Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(5): 8235–8249.
DOI: 10.3934/math.2022459
Received: 23 November 2021
Revised: 06 February 2022
Accepted: 13 February 2022
Published: 25 February 2022

Research article

Jacobi forms over number fields from linear codes

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Abstract: We suggest a Jacobi form over a number field $\mathbb{Q}(\sqrt{5}, i)$; for obtaining this, we use a linear code C over $R := \mathbb{F}_4 + u\mathbb{F}_4$, where $u^2 = 0$. We introduce MacWilliams identities for both complete weight enumerator and symmetrized weight enumerator in higher genus $g \geq 1$ of a linear code over R. Finally, we give invariants via a self-dual code of even length over R.

Keywords: Jacobi form; Frobenius ring; linear code; self-dual code; invariant
Mathematics Subject Classification: 11F50, 94B05, 05E99

1. Introduction

There have been many developments in invariant theory and coding theory with various applications [5, 13, 16–21, 27]. There are connections between invariants and weight enumerators for self-dual codes over various rings and fields [1–4, 7–12, 14]. Gleason [17] says the weight enumerators for binary Type II codes are invariant by a certain finite group of order 192. In [6], the authors prove that elliptic modular forms can be obtained from homogeneous weight enumerators of binary Type II codes by using specific Jacobi theta series. For a finite ring $\mathbb{Z}_{2m}$, a Jacobi form of the full Jacobi group is suggested by complete weight enumerator of Type II codes over the ring [10]. Recently, in [23], the authors determine some Jacobi forms from Type II codes over $\mathbb{Z}_{2^m}$, and they also use shadow codes. Bannai et al. [2], construct Hermitian modular form by using Type II codes over $\mathbb{F}_2 + u\mathbb{F}_2$ with $u^2 = 0$. They suggest invariants concerning the symmetrized biweight enumerators, i.e., the genus is equal to 2, of Type II codes over the ring. In many works, a Jacobi form is studied over totally real fields (see [3, 11, 25]). For this reason, in this work, we figure out a Jacobi form which is not over a totally real field. We establish a connection between Jacobi forms and codes over $R = \mathbb{F}_4 + u\mathbb{F}_4$ ($u^2 = 0$). Since the ring R is described using the ring of integers in the quartic field $K = \mathbb{Q}(\sqrt{5}, i)$, we should consider Jacobi forms over the field K, which is not totally real. The notion of Jacobi forms over arbitrary number fields was defined by Skogman in [24]. In this paper, we add examples of Jacobi forms over K by modifying theta series of the lattices associated to codes over R.

We focus on the ring $\mathbb{F}_4 + u\mathbb{F}_4$ with $u^2 = 0$ in this paper. This ring is a finite commutative local Frobenius ring of order 16. A Frobenius ring is one of the most significant part in coding theory since we get the result that $|C||C^{\perp}| = |R^n|$ for a code C of length n over a Frobenius ring; this fact is connected to MacWilliams identity directly. Furthermore, for rings of order 16, we already know about their generating characters from [13]. This means that we can adjust this information to our studies when finding MacWilliams identity. MacWilliams identity is one of significant results in coding theory that describes how the weight enumerator of a linear code and the weight enumerator of the dual code relate to each other. MacWilliams identity give various application in coding theory. For example, in [13], the author presents an upper bound for the minimum distances of divisible codes through their dual distances. In [21], the authors study Type II codes over $\mathbb{F}_4 + u\mathbb{F}_4$ and in particular the Gray map, the Lee weight, the construction of lattices and invariants. Actually, in [21], there are no results for invariants in higher genus.

In this paper, we suggest a Jacobi form from a linear code C over $R := \mathbb{F}_4 + u\mathbb{F}_4$, where $u^2 = 0$ (Theorem 3.1). This Jacobi form is not over totally real field, and it is related to complete weight enumerator of the code C. We introduce MacWilliams identities for both, complete weight enumerator and symmetrized weight enumerator in higher genus $g \geq 1$ of a linear code over R. Finally, we give invariants via a self-dual code of even length over R (Theorem 4.4).

