for j odd.

Remark 3.2: In general, when the compatibility conditions (3.1) are not necessarily satisfied, we have the following regularity results:

$$|u_l^0|_{L^2(-1,0)} \leqslant \kappa |f|_{L^\infty(-1,0)}, \tag{3.15a}$$

$$|u_l^j|_{L^2(-1,0)} \leqslant \kappa_j |u_l^{j-2}|_{W^{2,\infty}(-1,0)} \quad \text{for all even } j \geqslant 2. \tag{3.15b}$$

Indeed, from condition (1.2) we find that

$$\frac{x}{b(x)} \to \frac{1}{b_x(0)} \quad \text{as } x \to 0, \tag{3.16}$$

and hence

$$\left| \frac{x}{b(x)} \right| \leqslant \kappa, \quad \forall\, x \in [-1,1]. \tag{3.17}$$

We thus infer from the explicit expression (3.3a) that

$$|u_l^0(x)| \leqslant |f|_{L^\infty(-1,0)} \left| \int_{-1}^{x} \frac{s}{b(s)} \frac{ds}{s} \right| \leqslant -\kappa \ln(-x) |f|_{L^\infty(-1,0)}, \quad \forall\, x \in [-1,0), \tag{3.18}$$

and hence

$$|u_l^0|_{L^2(-1,0)} \leqslant \kappa |\ln(|x|)|_{L^2} |f|_{L^\infty} \leqslant \kappa |f|_{L^\infty(-1,0)}; \tag{3.19}$$

the estimates (3.15b) for u^j, $j \geqslant 2$ even, are similarly obtained. The regularity properties for u_r^j can be similarly deduced.

A. Interior layers $\theta_r^j, \theta_l^j, \xi^j$

Assuming enough compatibility conditions, we can guarantee, as in Lemma 3.1, that $|u_l^j(0^-)|$, $|u_r^j(0^+)|$, $|u_{lx}^j(0^-)|$, $|u_{rx}^j(0^+)| \leqslant \kappa_j$. In general, $u_l^j(0^-) \neq u_r^j(0^+)$. To resolve these discrepancies at $x=0$, using the stretched variable $\bar{x}=x/\epsilon$, we introduce the functions $\theta_l^j(\bar{x})$, and $\theta_r^j(\bar{x})$ which are defined as the solutions of the same equations (2.4) respectively on $(-\infty,0)$, and $(0,\infty)$ with the following boundary conditions:[3]

$$\theta_r^j(\bar{x}) = u_l^j(0^-), \quad \theta_{r\bar{x}}^j(\bar{x}) = \epsilon \theta_{rx}^j = \epsilon u_{lx}^j(0^-) \quad \text{at } \bar{x}=0, \tag{3.20a}$$

$$\theta_l^j(\bar{x}) = u_r^j(0^+), \quad \theta_{l\bar{x}}^j(\bar{x}) = \epsilon \theta_{lx}^j = \epsilon u_{rx}^j(0^+) \quad \text{at } \bar{x}=0, \tag{3.20b}$$

which allow us to determine the θ_l^j, θ_r^j explicitly. Notice that $u_r^j = u_l^j = 0$ for j odd. In particular, for $j=0,1$, we find

$$\theta_r^0 = \epsilon u_{lx}^0(0^-) \int_0^{\bar{x}} \exp\left(-\frac{b_1 s^2}{2}\right) ds + u_l^0(0^-), \tag{3.21a}$$

$$\theta_r^1 = -\epsilon u_{lx}^0(0^-) b_2 3^{-1} \int_0^{\bar{x}} s^3 \exp\left(-\frac{b_1 s^2}{2}\right) ds. \tag{3.21b}$$

Here we note that as $\bar{x} \to \infty$,

$$\theta_r^0 \to \epsilon u_{lx}^0(0^-) c_{r,0} + u_l^0(0^-) =: c_{r,\infty}^0(\epsilon), \tag{3.22a}$$

$$\theta_r^1 \to -\epsilon u_{lx}^0(0^-) b_2 3^{-1} c_{r,1} =: c_{r,\infty}^1(\epsilon), \tag{3.22b}$$

where $c_{r,0} = \int_0^\infty \exp(-b_1 s^2/2) ds$, $c_{r,1} = \int_0^\infty s^3 \exp(-b_1 s^2/2) ds$.

We denote by $\varphi \cup \psi$ the function on $(-1,1)$ equal to (the restriction of) φ on $(-1,0)$ and to (the restriction of) ψ on $(0,1)$ and consider the functions $u_l^j \cup \theta_r^j$ and $\theta_l^j \cup u_r^j$. Note that due to (3.20), these functions belong to $C^1([-1,1])$ and to $H^2(-1,1)$; see Figs. 2(b) and 2(e).

[3]These boundary conditions provide smooth (i.e., C^1) matching of $u_l^j(x)$ with $\theta_r^j(x/\epsilon)$ (respectively, of $u_r^j(x)$ with $\theta_l^j(x/\epsilon)$). Note that the interior layers θ_l^j, θ_r^j are independent of the fact that $f=0$ (as in Sec. II) or not. When $f \neq 0$, the outer solutions u_l^j, u_r^j only are affected.

We now estimate the interior layers θ_r^j, θ_l^j. Assuming the compatibility conditions (3.1) with $N=2k$, we infer from (3.14) that $|u_l^j(0^-)|$, $|u_r^j(0^+)|$, $|u_{lx}^j(0^-)|$, $|u_{rx}^j(0^+)| \leqslant \kappa_j$ for $0 \leqslant j \leqslant 2k+1$. We derive a relation similar to (2.7) but we need to take into account the boundary conditions (3.20). More precisely, we claim that for $\bar{x} \in [0,\infty)$,

$$\theta_{r\bar{x}}^j = \epsilon P_{3j}(\bar{x}) \exp\left(-\frac{b_1 \bar{x}^2}{2}\right), \quad \forall j \geqslant 0, \tag{3.23}$$

where the $P_{3j}(\bar{x})$ are as in (2.7).

For $j=0$, we easily deduce (3.23) from (3.21a). Now suppose that (3.23) holds for $j \leqslant n$. For $j=n+1 \leqslant 2k+1$, we find that as for (2.8)

$$-\left\{\theta_{r\bar{x}}^{n+1} \exp\left(\frac{b_1 \bar{x}^2}{2}\right)\right\}_{\bar{x}} = \epsilon P_{3n+2}(\bar{x}), \tag{3.24}$$

and by the boundary conditions (3.20), namely $\theta_{r\bar{x}}^{n+1}(\bar{x}=0) = \epsilon u_{lx}^{n+1}(x=0^-)$, we find

$$\theta_{r\bar{x}}^{n+1} = \epsilon P_{3(n+1)}(\bar{x}) \exp\left(-\frac{b_1 \bar{x}^2}{2}\right). \tag{3.25}$$

The following pointwise and norm estimates can be derived.

Lemma 3.2: Assume that the compatibility conditions (3.1) hold with $N=2k$, $k \geqslant 0$. Then there exist positive constants κ_{jm} and c such that for $x \in [0,1]$, $0 \leqslant j \leqslant 2k+1$,

$$\left|\frac{\mathrm{d}^m \theta_r^j}{\mathrm{d}x^m}\right| \leqslant \kappa_{jm} \begin{cases} 1 & \text{for } m=0 \\ \epsilon^{-m+1} \exp\left(-c\dfrac{x}{\epsilon}\right) & \text{for } m \geqslant 1. \end{cases} \tag{3.26}$$

Consequently, for $\sigma \in [0,1)$,

$$|\theta_r^j|_{H^m(\sigma,1)} \leqslant \kappa_{jm} \begin{cases} 1 & \text{for } m=0 \\ \epsilon^{-m+3/2} \exp\left(-c\dfrac{\sigma}{\epsilon}\right) & \text{for } m \geqslant 1, \end{cases} \tag{3.27}$$

and for $m \geqslant 0$,

$$|\theta_r^j|_{H^m(0,1)} \leqslant \kappa_{jm}(1 + \epsilon^{-m+3/2}). \tag{3.28}$$

Furthermore, there exist constants $c_{r,\infty}^j(\epsilon)$ with $|c_{r,\infty}^j(\epsilon)| \leqslant \kappa_j$ such that for $j \geqslant 0$,

$$\theta_r^j(\bar{x}) \to c_{r,\infty}^j(\epsilon) \quad \text{as } \bar{x} \to \infty. \tag{3.29}$$

Proof: Thanks to (3.23) we proceed as in (2.18) (2.19) to derive (3.26) for $m \geqslant 1$, $0 \leqslant j \leqslant 2k+1$. The case $m=0$ follows observing that

$$|\theta_r^j| = \left|\int_0^{\bar{x}} \theta_{r\bar{x}}^j(s)\mathrm{d}s + u_l^j(0^-)\right| \leqslant \kappa_j \epsilon \int_0^{\bar{x}} \exp(-cs)\mathrm{d}s + |u_l^j(0^-)| \leqslant \kappa_j. \tag{3.30}$$

The norm estimates (3.27) and (3.28) easily follow from (3.26).