2. Preliminaries

Throughout this paper, we use the following notations.

Notation

K	an algebraic number field
r_1	the number of real embeddings of K
r_2	the number of conjugate pairs of complex embeddings of K
δ_K	the different of K
$\mathcal{O}_K$	the ring of integers of K
$\mathcal{Q}$	the full ring of quaternions$\{x + y\kappa : x, y \in \mathbb{C},\ \kappa^2 = -1,\ a\kappa = \kappa\bar{a},\ \forall a \in \mathbb{C}\}$
$\mathfrak{h}_{\mathcal{Q}}$	$\{x + y\kappa \in \mathcal{Q} : y \in \mathbb{R}_{>0}\}$
$\|\|u + v\kappa\|\|_{\mathbb{C}}$	$u + iv$ for $u + v\kappa \in \mathcal{Q}$
$\|\|u + v\kappa\|\|_{\overline{\mathbb{C}}}$	$\overline{u} + i\overline{v}$ for $u + v\kappa \in \mathcal{Q}$
R	a Frobenius ring $\mathbb{F}_4 + u\mathbb{F}_4$, where $u^2 = 0$
χ	the generating character of R
wt_L	the Lee weight in $\mathbb{F}_4$
$\hat{w}t_L$	the Lee weight in R
N	the cardinality of R
${}^tM$	the transpose of a matrix M
I_m	an $m \times m$ identity matrix

$diag(v_1,\ldots,v_m)$	an $m\times m$ diagonal matrix, where an (i,i)-th component is v_i $(1\le i\le m)$
$GL(m,F)$	the general linear group of degree m over F
$A\bigotimes B$	the Kronecker product of two matrices A and B
$e[z]$	$e^{2\pi iz}$

Let K be an algebraic number field, and Q be the full ring of quaternions $\{x+y\kappa : x,y\in\mathbb{C},\ \kappa^2=-1,\ a\kappa=\kappa\overline{a},\ \forall a\in\mathbb{C}\}$; the set of all the elements of Q is equal to $\{a+bi+cj+d\kappa : a,b,c,d\in\mathbb{R},\ i^2=j^2=\kappa^2=-1,\ ij=\kappa,\ j\kappa=i,\ ji=-\kappa\}$ (see [22, p. 220]). For an element $\alpha\in K$, we denote the real conjugates of α by $\alpha^{(1)},\ldots,\alpha^{(r_1)}$, and the complex conjugates of α by $\alpha^{(r_1+1)},\ldots,\alpha^{(r_1+2r_2)}$, where $\alpha^{(j+r_2)}=\overline{\alpha^{(j)}}$ for $r_1+1\le j\le r_1+r_2$.

We set $\Gamma^J(K)=\Gamma\ltimes O_K^2$, where $\Gamma=SL(2,O_K)$; the group $\Gamma^J(K)$ is called the *Jacobi group of K*. The group law of $\Gamma^J(K)$ is given by

$$(A,X)\cdot(B,Y)=(AB,XB+Y)\quad (A,B\in\Gamma,\ X,Y\in O_K^2).$$

Let $\mathfrak{h}$ be the upper half plane, and $\mathfrak{h}_Q=\{x+y\kappa\in Q\mid x\in\mathbb{C},y\in\mathbb{R}^+\}$ be the quaternionic upper half plane. Let $\mathcal{H}$ be the space $\mathfrak{h}^{r_1}\times\mathfrak{h}_Q^{r_2}\times\mathbb{C}^{r_1}\times Q^{r_2}$; an element of $\mathcal{H}$ is written as $(\vec{\tau},\vec{z}):=(\tau_1,\ldots,\tau_{r_1+r_2},z_1,\ldots,z_{r_1+r_2})$.