To prove (3.29), thanks to (3.30) we can extract a sequence $p_n \to \infty$ and find an accumulation point $c_{r,\infty}^j(\epsilon)$ (note $|\theta_r^j(\bar{x})| \leqslant \kappa_j$) such that $\theta_r^j(p_n) \to c_{r,\infty}^j(\epsilon)$; (3.29) easily follows observing that

$$\theta_r^j(\bar{x}) - \theta_r^j(p_n) = \int_{p_n}^{\bar{x}} \theta_{r\bar{x}}^j(s)\mathrm{d}s, \tag{3.31}$$

and, letting $p_n \to \infty$,

$$|\theta_r^j(\bar{x}) - c_{r,\infty}^j(\epsilon)| = \left| \int_{\bar{x}}^{\infty} \theta_{r\bar{x}}^j(s)\mathrm{d}s \right| \leqslant \kappa\epsilon \exp(-c\bar{x}). \quad (3.32)$$

Remark 3.3: We can similarly perform the analysis for θ_l^j and derive the pointwise and norm estimates as above. Here we denote by $c_{l,\infty}^j$ the limit of θ_l^j as $\bar{x}\to-\infty$.

By our constructions, we then notice that from (3.32) the functions $g^j := -(u_l^j \cup \theta_r^j) - (\theta_l^j \cup u_r^j)$ attain the values $-\theta_l^j = -c_{l,\infty}^j(\epsilon) + e.s.t.$ at $x=-1$ and $-\theta_r^j = -c_{r,\infty}^j(\epsilon) + e.s.t.$ at $x=1$ (see (3.2) and (3.32)). To remedy these discrepancies between g^j and u^ϵ at the boundaries $x=-1,1$ (we recall that $u^\epsilon(-1)=u^\epsilon(1)=0$), we introduce interior layers ζ^j similar to θ^j but we use different boundary conditions: the $\zeta^j = \zeta^j(\bar{x})$ satisfy (2.4) and

$$\zeta^j = -\theta_l^j \quad \text{at } x=-1, \quad \zeta^j = -\theta_r^j \quad \text{at } x=1 \quad \text{for } j \geqslant 0. \quad (3.33)$$

As before we are able to obtain explicit solutions: we set

$$\vartheta^j(\alpha,\beta) = \bar{\theta}^j, \quad j=0,1, \quad (3.34a)$$

where the $\bar{\theta}^j$ are as in (2.6); then by (3.32) and some elementary calculations, we find:

$$\zeta^0 = \vartheta^0(-c_{l,\infty}^0(\epsilon), -c_{r,\infty}^0(\epsilon)) + e.s.t., \quad (3.34b)$$

$$\zeta^1 = \vartheta^1(-c_{l,\infty}^0(\epsilon), -c_{r,\infty}^0(\epsilon)) + \vartheta^0(-c_{l,\infty}^1(\epsilon), -c_{r,\infty}^1(\epsilon)) + e.s.t. \quad (3.34c)$$

Proceeding as in Lemma 2.1 we obtain for the ζ^j the same estimates as (2.15)–(2.17) (but "$j=0$ and $m=0$," "$j \geqslant 1$ or $m \geqslant 1$," respectively, being replaced by "$m=0$," "$m \geqslant 1$"). Here we used the fact that in (2.7)–(2.11) for $\bar{\theta}^j = \bar{\zeta}^j$ (approximate form of ζ^j),

$$D_{n+1} = -c_{l,\infty}^{n+1}(\epsilon), \quad C_{n+1}c_0 = -c_{r,\infty}^{n+1}(\epsilon) + c_{l,\infty}^{n+1}(\epsilon) - \int_{-\infty}^{\infty} P_{3(n+1)}(s)\exp\left(-\frac{b_1 s^2}{2}\right)\mathrm{d}s$$

and $|c_{r,\infty}^j(\epsilon)|$, $|c_{l,\infty}^j(\epsilon)| \leqslant \kappa_j$.

B. Asymptotic errors

Let

$$w_{\epsilon n} = u^\epsilon - \xi_{\epsilon n} - \eta_{\epsilon n} - \zeta_{\epsilon n}, \quad (3.35a)$$

where

$$\xi_{\epsilon n} = \sum_{j=0}^{2n} \epsilon^j (u_l^j \cup \theta_r^j), \quad \eta_{\epsilon n} = \sum_{j=0}^{2n} \epsilon^j (\theta_l^j \cup u_r^j), \quad \zeta_{\epsilon n} = \sum_{j=0}^{2n} \epsilon^j \zeta^j. \quad (3.35b)$$

From the outer expansions (2.1) and the interior layers $\theta_r^j, \theta_l^j, \zeta^j$, after some elementary calculations, we find that

$$L_\epsilon w_{\epsilon n} = R_2^n + R_3^n + R_4^n + e.s.t. \text{ in } \Omega, \quad (3.36a)$$

$$w_{\epsilon n}(-1) = w_{\epsilon n}(1) = 0, \quad (3.36b)$$

where

$$R_2^n = \epsilon^{2n+2}(u_{lxx}^{2n} \cup u_{rxx}^{2n}); \quad \text{note } u_l^{2n-1} = u_r^{2n-1} = 0, \quad (3.36c)$$

$$R_3^n = \sum_{j=0}^{2n} \epsilon^j (\theta_{lx}^j \cup \theta_{rx}^j) R^{j,2n}(b), \quad R_4^n = \sum_{j=0}^{2n} \epsilon^j \zeta_x^j R^{j,2n}(b); \tag{3.36d}$$

the $R^{j,2n}(b)$ are as in (2.21c).

Notice that from (3.14) $|u_{lxx}^j(0^-)|$, $|u_{rxx}^j(0^+)| \leqslant \kappa_j$, $0 \leqslant j \leqslant 2n+1$ if we take $N=2n+1$ in the compatibility conditions (3.1).

We then estimate the L^2-norms of R_2^n, R_3^n, and R_4^n as follows. We first easily find that

$$|R_2^n|_{L^2(\Omega)} \leqslant \kappa_n \epsilon^{2n+2}. \tag{3.37}$$

Using (2.22) and the pointwise estimates (3.26), we find as for (2.23):

$$|R_3^n| \leqslant \kappa_n \epsilon^{2n+2} \sum_{j=0}^{2n} |\bar{x}|^{2n+2-j} (|\theta_{rx}^j| \chi_{(0,1]} + |\theta_{lx}^j| \chi_{[-1,0)}) \leqslant \kappa_n \epsilon^{2n+2} \exp\left(-\frac{c|x|}{2\epsilon}\right), \tag{3.38}$$

where $\chi_A(x)$ is the characteristic function of the set A, and

$$|R_3^n|_{L^2(\Omega)} \leqslant \kappa \epsilon^{2n+5/2}, \quad |R_4^n|_{L^2(\Omega)} \leqslant \kappa \epsilon^{2n+3/2}. \tag{3.39}$$

We thus obtain the following theorem. Here and after it is convenient to introduce a notation $\phi = \phi(\epsilon, m)$ meaning:

$$\phi(\epsilon, m) = \begin{cases} 1 & \text{for } m = 0 \\ \epsilon^{-1} & \text{for } m = 1 \\ \epsilon^{-3} & \text{for } m = 2. \end{cases} \tag{3.40}$$

Theorem 3.1: *Assume that the compatibility conditions (3.1) hold with $N=2n+1$. Let u^ϵ be the solution of (1.1) with $\alpha=\beta=0$. Then there exists a constant $\kappa_n > 0$ independent of ϵ such that for $m=0,1,2$,*

$$\|u^\epsilon - \xi_{\epsilon n} - \eta_{\epsilon n} - \zeta_{\epsilon n}\|_{H^m(\Omega)} \leqslant \kappa_n \epsilon^{2n+3/2} \phi(\epsilon, m), \tag{3.41}$$

where $\xi_{\epsilon n}$, $\eta_{\epsilon n}$, and $\zeta_{\epsilon n}$ are as in (3.35b) and ϕ is as in (3.40).