The group $\Gamma^J(K)$ acts on $\mathcal{H}$, and the action is given as follows: For elements $\left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right)\in SL(2,O_K)$ and $[\lambda,\mu]\in O_K^2$,

$$\begin{aligned}&\left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right)\circ(\vec{\tau},\vec{z})\\ &=\Bigg(\frac{\alpha^{(1)}\tau_1+\beta^{(1)}}{\gamma^{(1)}\tau_1+\delta^{(1)}},\ldots,\frac{\alpha^{(r_1)}\tau_{r_1}+\beta^{(r_1)}}{\gamma^{(r_1)}\tau_{r_1}+\delta^{(r_1)}},(\alpha^{(r_1+1)}\tau_{r_1+1}+\beta^{(r_1+1)})(\gamma^{(r_1+1)}\tau_{r_1+1}+\delta^{(r_1+1)})^{-1},\\ &\quad\ldots,(\alpha^{(r_1+r_2)}\tau_{r_1+r_2}+\beta^{(r_1+r_2)})(\gamma^{(r_1+r_2)}\tau_{r_1+r_2}+\delta^{(r_1+r_2)})^{-1},\frac{z_1}{\gamma^{(1)}\tau_1+\delta^{(1)}},\ldots,\\ &\quad\frac{z_{r_1}}{\gamma^{(r_1)}\tau_{r_1}+\delta^{(r_1)}},(\gamma^{(r_1+1)}\tau_{r_1+1}+\delta^{(r_1+1)})^{-1}z_{r_1+1},\ldots,(\gamma^{(r_1+r_2)}\tau_{r_1+r_2}+\delta^{(r_1+r_2)})^{-1}z_{r_1+r_2}\Bigg),\end{aligned}$$

and

$[\lambda,\mu]\circ(\vec{\tau},\vec{z})=(\vec{\tau},z_1+\tau_1\lambda^{(1)}+\mu^{(1)},\ldots,z_{r_1+r_2}+\tau_{r_1+r_2}\lambda^{(r_1+r_2)}+\mu^{(r_1+r_2)})$.

For $\gamma,\delta,\lambda\in K$ and $(\vec{\tau},\vec{z})\in\mathcal{H}$, set

$$\begin{aligned}\mathcal{T}_{\mathcal{R}}({}^tM\vec{z}(\gamma\vec{\tau}+\delta)^{-1}\gamma\vec{z}M)&=\sum_{j=1}^{r_1}\widetilde{m}^{(j)}\frac{\gamma^{(j)}z_j^2}{\gamma^{(j)}\tau_j+\delta^{(j)}}\\ &\quad+\sum_{j=r_1+1}^{r_1+r_2}\|{}^t\vec{m}^{(j)}(u_j+\overline{v_j}\kappa)(\gamma^{(j)}\tau_j+\delta^{(j)})^{-1}\gamma^{(j)}(u_j+v_j\kappa)\vec{m}^{(j)}\|_{\mathbb{C}}\\ &\quad+\sum_{j=r_1+1}^{r_1+r_2}\|{}^t\vec{m}^{(j)}(u_j+\overline{v_j}\kappa)(\gamma^{(j)}\tau_j+\delta^{(j)})^{-1}\gamma^{(j)}(u_j+v_j\kappa)\vec{m}^{(j)}\|_{\overline{\mathbb{C}}},\end{aligned}$$

$$\mathcal{T}_{\mathcal{R}}({}^tM(\lambda\vec{\tau}\lambda+2\lambda\vec{z})M)=\sum_{j=1}^{r_1}\widetilde{m}^{(j)}({\lambda^{(j)}}^2\tau_j+2\lambda^{(j)}z_j)$$

$$
+\sum_{j=r_1+1}^{r_1+r_2} \|{}^t\vec{m}^{(j)}(\lambda^{(j)}\tau_j\lambda^{(j)}+2\lambda^{(j)}z_j)\vec{m}^{(j)}\|_{\mathbb{C}}
$$
$$
+\sum_{j=r_1+1}^{r_1+r_2} \|{}^t\vec{m}^{(j)}(\lambda^{(j)}\tau_j\lambda^{(j)}+2\lambda^{(j)}z_j)\vec{m}^{(j)}\|_{\overline{\mathbb{C}}},
$$