Proof: From (3.37) and (3.39), the right-hand side of (3.36a) is majorized by $\kappa_n \epsilon^{2n+3/2}$ in the L^2-norm. The lemma follows applying Lemma 1.1 to Eq. (3.36) with $u = w_{\epsilon n}$. □

IV. ASYMPTOTIC ANALYSIS III: *f*, *b* NONCOMPATIBLE, $\alpha=\beta=0$

We now want to remove the compatibility conditions (3.1). For that purpose, we decompose f into $\hat{f}$ and B_j, as explained in the following, with

$$\hat{f} = f - \sum_{k=0}^{N} \gamma_k B_k(x), \tag{4.1a}$$

where

$$B_0 = b_x(x), \quad B_1 = b(x), \tag{4.1b}$$

$$B_{k+2} = b(x) \int_0^x B_k(s) \mathrm{d}s, \quad k \geqslant 0. \tag{4.1c}$$

Note that since $\mathrm{d}^i B_k / \mathrm{d}x^i(0) = 0$ for $i < k$ and $\mathrm{d}^i B_k / \mathrm{d}x^i(0) \neq 0$ for $i = k$ (recall $b(0)=0$ and $b_x(0) \geqslant 1$), we can recursively find all the γ_k, $k \geqslant 0$ so that the compatibility conditions (3.1) for $f = \hat{f}$ holds for $0 \leqslant i \leqslant N$ (e.g., $\gamma_0 = f(0) b_x(0)^{-1}$, $\gamma_1 = (f_x(0) - \gamma_0 b_{xx}(0)) b_x(0)^{-1}$), that is the first $N+1$ terms of the Taylor series expansion of $\hat{f}$ vanish. Hence, for $\hat{f}$, the asymptotic analysis of Sec. III applies, f

being replaced by $\hat{f}$. We thus only have to consider the cases of $f=B_k$, $k\geqslant 0$ which is now our task in this section.

We consider throughout this section the case $f=B_k(x)$, $k\geqslant 0$, $\alpha=\beta=0$. As indicated before, the compatibility conditions (3.1) are not satisfied in this case (at least for $i\geqslant k$; note that $\mathrm{d}^i f/\mathrm{d}x^i(0)=\mathrm{d}^i B_k/\mathrm{d}x^i(0)\neq 0$ for $i=k$) and the outer solutions u_l^j, u_r^j may display singularities at $x=0$ which we now describe. But we will observe that only for $f=B_k(x)$, $k=2J$, $J\geqslant 0$ integer, the outer solutions are singular at $x=0$. For $f=B_k(x)$, $k=2J+1$, it turns out that the outer solutions are bounded in the neighborhood of $x=0$ and this enables us to perform the same asymptotic analysis as in Sec. III.

Hence let us first examine the simper case where $f=B_k$ and k is odd. We claim that for $f=B_{2J+1}(x), \forall j, \forall m\geqslant 0$,

$$\left|\frac{\mathrm{d}^m u_l^j}{\mathrm{d}x^m}(0^-)\right|, \quad \left|\frac{\mathrm{d}^m u_r^j}{\mathrm{d}x^m}(0^+)\right| \leqslant \kappa_{jm}. \tag{4.2}$$

Indeed, since $f=B_{2J+1}$, we can recursively perform the following calculations. For $x\in[-1,0)$, and $d=0,\ldots,J$:

$$u_{lx}^0 = -f/b = -\int_0^x B_{2J-1}(s)\mathrm{d}s, \tag{4.3a}$$

$$u_{lx}^{2d} = -u_{lxx}^{2d-2}/b = (-1)^{d+1}\int_0^x B_{2J-1-2d}(s)\mathrm{d}s, \quad d=1,\ldots,J-2, \tag{4.3b}$$

$$u_{lx}^{2(J-1)} = -u_{lxx}^{2(J-2)}/b = (-1)^J\int_0^x B_1(s)\mathrm{d}s = (-1)^J\int_0^x b(s)\mathrm{d}s, \tag{4.3c}$$

$$u_{lx}^{2J} = -u_{lxx}^{2(J-1)}/b = (-1)^{J+1}, \tag{4.3d}$$

and $u_{lx}^j=0$ for all $j\geqslant 2J+2$; recall that $u_l^j=0$ for j odd. Hence, all of the right-hand sides of (4.3) are smooth and thus our claim follows. The estimates for u_r^j can be similarly deduced.

Thanks to (4.2), we can perform for $f=B_{2J+1}$ the same asymptotic analysis as we have done in Sec. III. The asymptotic errors are thus similarly deduced leading to Theorem 4.1.

Theorem 4.1: *Let u^ϵ be the solution of (1.1) with $f=B_{2J+1}$, $J\geqslant 0$, $\alpha=\beta=0$. Then there exists a constant $\kappa_n>0$ independent of ϵ such that for all $n\geqslant 0$, $m=0,1,2$,*

$$\|u^\epsilon-\xi_{\epsilon n}-\eta_{\epsilon n}-\zeta_{\epsilon n}\|_{H^m(\Omega)} \leqslant \kappa_n\epsilon^{2n+3/2}\phi(\epsilon,m), \tag{4.4}$$

where $\xi_{\epsilon n}$, $\eta_{\epsilon n}$, and $\zeta_{\epsilon n}$ are as in (3.35b) and ϕ is as in (3.40).

Remark 4.1: From Lemma 2.1 for the ζ^j and Lemma 3.2 we find that

$$\begin{aligned}|\zeta^j(\bar{x})+\theta_r^j(\bar{x})| &\leqslant |\zeta^j-\zeta^j(x=1)|+|\theta_r^j-\theta_r^j(x=1)| \leqslant \left|\int_{\bar{x}}^\infty \zeta_{\bar{x}}^j(s)\mathrm{d}s\right| + \left|\int_{\bar{x}}^\infty \theta_{r\bar{x}}^j(s)\mathrm{d}s\right| \\ &\leqslant \kappa_j(1+\epsilon)\exp(-c\bar{x}) \quad \text{for } \bar{x}>0.\end{aligned} \tag{4.5}$$

The estimates for $|\zeta^j(\bar{x})+\theta_l^j(\bar{x})|$, $\bar{x}<0$, can be similarly deduced. In particular, we can conclude that for the case f satisfying the compatibility conditions (3.1) with $N=1$ or for $f=B_{2J+1}$ (not satisfying the compatibility conditions (3.1)), thanks to (4.5), Theorems 3.1 and 4.1, we have

$$|u^\epsilon - u^0|_{L^2(-1,1)} \leqslant |u^\epsilon - u_l^0|_{L^2(-1,0)} + |u^\epsilon - u_r^0|_{L^2(0,1)} \leqslant \kappa|u^\epsilon - u_l^0 \cup \theta_r^0 - \theta_l^0 \cup u_r^0 - \zeta^0|_{L^2(-1,1)} + \kappa\epsilon^{1/2}$$
$$\leqslant \kappa\epsilon^{1/2}, \tag{4.6}$$

and thus $u^\epsilon = u^0 + O(\epsilon^{1/2})$ in $L^2(\Omega)$ where $u^0 = u_l^0 \cup u_r^0$, u_l^0 and u_r^0 as in (3.3a). As in Remark 2.2, we can, for $n=0$, write the left-hand side of (4.4) as $\|u^\epsilon - u^0 - (\xi_{\epsilon 0} + \eta_{\epsilon 0} + \zeta_{\epsilon 0} - u^0)\|_{H^m(\Omega)}$, where the corrector $\xi_{\epsilon 0} + \eta_{\epsilon 0} + \zeta_{\epsilon 0} - u^0$ vanishes at $x=-1$ and 1, and is discontinuous at $x=0$.

We now analyze the more involved case where $f = B_{2J}$. Before we proceed with the asymptotic analysis, we need to measure the singularities of u_l^j, u_r^j near $x=0$ which are provided by the following lemma.

Lemma 4.1: For j even $\geqslant 0$, if $f = B_{2J}$, $J \geqslant 0$, the outer solutions $u^j = u_l^j \chi_{[-1,0)} + u_r^j \chi_{(0,1]}$ are estimated as follows: there exists a positive constant κ_{jm} independent of x such that, for $|x| > 0$,

$$\left|\frac{\mathrm{d}^m u^j}{\mathrm{d}x^m}(x)\right| \leqslant \kappa_{jm} \begin{cases} 1 & \text{for } m \geqslant 0 \text{ and } j \leqslant 2(J-1) \\ -\ln(|x|) & \text{for } m = 0 \text{ and } j = 2J \\ |x|^{-(j-2J)} & \text{for } m = 0 \text{ and } j \geqslant 2(J+1) \\ |x|^{-(j+m-2J)} & \text{for } m \geqslant 1 \text{ and } j \geqslant 2J. \end{cases} \tag{4.7}$$

Proof: We claim that for j even, $j \geqslant 2J, m \geqslant 1$,

$$\left|\frac{\mathrm{d}^m u^j}{\mathrm{d}x^m}(x)\right| \leqslant \kappa_{jm}|x|^{-(j+m-2J)}. \tag{4.8}$$

Indeed, let $f = B_0(x) = b_x(x)$, i.e., $J=0$. We then use two inductions on j and m. We first verify (4.8) for $j=0$ as follows. For $j=0$, $m=1$, from the outer equation (2.1a), we verify that

$$|u_x^0| \leqslant \left|\frac{b_x}{b}\right| \leqslant \left|\frac{x b_x}{b}\right|\left|\frac{1}{x}\right| \leqslant \frac{\kappa}{|x|}. \tag{4.9}$$

We assume that (4.8) is valid for $j=0$, $m \leqslant s$. We then verify that (4.8) holds for $j=0$, $m=s+1$. Differentiating (2.1a) s times in x, we find

$$-b\frac{\mathrm{d}^{s+1}u^0}{\mathrm{d}x^{s+1}} = \sum_{r=1}^{s} \binom{s}{r} \frac{\mathrm{d}^r b}{\mathrm{d}x^r}\frac{\mathrm{d}^{s-r+1}u^0}{\mathrm{d}x^{s-r+1}} + \frac{\mathrm{d}^{s+1}b}{\mathrm{d}x^{s+1}}. \tag{4.10}$$

Hence it is not hard to find that

$$\left|\frac{\mathrm{d}^{s+1}u^0}{\mathrm{d}x^{s+1}}\right| \leqslant \kappa_s |b|^{-1}\left\{\sum_{r=1}^{s} |x|^{-(s-r+1)} + 1\right\} \leqslant \kappa_s |x|^{-(s+1)}. \tag{4.11}$$

We thus verified (4.8) for $j=0$, $m \geqslant 1$ when $J=0$. We now assume for $J=0$ that for all even $j \leqslant 2n$, $m \geqslant 1$, the claim (4.8) is valid. We then verify the case $j = 2(n+1)$ as follows. From the outer equation (2.1c) we find that the case $j=2(n+1)$, $m=1$ is valid observing that

$$\left|\frac{\mathrm{d}u^{2(n+1)}}{\mathrm{d}x}\right| \leqslant |b|^{-1}\left|\frac{\mathrm{d}^2 u^{2n}}{\mathrm{d}x^2}\right| \leqslant \kappa_n |b|^{-1}|x|^{-2(n+1)} \leqslant \kappa_n |x|^{-2(n+1)-1}. \tag{4.12}$$

Assume that (4.8) is valid for $j=2(n+1)$, $m \leqslant s$. For $j=2(n+1)$, $m=s+1$, as for (4.10), we find with (2.1c) that

$$-b\frac{\mathrm{d}^{s+1}u^{2(n+1)}}{\mathrm{d}x^{s+1}} = \sum_{r=1}^{s} \binom{s}{r} \frac{\mathrm{d}^r b}{\mathrm{d}x^r}\frac{\mathrm{d}^{s-r+1}u^{2(n+1)}}{\mathrm{d}x^{s-r+1}} + \frac{\mathrm{d}^{s+2}u^{2n}}{\mathrm{d}x^{s+2}}. \tag{4.13}$$

Hence

$$\left| \frac{\mathrm{d}^{s+1}u^{2(n+1)}}{\mathrm{d}x^{s+1}} \right| \leqslant \kappa_s |b|^{-1} \left\{ \sum_{r=1}^{s} |x|^{-(s-r+1+2(n+1))} + |x|^{-(s+2+2n)} \right\} \leqslant \kappa_s |x|^{-(s+1+2(n+1))}.$$

We thus proved that the claim (4.8) is valid for all even $j \geqslant 0$, $m \geqslant 1$ when $J=0$.

We now consider the case $f=B_{2J}$, $J \geqslant 0$. We first recursively find: for $x \in [-1,0) \cup (0,1]$, and $d=0,\ldots,J$,

$$u_x^0 = -f/b = -\int_0^x B_{2(J-1)}(s)\mathrm{d}s, \tag{4.14a}$$

$$u_x^{2d} = -u_{xx}^{2(d-1)}/b = (-1)^{d+1}\int_0^x B_{2(J-1)-2d}(s)\mathrm{d}s, \quad d=1,\ldots,J-2, \tag{4.14b}$$

$$u_x^{2(J-1)} = -u_{xx}^{2(J-2)}/b = (-1)^J \int_0^x B_0(s)\mathrm{d}s = (-1)^J \int_0^x b_x(s)\mathrm{d}s, \tag{4.14c}$$

$$u_x^{2J} = -u_{xx}^{2(J-1)}/b = (-1)^{J+1} b_x/b. \tag{4.14d}$$

Hence the analysis for $j=2J$ is repeating that for $j=0$ and thus (4.8) follows.

For $m=0$, j even, $j \geqslant 2J$, we notice from (4.8) that, for $x>0$,

$$|u^j| \leqslant \int_x^1 |u_x^j(s)|\mathrm{d}s \leqslant \kappa_j \int_x^1 s^{-(j+1-2J)}\mathrm{d}s \leqslant \kappa_j \begin{cases} -\ln(x) & \text{for } j=2J, \\ x^{-(j-2J)} & \text{for } j \geqslant 2J+1; \end{cases} \tag{4.15}$$

the case $x<0$ follows similarly.

For $m \geqslant 0$, j even, $0 \leqslant j \leqslant 2(J-1)$, the right-hand sides of (4.14a), (4.14b), and (4.14c) are smooth and thus

$$\left| \frac{\mathrm{d}^m u^j}{\mathrm{d}x^m}(x) \right| \leqslant \kappa_{jm} \quad \text{for } |x|>0. \tag{4.16}$$

Hence the lemma follows. □

Since $\mathrm{d}^i f/\mathrm{d}x^i(0) = \mathrm{d}^i B_{2J}/\mathrm{d}x^i(0) = 0$, $i=0,\ldots,2J-1$, we conclude that Theorem 3.1 holds with $f=B_{2J}$, $J \geqslant 0$ for $n \leqslant J-1$. But for $n \geqslant J$, from Lemma 4.1 we observe the logarithmic or power singularities at $x=0$ due to u^j, $j \geqslant 2J$. To handle these singularities, we introduce the interior layers as follows.

A. Interior layers $\tilde{\theta}_r^j, \tilde{\theta}_l^j, \tilde{\xi}^j$

Similar to θ_r^j, θ_l^j, we define the interior layers $\tilde{\theta}_r^j, \tilde{\theta}_l^j$ but, to avoid the singularities of u^j at $x=0$ as indicated in Lemma 4.1, we this time match the $\tilde{\theta}_r^j$ to the u_l^j at $x=-\epsilon$ and the $\tilde{\theta}_l^j$ to the u_r^j at $x=\epsilon$. The interior layers $\tilde{\theta}_l^j(\bar{x})$, $\tilde{\theta}_r^j(\bar{x})$ satisfy the interior layer equations (2.4) but on $(-\infty,1)$, $(-1,\infty)$, respectively, with boundary conditions:

$$\tilde{\theta}_r^j(\bar{x}) = u_l^j(x), \quad \tilde{\theta}_{r\bar{x}}^j(\bar{x}) = \epsilon\tilde{\theta}_{rx}^j = \epsilon u_{lx}^j(x) \text{ at } x=-\epsilon \ (\bar{x}=-1), \tag{4.17a}$$

$$\tilde{\theta}_l^j(\bar{x}) = u_r^j(x), \quad \tilde{\theta}_{l\bar{x}}^j(\bar{x}) = \epsilon\tilde{\theta}_{lx}^j = \epsilon\tilde{\theta}_{rx}^j(x) \text{ at } x=\epsilon \ (\bar{x}=1). \tag{4.17b}$$

We are then able to find explicit solutions $\tilde{\theta}_r^j, \tilde{\theta}_l^j$ and in particular, for $j=0,1$, we find that

$$\tilde{\theta}_r^0 = \exp\left(\frac{b_1}{2}\right)\epsilon u_{lx}^0(-\epsilon)\int_{-1}^{\bar{x}} \exp\left(-\frac{b_1 s^2}{2}\right)\mathrm{d}s + u_l^0(-\epsilon), \tag{4.18a}$$

$$\tilde{\theta}_r^1 = \exp\left(\frac{b_1}{2}\right)\epsilon u_{lx}^0(-\epsilon) b_2 3^{-1}\int_{-1}^{\bar{x}} (1+s^3)\exp\left(-\frac{b_1 s^2}{2}\right)\mathrm{d}s. \tag{4.18b}$$

Notice that as $\bar{x}\to\infty$, we easily find that

$$\tilde{\theta}_r^0 \to \exp\left(\frac{b_1}{2}\right)\epsilon u_{lx}^0(-\epsilon)\tilde{c}_{r,0} + u_l^0(-\epsilon) =: \tilde{c}_{r,\infty}^0(\epsilon), \tag{4.19a}$$

$$\tilde{\theta}_r^1 \to \exp\left(\frac{b_1}{2}\right)\epsilon u_{lx}^0(-\epsilon) b_2 3^{-1}(\tilde{c}_{r,0}+\tilde{c}_{r,1}) =: \tilde{c}_{r,\infty}^1(\epsilon), \tag{4.19b}$$

where $\tilde{c}_{r,0}=\int_{-1}^{\infty}\exp(-b_1 s^2/2)\mathrm{d}s$, $\tilde{c}_{r,1}=\int_{-1}^{\infty}s^3\exp(-b_1 s^2/2)\mathrm{d}s$.