where

- $z_j = u_j + v_j\kappa$ for $j = r_1+1,\dots,r_1+r_2$.
- $\vec{m}^{(i)}$: vectors in $\mathbb{C}^{r_1+2r_2}$ $(1 \le i \le r_1+2r_2)$ such that $\vec{m}^{(j+r_2)} = \overline{\vec{m}^{(j)}}$ for $j = r_1+1,\dots,r_1+r_2$.
- $M = (\vec{m}^{(1)} \quad \cdots \quad \vec{m}^{(r_1+2r_2)})$: an $(r_1+2r_2)\times(r_1+2r_2)$-matrix over $\mathbb{C}$.
- $\widetilde{m}^{(j)} = {}^t\vec{m}^{(j)}\vec{m}^{(j)}$.

Next, we need to present a multiplier system for $SL(2,O_K)$. For doing this, we set the following: For $A = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in SL(2,O_K)$ and $\vec{\tau} = (\tau_1,\dots,\tau_{r_1+r_2}) \in \mathfrak{h}^{r_1}\times\mathfrak{h}_Q^{r_2}$,

$$
A\circ\vec{\tau} = \Bigg(\frac{\alpha^{(1)}\tau_1+\beta^{(1)}}{\gamma^{(1)}\tau_1+\delta^{(1)}},\dots,\frac{\alpha^{(r_1)}\tau_{r_1}+\beta^{(r_1)}}{\gamma^{(r_1)}\tau_{r_1}+\delta^{(r_1)}},(\alpha^{(r_1+1)}\tau_{r_1+1}+\beta^{(r_1+1)})(\gamma^{(r_1+1)}\tau_{r_1+1}+\delta^{(r_1+1)})^{-1},
$$
$$
\dots,(\alpha^{(r_1+r_2)}\tau_{r_1+r_2}+\beta^{r_1+r_2})(\gamma^{(r_1+r_2)}\tau_{r_1+r_2}+\delta^{r_1+r_2})^{-1}\Bigg),
$$

and

$$
\mathcal{J}(A,\vec{\tau}) = \mathcal{N}(\gamma\vec{\tau}+\delta),
$$

where

$$
\mathcal{N}(\gamma\vec{\tau}+\delta) := \prod_{j=1}^{r_1}(\gamma^{(j)}\tau_j+\delta^{(j)})\prod_{j=r_1+1}^{r_1+r_2}(|\gamma^{(j)}x_j+\delta^{(j)}|^2+y_j^2|\gamma^{(j)}|^2),
$$

and $\tau_j = x_j + y_j\kappa$ for $j = r_1+1,\dots,r_1+r_2$. A *multiplier system* for $SL(2,O_K)$ is a function $\chi : SL(2,O_K)\to\mathbb{C}$ such that

$$
\chi(AB)\mathcal{J}(AB,\vec{\tau})^{\frac{1}{2}} = \chi(A)\mathcal{J}(A,B\circ\vec{\tau})^{\frac{1}{2}}\chi(B)\mathcal{J}(B,\vec{\tau})^{\frac{1}{2}}
$$

for all $A, B \in SL(2,O_K)$ and $\vec{\tau}\in\mathfrak{h}^{r_1}\times\mathfrak{h}_Q^{r_2}$.

The next definition is about a *Jacobi form* of weight k and index m with an index vector for a number field.

Definition 2.1. *[22, 24] Let K be an algebraic number field, $k \in \frac{1}{2}\mathbb{Z}$, and $m \in O_K$. Let χ be a multiplier system for $SL(2,O_K)$, and $\vec{m}$ be a vector in $\mathbb{C}^n$ such that ${}^t\vec{m}^{(j)}\vec{m}^{(j)} = m^{(j)}$ for $j = 1,\dots,n$. A* Jacobi form *of weight k, index m, index vector $\vec{m}$ and the multiplier system χ for the number field K is a function $\Phi : \mathcal{H}\to\mathbb{C}$ satisfying*