As before, we denote by $u_l^j \sqcup \tilde{\theta}_r^j$ (respectively $\tilde{\theta}_l^j \sqcup u_r^j$) the function on $(-1,1)$ equal to u_l^j (respectively, $\tilde{\theta}_l^j$) on $(-1,-\epsilon)$ (respectively, $(-1,\epsilon)$) and to $\tilde{\theta}_r^j$ (respectively, u_r^j) on $(-\epsilon,1)$ (respectively, $(\epsilon,1)$). Note that due to (4.17), these functions belong to $C^1([-1,1])$ and to $H^2(-1,1)$; see Figs. 2(c) and 2(f):

We now estimate the interior layers $\tilde{\theta}_r^j, \tilde{\theta}_l^j$. We first claim that for $\bar{x}\in[-1,\infty)$,

$$\tilde{\theta}_{r\bar{x}}^j = P_{3j}(\bar{x})\exp\left(-\frac{b_1\bar{x}^2}{2}\right), \quad \forall\, j \geq 0, \tag{4.20}$$

where $P_{3j}(\bar{x})$ is a polynomial in $\bar{x}$ of degree $3j$ with (unlike before) the absolute value of its coefficients bounded by $\kappa_j(\epsilon+\epsilon^{-(j-2J)})$.

To prove this claim, we first notice that $u_l^j=0$ for j odd and that from (4.7), for j even, we have

$$|u_l^j(-\epsilon)| \leq \kappa_j\begin{cases}1+\epsilon^{-(j-2J)} & \text{for } j\neq 2J,\\ -\ln(\epsilon) & \text{for } j=2J,\end{cases} \qquad |u_{lx}^j(-\epsilon)| \leq \kappa_j(1+\epsilon^{-(j+1-2J)}). \tag{4.21}$$

For $j=0$, the claim (4.20) is easily verified from (4.18); we notice that $P_0(\bar{x})=e^{b_1/2}\epsilon u_{lx}^0(-\epsilon)$ and from (4.21) we find that $|P_0(\bar{x})|$ is bounded by $\kappa_0(\epsilon+\epsilon^{2J})$. Assume that (4.20) holds for $j\leq n$. For $j=n+1$, from Eq. (2.4c) with θ^j being replaced by $\tilde{\theta}_r^j$, we find as in (2.8) that

$$-\left\{\tilde{\theta}_{r\bar{x}}^{n+1}\exp\left(\frac{b_1\bar{x}^2}{2}\right)\right\}_{\bar{x}} = \sum_{k=0}^{n} b_{n-k+2}\bar{x}^{n-k+2}P_{3k}(\bar{x}) = P_{3n+2}(\bar{x}). \tag{4.22}$$

By our assumption the absolute values of the coefficients of the $P_{3k}(\bar{x})$, $k=0,\ldots,n$, are bounded by $\kappa_n(\epsilon+\epsilon^{-(n-2J)})$, and so are those of $P_{3n+2}(\bar{x})$ in (4.22) and $P_{3(n+1)}(\bar{x})$ in (4.23) and (4.24). From (4.22) we thus find that for a constant C_{n+1} independent of ϵ,

$$\tilde{\theta}_{r\bar{x}}^{n+1} = (P_{3(n+1)}(\bar{x})+C_{n+1})\exp\left(-\frac{b_1\bar{x}^2}{2}\right). \tag{4.23}$$

Hence, by (4.21) and the boundary conditions (4.17) at $\bar{x}=-1$, we find that

$$|C_{n+1}| \leq |P_{3(n+1)}(\bar{x}=-1)| + \exp\left(\frac{b_1}{2}\right)\epsilon|u_{lx}^{n+1}(-\epsilon)| \leq \kappa_{n+1}(\epsilon+\epsilon^{-(n+1-2J)}). \tag{4.24}$$

Therefore the absolute values of the coefficients in the polynomial $P_{3(n+1)}(\bar{x})+C_{n+1}$ corresponding to the $\tilde{\theta}_{r\bar{x}}^{n+1}$ are bounded by $\kappa_{n+1}(\epsilon+\epsilon^{-(n+1-2J)})$ as we want. We thus verified our claim (4.20) for all $j\geq 0$.

The following pointwise and norm estimates can be derived.

Lemma 4.2: Let $j\geqslant 0$, $f=B_{2J}$, $J\geqslant 0$ *and*

$$\varrho(\epsilon,j,J)=\begin{cases}1+\epsilon^{-(j-2J)} & \text{for } j\neq 2J\\ -\ln(\epsilon) & \text{for } j=2J.\end{cases} \tag{4.25}$$

Then there exist positive constants κ_{jm} *and* c *such that for* $x\in[-\epsilon,1]$,

$$\left|\frac{\mathrm{d}^m\tilde{\theta}_r^j}{\mathrm{d}x^m}\right|\leqslant \kappa_{jm}\begin{cases}\varrho(\epsilon,j,J) & \text{for } m=0\\ (\epsilon+\epsilon^{-(j-2J)})\epsilon^{-m}\exp\left(-c\dfrac{|x|}{\epsilon}\right) & \text{for } m\geqslant 1,\end{cases} \tag{4.26}$$

and for $\sigma\in[-\epsilon,1)$,

$$|\tilde{\theta}_r^j|_{H^m(\sigma,1)}\leqslant \kappa_{jm}\begin{cases}\varrho(\epsilon,j,J) & \text{for } m=0\\ (\epsilon+\epsilon^{-(j-2J)})\epsilon^{-m+1/2}\exp\left(-c\dfrac{|\sigma|}{\epsilon}\right) & \text{for } m\geqslant 1.\end{cases} \tag{4.27}$$

In particular, for $m\geqslant 0$,

$$|\tilde{\theta}_r^j|_{H^m(-\epsilon,1)}\leqslant \kappa_{jm}(\varrho(\epsilon,j,J)+(\epsilon+\epsilon^{-(j-2J)})\epsilon^{-m+1/2}). \tag{4.28}$$

Furthermore, there exist constants $\tilde{c}_{r,\infty}^j(\epsilon)$ *with* $|\tilde{c}_{r,\infty}^j(\epsilon)|\leqslant \kappa_j\varrho(\epsilon,j,J)$ *such that*

$$\tilde{\theta}_r^j(\bar{x})\to \tilde{c}_{r,\infty}^j(\epsilon)\quad \text{as } \bar{x}\to\infty. \tag{4.29}$$

Proof: We derive (4.26), $m\geqslant 1$, from (4.20) as we did for (2.18)–(2.19). For $m=0$, we notice that from (4.20) and (4.21)

$$|\tilde{\theta}_r^j|=\left|\int_{-1}^{\bar{x}}\tilde{\theta}_{r\bar{x}}^j(s)\mathrm{d}s+u_l^j(-\epsilon)\right|\leqslant \kappa_j(\epsilon+\epsilon^{-(j-2J)})\int_{-1}^{\infty}\exp\left(-\frac{b_1s^2}{4}\right)\mathrm{d}s$$
$$+\kappa_j\begin{cases}1+\epsilon^{-(j-2J)} & \text{for } j \text{ even}\neq 2J\\ -\ln(\epsilon) & \text{for } j=2J\\ 0 & \text{for } j \text{ odd}\end{cases}, \tag{4.30}$$

and thus (4.26) follows.

The norm estimates (4.27) and (4.28) are deduced directly from (4.26).

The convergence (4.29) follows as in (3.31) and (3.32). □

Remark 4.2: We can similarly perform the analysis for $\tilde{\theta}_l^j$ and derive the pointwise and norm estimates as above. Here we denote by $\tilde{c}_{l,\infty}^j$ the limit of $\tilde{\theta}_l^j$ as $\bar{x}\to-\infty$.

By our constructions, as in the analysis of θ_r^j and θ_l^j, we then notice that, for $j\geqslant 0$, the function $\tilde{g}^j:=-(u_l^j\sqcup\tilde{\theta}_r^j)-(\tilde{\theta}_l^j\sqcup u_r^j)$ attain the values $-\tilde{\theta}_l^j=-\tilde{c}_{l,\infty}^j(\epsilon)+e.s.t.$ at $x=-1$ and $-\tilde{\theta}_r^j=-\tilde{c}_{r,\infty}^j(\epsilon)+e.s.t.$ at $x=1$. To remedy these discrepancies between $\tilde{g}^j$ and u^ϵ at the boundaries $x=-1,1$ (recall that $u^\epsilon(-1)=u^\epsilon(1)=0$), we introduce interior layers $\tilde{\zeta}^j$ similar to θ^j but we use different boundary conditions as follows: the $\tilde{\zeta}^j=\tilde{\zeta}^j(\bar{x})$ satisfy (2.4) and

$$\tilde{\zeta}^j=-\tilde{\theta}_l^j\quad \text{at } x=-1,\quad \tilde{\zeta}^j=-\tilde{\theta}_r^j\quad \text{at } x=1 \text{ for } j\geqslant 0. \tag{4.31}$$

As before we are able to obtain explicit solutions. In particular,

$$\tilde{\zeta}^0=\vartheta^0(-\tilde{c}_{l,\infty}^0(\epsilon),-\tilde{c}_{r,\infty}^0(\epsilon))+e.s.t., \tag{4.32a}$$

$$\tilde{\zeta}^1 = \vartheta^1(-\tilde{c}^0_{l,\infty}(\epsilon), -\tilde{c}^0_{r,\infty}(\epsilon)) + \vartheta^0(-\tilde{c}^1_{l,\infty}(\epsilon), -\tilde{c}^1_{r,\infty}(\epsilon)) + e.s.t., \tag{4.32b}$$

where ϑ^0 and ϑ^1 are as in (3.34a).

The pointwise and norm estimates for $\tilde{\zeta}^j$ follow below:

Lemma 4.3: For $j \geqslant 0$, there exist positive constants κ_{jm} and c such that

$$\left|\frac{\mathrm{d}^m \tilde{\zeta}^j}{\mathrm{d}x^m}\right| \leqslant \kappa_{jm} \varrho(\epsilon,j,J) \begin{cases} 1 & \text{for } m=0 \\ \epsilon^{-m} \exp\left(-c\dfrac{|x|}{\epsilon}\right) & \text{for } m \geqslant 1, \end{cases} \tag{4.33}$$

and for $\sigma \in [0,1)$,

$$|\tilde{\zeta}^j|_{H^m((-1,-\sigma)\cup(\sigma,1))} \leqslant \kappa_{jm} \varrho(\epsilon,j,J) \begin{cases} 1 & \text{for } m=0 \\ \epsilon^{-m+1/2} \exp\left(-c\dfrac{\sigma}{\epsilon}\right) & \text{for } m \geqslant 1. \end{cases} \tag{4.34}$$

In particular, for $m \geqslant 0$,

$$|\tilde{\zeta}^j|_{H^m(-1,1)} \leqslant \kappa_{jm} \varrho(\epsilon,j,J)(1+\epsilon^{-m+1/2}). \tag{4.35}$$

Proof: We similarly find that Lemma 2.1 is valid with θ^j being replaced by $\tilde{\zeta}^j$. But we need to take into account the boundary conditions (4.31). Using the approximate form of $\tilde{\zeta}^j$ as for $\bar{\theta}^j$, (4.32a) and the induction in (2.7)–(2.11) with the boundary conditions (4.31), namely

$$D_{n+1} = -\tilde{c}^{n+1}_{l,\infty}(\epsilon), \quad C_{n+1}c_0 = -\tilde{c}^{n+1}_{r,\infty}(\epsilon) + \tilde{c}^{n+1}_{l,\infty}(\epsilon) - \int_{-\infty}^{\infty} P_{3(n+1)}(s)\exp\left(-\frac{b_1 s^2}{2}\right)\mathrm{d}s,$$

the lemma then follows observing that $|\tilde{c}^j_{r,\infty}(\epsilon)|$, $|\tilde{c}^j_{l,\infty}(\epsilon)| \leqslant \kappa_j \varrho(\epsilon,j,J)$ and the absolute values of the coefficients in $P_{3(n+1)}(s)$ above are, by induction arguments, bounded by $\kappa_n \varrho(\epsilon,n,J)$. □

B. Asymptotic errors

Let

$$w_{\epsilon n} = u^\epsilon - \tilde{\xi}_{\epsilon n} - \tilde{\eta}_{\epsilon n} - \tilde{\zeta}_{\epsilon n}, \tag{4.36a}$$

where

$$\tilde{\xi}_{\epsilon n} = \sum_{j=0}^{2n} \epsilon^j (u^j_l \sqcup \tilde{\theta}^j_r), \quad \tilde{\eta}_{\epsilon n} = \sum_{j=0}^{2n} \epsilon^j (\tilde{\theta}^j_l \sqcup u^j_r), \quad \tilde{\zeta}_{\epsilon n} = \sum_{j=0}^{2n} \epsilon^j \tilde{\zeta}^j. \tag{4.36b}$$

After some elementary calculations, we find that

$$L_\epsilon w_{\epsilon n} = R^n_5 + R^n_6 + R^n_7 + R^n_8 + e.s.t. \text{ in } \Omega, \tag{4.37a}$$

$$w_{\epsilon n}(-1) = w_{\epsilon n}(1) = 0, \tag{4.37b}$$

where

$$R^n_5 = \epsilon^{2n+2} u^{2n}_{lxx} \sqcup \left(\sum_{j=0}^{2n} \epsilon^j \tilde{\theta}^j_{rx} R^{j,2n}(b)\right), \quad R^n_6 = \left(\sum_{j=0}^{2n} \epsilon^j \tilde{\theta}^j_{lx} R^{j,2n}(b)\right) \sqcup \epsilon^{2n+2} u^{2n}_{rxx}, \tag{4.37c}$$

$$R_7^n = \sum_{j=0}^{2n} \epsilon^j \tilde{\zeta}_x^j R^{j,2n}(b), \quad R_8^n = f(x)\chi_{[-\epsilon,\epsilon]}(x); \tag{4.37d}$$

the $R^{j,2n}(b)$ are as in (2.21c). We then deduce from Lemma 4.1 that for $x \in [-1,-\epsilon]$,

$$|R_5^n(x)| \leq \kappa_n \epsilon^{2n+2}(1 + |x|^{-(2n+2-2J)}), \tag{4.38}$$

and from Lemma 4.2, similar to (2.22)–(2.23), for $x \in [-\epsilon, 1]$,

$$|R_5^n(x)| \leq \sum_{j=0}^{2n} \epsilon^j \epsilon^{2n+2-j}|\bar{x}|^{2n+2-j}(1 + \epsilon^{-(j+1-2J)})\exp(-c|\bar{x}|) \leq \kappa_n(\epsilon^{2n+2} + \epsilon^{2J+1})\exp\left(-\frac{c|\bar{x}|}{2}\right). \tag{4.39}$$

Hence

$$|R_5^n|_{L^2(\Omega)} \leq \kappa|R_5^n|_{L^2(-1,-\epsilon)} + \kappa|R_5^n|_{L^2(-\epsilon,1)} \leq \kappa_n(\epsilon^{2J+1/2} + \epsilon^{2n+2}), \tag{4.40}$$

and similarly,

$$|R_6^n|_{L^2(\Omega)} \leq \kappa_n(\epsilon^{2J+1/2} + \epsilon^{2n+2}). \tag{4.41}$$

As for (2.23) we find that

$$|R_7^n|_{L^2(\Omega)} \leq \kappa_n \varrho(\epsilon,2n,J)\epsilon^{2n+3/2} \leq \kappa_n \begin{cases} \epsilon^{2J+3/2} + \epsilon^{2n+3/2} & \text{for } n \neq J \\ -\ln(\epsilon)\epsilon^{2J+3/2} & \text{for } n = J \end{cases} \leq \kappa_n(\epsilon^{2J+1/2} + \epsilon^{2n+3/2}). \tag{4.42}$$

Since $f=B_{2J}$, $\mathrm{d}^m f/\mathrm{d}x^m(0)=0$, $m=0,\ldots,2J-1$, and hence from the Taylor theorem $|f(x)| \leq \kappa|x|^{2J}$. We thus find

$$|R_8^n|_{L^2(\Omega)} \leq \kappa_n\left[\int_{-\epsilon}^{\epsilon} |x|^{4J}\mathrm{d}x\right]^{1/2} \leq \kappa_n \epsilon^{2J+1/2}. \tag{4.43}$$

Therefore the following theorem has been proved.

Theorem 4.2: *Let u^ϵ be the solution of (1.1) with $f=B_{2J}$, $J \geq 0$, $\alpha=\beta=0$. As $\epsilon \to 0$, u^ϵ is singular near $x=0$, its singularity being carried by the interior layers $\tilde{\theta}_l^j$, $\tilde{\theta}_r^j$, and $\tilde{\zeta}^j$. Furthermore, there exists a constant $\kappa_n > 0$ independent of ϵ such that for $m=0,1,2$,*

$$\|u^\epsilon - \tilde{\xi}_{\epsilon n} - \tilde{\eta}_{\epsilon n} - \tilde{\zeta}_{\epsilon n}\|_{H^m(\Omega)} \leq \kappa_n(\epsilon^{2J+1/2} + \epsilon^{2n+3/2})\phi(\epsilon,m), \tag{4.44}$$

where $\tilde{\xi}_{\epsilon n}$, $\tilde{\eta}_{\epsilon n}$, and $\tilde{\zeta}_{\epsilon n}$ are as in (4.36b) and ϕ is as in (3.40).

Proof: Using (4.40)–(4.43), the right-hand side of (4.37a) is majorized by $\kappa_n(\epsilon^{2J+1/2} + \epsilon^{2n+3/2})$ in the L^2-norm. The lemma follows applying Lemma 1.1 to Eq. (4.37) with $u=w_{\epsilon n}$. □

Remark 4.3: We note from Theorem 4.2 that increasing both J and n improves the asymptotic errors.

Remark 4.4: We infer from Theorem 4.2 that the solution u^ϵ corresponding to $f=B_0(x)$ possesses the most severe singularities. In this case, we have $f=B_0(x)=b_x(x)$ and $u_l^0=-\ln(|b(x)|/|b(-1)|)$ for $x \in [-1,0)$ and $u_r^0=-\ln(|b(x)|/|b(1)|)$ for $x \in (0,1]$. We can then verify that $u^\epsilon \to u^0$ in $L^2(\Omega)$ as $\epsilon \to 0$, where $u^0=u_l^0 \sqcup u_r^0$. Indeed, from Theorem 4.2

$$|u^\epsilon - u_l^0 \sqcup \tilde{\theta}_r^0 - \tilde{\theta}_l^0 \sqcup u_r^0 - \tilde{\zeta}^0|_{L^2(\Omega)} \leq \kappa \epsilon^{1/2}. \tag{4.45}$$

Notice that from (4.31) and (4.33) we find that for $\bar{x} > 0$,

$$|\tilde{\zeta}^0(\bar{x}) + \tilde{c}^0_{r,\infty}(\epsilon)| \leq \left| \int_{\bar{x}}^{\infty} \tilde{\zeta}^0_{\bar{x}}(s)\mathrm{d}s \right| + e.s.t. \leq -\kappa \ln(\epsilon)\exp(-c\bar{x}), \tag{4.46}$$

from (4.26), (4.29) for $\bar{x}>0$,

$$|\tilde{\theta}^0_r - \tilde{c}^0_{r,\infty}(\epsilon)| = \left| \int_{\bar{x}}^{\infty} \tilde{\theta}^0_{r\bar{x}}(s)\mathrm{d}s \right| \leq -\kappa \exp(-c\bar{x}), \tag{4.47}$$

and thus

$$|\tilde{\theta}^0_r + \tilde{\zeta}^0| \leq |\tilde{\theta}^0_r - \tilde{c}^0_{r,\infty}(\epsilon)| + |\tilde{\zeta}^0 + \tilde{c}^0_{r,\infty}(\epsilon)| \leq -\kappa \ln(\epsilon)\exp(-c\bar{x}). \tag{4.48}$$

Similarly for $\bar{x}<0$, $|\tilde{\theta}^0_l + \tilde{\zeta}^0| \leq -\kappa \ln(\epsilon)\exp(c\bar{x})$. Hence for $0<\lambda<1$,

$$|u^\epsilon - u^0_l|_{L^2(-1,-\epsilon^\lambda)} \leq |u^\epsilon - u^0_l \sqcup \tilde{\theta}^0_r - \tilde{\theta}^0_l \sqcup u^0_r - \tilde{\zeta}^0|_{L^2(-1,-\epsilon^\lambda)} - \kappa\epsilon^{1/2}\ln(\epsilon)\exp(-c\epsilon^{\lambda-1}). \tag{4.49}$$

From (4.45) we find that $|u^\epsilon - u^0_l|_{L^2(-1,-\epsilon^\lambda)} \leq \kappa\epsilon^{1/2}$, similarly $|u^\epsilon - u^0_r|_{L^2(\epsilon^\lambda,1)} \leq \kappa\epsilon^{1/2}$. As in Remark 2.2, we can, for $n=0$, write the left-hand side of (4.44) as $\|u^\epsilon - u^0 - (\tilde{\xi}_{\epsilon 0} + \tilde{\eta}_{\epsilon 0} + \tilde{\zeta}_{\epsilon 0} - u^0)\|_{H^m(\Omega)}$, where the corrector $\tilde{\xi}_{\epsilon 0} + \tilde{\eta}_{\epsilon 0} + \tilde{\zeta}_{\epsilon 0} - u^0$ vanishes at $x=-1$ and 1, and is discontinuous at $x=0$.

Remark 4.5: Combining the results of Secs. II–IV we are able to consider the case where f and b are noncompatible and α and β are arbitrary. We just write $u^\epsilon = u^\epsilon_1 + u^\epsilon_2 + u^\epsilon_3$ with

$$L_\epsilon u^\epsilon_1 = 0, \quad u^\epsilon_1(-1) = \alpha, \quad u^\epsilon_1(1) = \beta,$$

$$L_\epsilon u^\epsilon_2 = f - \sum_{k=0}^{N} \gamma_k B_k(x), \quad u^\epsilon_2(-1) = u^\epsilon_2(1) = 0, \tag{4.50}$$

$$L_\epsilon u^\epsilon_3 = \sum_{k=0}^{N} \gamma_k B_k(x), \quad u^\epsilon_3(-1) = u^\epsilon_3(1) = 0.$$

The asymptotic behavior of the solutions of each problem is analyzed, respectively, in Secs. II–IV.

V. EXAMPLES

Before we present some applications of the results above, we start with the following useful theorem:

Theorem 5.1: *Assume that $b(x)=-b(-x)$ in the neighborhood of $x=0$ and the following compatibility conditions hold:*[4]

$$\frac{\mathrm{d}^{2i}f}{\mathrm{d}x^{2i}}(0) = 0 \quad \textit{for } 0 \leq i \leq M. \tag{5.1}$$

Let u^ϵ be the solution of (1.1) with $\alpha=\beta=0$. Then there exists a constant $\kappa_n>0$ independent of ϵ such that for $n \leq M$, $m=0,1,2$,

$$\|u^\epsilon - \xi_{\epsilon n} - \eta_{\epsilon n} - \zeta_{\epsilon n}\|_{H^m(\Omega)} \leq \kappa_n \epsilon^{2n+3/2}\phi(\epsilon,m), \tag{5.2}$$

where $\xi_{\epsilon n}$, $\eta_{\epsilon n}$, and $\zeta_{\epsilon n}$ are as in (3.35b) and ϕ is as in (3.40).

Proof: Let

[4]If f is odd (i.e., $f(x)=-f(-x)$) in the neighborhood of $x=0$, (5.1) is obviously satisfied.

$$\hat{f} = f - \sum_{J=0}^{M} \gamma_J B_{2J+1}(x). \tag{5.3}$$

Since $d^l B_{2J+1}/dx^l(0)=0$ for $l<2J+1$ and $\neq 0$ for $l=2J+1$, we may choose γ_J's so that $d^l\hat{f}/dx^l(0)=0$ for $l=2J+1$, $J=0,\ldots,M$. Notice also that since $B_1(x)=b(x)$ is odd in the neighborhood of $x=0$, so is $B_{2J+1}(x)$, $J\geq 0$ and hence $d^l B_{2J+1}/dx^l(0)=0$, for l even. Thus thanks to (5.1), $d^l\hat{f}/dx^l(0)=0$ for l even and hence we have $d^l\hat{f}/dx^l(0)=0$ for $l=0,1,\ldots,2M+1$.

The theorem follows from Theorem 3.1 and Theorem 4.1 observing that for $n\leq M$,

$$\|u^\epsilon - \xi_{\epsilon n} - \eta_{\epsilon n} - \zeta_{\epsilon n}\|_{H^m(\Omega)} \leq \|u^{\epsilon,\hat{f}} - \xi_{\epsilon n}^{\hat{f}} - \eta_{\epsilon n}^{\hat{f}} - \zeta_{\epsilon n}^{\hat{f}}\|_{H^m(\Omega)} + \sum_{J=0}^{M} \|u^{\epsilon,J} - \xi_{\epsilon n}^J - \eta_{\epsilon n}^J - \zeta_{\epsilon n}^J\|_{H^m(\Omega)}. \tag{5.4}$$

Here $u^\epsilon = u^{\epsilon,\hat{f}} + \sum_{J=0}^M u^{\epsilon,J}$; $u^{\epsilon,\hat{f}}$, $u^{\epsilon,J}$ are the solutions corresponding to $\hat{f}$, $\gamma_J B_{2J+1}(x)$, respectively; $\xi_{\epsilon n}^{\hat{f}}$, $\eta_{\epsilon n}^{\hat{f}}$, $\zeta_{\epsilon n}^{\hat{f}}$ are the asymptotic expansions corresponding to $\hat{f}$ and $\xi_{\epsilon n}^J$, $\eta_{\epsilon n}^J$, $\zeta_{\epsilon n}^J$ to $\gamma_J B_{2J+1}(x)$. □

We now show how to apply the Theorem 5.1 and the asymptotic expansions introduced in Secs. II–IV to some simple examples.

Example 5.1: For $b=x$, $f=\sin x$, $\alpha=\beta=0$, we notice that b is odd and $d^m f/dx^m(0)=0$ for m even. From Theorem 5.1 we easily find that (5.2) holds for all $n\geq 0$.

Example 5.2: For $b=\sin x$, $f=x$, $\alpha=\beta=0$, we find that b is odd and $d^m f/dx^m(0)=0$, for $1\neq m\geq 0$. We then easily see that in this case (5.2) holds for all $n\geq 0$.

Example 5.3: For $b=x$, $f=1-e^x$, $\alpha=\beta=0$, we find that b is odd and $f(0)=0$, $d^m f/dx^m(0)\neq 0$, for $m\geq 1$. In this case we find that (5.2) holds for $n=0$. If we utilize the asymptotic expansions corresponding to $B_2(x)=x^2$, introduced in Sec. IV, we can improve the asymptotic errors. We first decompose f into f_1, f_2:

$$f_1 = f - \gamma B_2(x) = 1 - e^x - \gamma x^2, \tag{5.5}$$

$$f_2 = \gamma B_2(x) = \gamma x^2. \tag{5.6}$$

We choose $\gamma=-1/2$ so that $f_1(0)=d^2 f_1/dx^2(0)=0$. From Theorem 5.1, we easily find that for $n=0,1$,

$$\|u^{\epsilon,1} - \xi_{\epsilon n} - \eta_{\epsilon n} - \zeta_{\epsilon n}\|_{H^m(\Omega)} \leq \kappa \epsilon^{2n+3/2}\phi(\epsilon,m), \tag{5.7}$$

where $u^{\epsilon,1}$ and $\xi_{\epsilon n}$, $\eta_{\epsilon n}$, $\zeta_{\epsilon n}$ are the solution and the asymptotic expansions corresponding to f_1 and $\phi(\epsilon,m)$ is given in (3.40). From Theorem 4.2 we find that

$$\|u^{\epsilon,2} - \tilde{\xi}_{\epsilon n} - \tilde{\eta}_{\epsilon n} - \tilde{\zeta}_{\epsilon n}\|_{H^m(\Omega)} \leq \kappa_n(\epsilon^{5/2} + \epsilon^{2n+3/2})\phi(\epsilon,m), \tag{5.8}$$

where $u^{\epsilon,2}$ and $\tilde{\xi}_{\epsilon n}$, $\tilde{\eta}_{\epsilon n}$, $\tilde{\zeta}_{\epsilon n}$ are the solution and the asymptotic expansions corresponding to $f_2=\gamma B_2(x)$. We thus find that for $n=0,1$, $m=0,1,2$,

$$\|u^\epsilon - \xi_{\epsilon n} - \eta_{\epsilon n} - \zeta_{\epsilon n} - \tilde{\xi}_{\epsilon n} - \tilde{\eta}_{\epsilon n} - \tilde{\zeta}_{\epsilon n}\|_{H^m(\Omega)} \leq \kappa(\epsilon^{5/2} + \epsilon^{2n+3/2})\phi(\epsilon,m). \tag{5.9}$$

Example 5.4: For $b=x$, $f=e^x$, $\alpha=\beta=0$, no compatibility condition is satisfied and singularities occur. We decompose f into f_1, f_2:

$$f_1 = f - \gamma B_0(x) = e^x - \gamma, \tag{5.10}$$

$$f_2 = \gamma B_0(x) = \gamma. \tag{5.11}$$

We choose $\gamma=1$ so that $f_1(0)=0$. From Theorem 5.1 as in Example 5.3 we find that for $n=0$,

$$\|u^{\epsilon,1}-u_l^0 \cup \theta_r^0-\theta_l^0 \cup u_r^0-\zeta^0\|_{H^m(\Omega)} \leq \kappa\epsilon^{3/2}\phi(\epsilon,m). \tag{5.12}$$

From Theorem 4.2 we find that

$$\|u^{\epsilon,2}-u_l^0 \sqcup \tilde{\theta}_r^0-\tilde{\theta}_l^0 \sqcup u_r^0-\tilde{\zeta}^0\|_{H^m(\Omega)} \leq \kappa\epsilon^{1/2}\phi(\epsilon,m). \tag{5.13}$$

We thus have

$$\|u^{\epsilon}-u_l^0 \cup \theta_r^{\theta}-\theta_l^0 \cup u_r^0-\zeta^0-u_l^0 \sqcup \tilde{\theta}_r^0-\tilde{\theta}_l^0 \sqcup u_r^0-\tilde{\zeta}^0\|_{H^m(\Omega)} \leq \kappa\epsilon^{1/2}\phi(\epsilon,m). \tag{5.14}$$

For $\alpha\neq 0$ or $\beta\neq 0$ in Examples 5.1–5.4, we just additionally apply Theorem 2.1.

Remark 5.1: In Example 5.4, as we did in Remarks 4.1 and 4.4, we can verify that $u^{\epsilon}\to u^0$ in $L^2(\Omega)$ as $\epsilon\to 0$, where

$$u^0=-\ln(|x|)+\begin{cases}-\displaystyle\int_{-1}^{x}\frac{e^s-1}{s}\mathrm{d}s\\ \displaystyle\int_{x}^{1}\frac{e^s-1}{s}\mathrm{d}s\end{cases}=\begin{cases}-\displaystyle\int_{-1}^{x}\frac{e^s}{s}\mathrm{d}s & \text{for } x\in[-1,0)\\ \displaystyle\int_{x}^{1}\frac{e^s}{s}\mathrm{d}s & \text{for } x\in(0,1].\end{cases} \tag{5.15}$$

VI. CONCLUDING REMARKS

In this article we have studied the turning points which appear in the linear equation (1.1). We have shown the diversity of situations which can occur, including the following: an internal (interior) boundary layer near the turning point which may be supplemented by boundary layers at the end points; or even the occurrence of logarithmic singularities (or negative power singularities) at such points. Expressed in an oversimplified way, the difficulty comes from the fact that the information propagates from the end points into the interior and they meet and possibly collide at the turning point.

We have systematically detected the singular terms (interior layers) due to the turning point at $x=0$ in Secs. II–IV as well as the outer solutions. To obtain the asymptotic errors in the H^m-spaces, $m=0,1,2$, we have smoothly (C^1) matched the outer and the interior layer solutions with the boundary conditions (3.20) and (4.17), which enables us to perform the global analysis on the whole domain $\Omega=(-1,1)$.

Using the standard asymptotic technique with regard to the small parameter ϵ we derived the outer solutions and the interior layer solutions which carry out the singularities or discontinuities of the outer solutions at $x=0$. Employing regularity results for the problem under consideration we obtained in the Sobolev context sharp asymptotic estimates of the error between the exact solution of (1.1) and the asymptotic expressions composed of the outer and interior layer solutions which are matched with H^2 (or C^1)-smoothness.

In the numerical simulations context, understanding turning point behaviors, e.g., monotone transition layers, spikes (see Desanti (1987a, b)), logarithmic singularities, one needs to either design sophisticated (irregular) meshes (see, e.g., Stynes (2005)) or one can utilize the singular functions (splines) which absorb the singularities due to the small ϵ. This approach was used in the context of singular perturbations in, e.g., Cheng and Temam (2002); Cheng *et al.* (2000); Jung and Temam (2005; 2006); and Jung (2005). The idea of utilizing explicit forms of singularities in numerical schemes was also used in different contexts in, e.g., Cai *et al.* (1989); and Hou and Wu (1997). It was shown in these articles that this procedure can save much computing time; these numerical issues will be addressed elsewhere.

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Note added in proofs: After this article went to press we learned of a number of relevant references related to turning point problems. The comparison of these articles with the present one is as follows. The references, Berger *et al.* (1984); Sun and Stynes (1994), discuss the following type of two-point boundary value problem:

$$-\epsilon^2 u^\epsilon_{xx} - b(x)u^\epsilon_x + c(x)u^\epsilon = f(x) \text{ in } (-1,1),$$

$$u^\epsilon(-1) = u^\epsilon(1) = 0,$$

where $b(x)$, $c(x)$, and $f(x)$ are smooth and $c(x) \geqslant \kappa_0 > 0$; $b(x)$ is allowed to have, respectively, a finite number of simple zeros in Berger *et al.* (1984) and a multiple zero in Sun and Stynes (1994). These two articles discuss the estimates of the solution u^ϵ and its derivatives and the uniformly convergent numerical methods: in Berger *et al.* (1984) a modified El-Mistikawy-Werle scheme which is an exponentially fitted finite difference scheme adapted to the differential operator and in Sun and Stynes (1994) finite elements scheme on nonuniform meshes (more refined meshes near the turning points). We note that due to the condition $c(x) > 0$ there is no compatibility issue in the limit porblem, i.e. when $\epsilon = 0$ we have $u^0(0) = f(0)/c(0)$ in case that $b(0) = 0$, and thus we only have interior layers due to singularities of cusp or spike types. On the other hand, our article covers both the compatible and noncompatible cases (note that $b(0)u^0(0) = 0 = f(0)$ for Eq. (1.1)) and thus our problems encompass a greater variety of behaviors, with more severe interior layers due to logarithmic and negative power singularities as well as cusps or spikes.

In Hemker (1977) and Han and Kellogg (1982; 1983), the authors suggest how to construct the Finite Element basis for uniformly approximating the solutions for the singularly perturbed problems in the one dimensional space; these articles are devoted to the boundary and interior layers in Hemker (1977) and the boundary layers in Han and Kellogg (1982; 1983). Hemker (1977) discusses the so-called exponentially fitted splines adapted to a given differential operator and Han and Kellogg (1982; 1983) construct classical polynomial elements spaces enriched with a singular function which absorbs the boundary layer singularities. The idea using the explicit form of singularities due to the small ϵ in the Finite Element basis can be extended to the turning point problems and these extensions will appear elsewhere.

The authors thank the anonymous referees of the article Jung and Teman for pointing these articles to their attention. The article Jung and Teman addresses a linear algebra problem; it does not address turning points issues.

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