$$
\Phi\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\circ(\vec{\tau},\vec{z})\right) = \chi\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right)\mathcal{N}(\gamma\vec{\tau}+\delta)^k e[\mathcal{T}_{\mathcal{R}}({}^tM\vec{z}(\gamma\vec{\tau}+\delta)^{-1}\gamma\vec{z}M)]\Phi(\vec{\tau},\vec{z})
$$

and

$$
\Phi([\lambda,\mu]\circ(\vec{\tau},\vec{z})) = e[-\mathcal{T}_{\mathcal{R}}({}^tM(\lambda\vec{\tau}\lambda+2\lambda\vec{z})M)]\Phi(\vec{\tau},\vec{z})
$$

for all $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)\in SL(2,O_K)$, $[\lambda,\mu]\in O_K^2$, $\vec{\tau}\in\mathfrak{h}^{r_1}\times\mathfrak{h}_Q^{r_2}$ and $\vec{z}\in\mathbb{C}^{r_1}\times Q^{r_2}$.

A *Frobenius ring* is a finite commutative ring R satisfying that the R-module R is injective. We consider the number field $\mathbb{Q}(\sqrt{5}, i)$ which has the ring of integers $\mathbb{Z}[\frac{1+\sqrt{5}}{2}, i]$. Let u be the residue class of $x+1$ in the quotient ring

$$R := \mathbb{F}_4 + u\mathbb{F}_4 = \mathbb{Z}[\frac{1+\sqrt{5}}{2}, x]/\langle 2, (x+1)^2\rangle,$$

i.e., $u^2 = 0$ in R. On the other side, the ring R is isomorphic to $\mathbb{F}_2[u, v]/\langle v^2 + v + 1, u^2\rangle$. This ring R is a finite commutative local Frobenius ring of order 16 (cf. [13]).

A code C of length n over R is an R-submodule of R^n, and an element $c = (c_1, \dots, c_n)$ in C is called a codeword in C. The dual code $C^\perp$ of C is $\{c \in R^n : c \cdot \hat{c} = 0 \text{ for all } \hat{c} \in C\}$ with respect to the Euclidean inner product. If $C \subseteq C^\perp$ (resp. $C = C^\perp$), then C is a self-orthogonal (resp. self-dual) code.

The Lee weight $wt_L(a)$ of an element a in $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$ is given as follows:

$$wt_L(a) = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{if } a = \omega \text{ or } \bar{\omega}, \\ 2 & \text{if } a = 1. \end{cases}$$

For a vector $w = (w_1, \dots, w_n) \in \mathbb{F}_4^n$, the Lee weight $wt_L(w)$ of w is $\sum_{i=1}^n wt_L(w_i)$.

Definition 2.2 gives the Lee weight of an element in $R = \mathbb{F}_4 + u\mathbb{F}_4$.

Definition 2.2. *For an element $\alpha = a + bu$ in R ($a, b \in \mathbb{F}_4$), the Lee weight $\hat{w}t_L(\alpha)$ of α in R is*

$$\hat{w}t_L(\alpha) = wt_L(b) + wt_L(a + b),$$

where wt_L is the Lee weight in $\mathbb{F}_4$. For a vector $v = (v_1, \dots, v_n)$ in R^n, the Lee weight $\hat{w}t_L(v)$ of v is $\sum_{i=1}^n \hat{w}t_L(v_i)$.

In the following Table 1, we suggest the Lee weights of all the elements in R.

Table 1. Lee weights of all the elements α of R.

α	$\hat{w}t_L(\alpha)$	α	$\hat{w}t_L(\alpha)$	α	$\hat{w}t_L(\alpha)$	α	$\hat{w}t_L(\alpha)$
0	0	1	2	ω	1	$\bar{\omega}$	1
u	4	$1+u$	2	$\omega+u$	3	$\bar{\omega}+u$	3
ωu	2	$1+\omega u$	2	$\omega+\omega u$	1	$\bar{\omega}+\omega u$	3
$\bar{\omega}u$	2	$1+\bar{\omega}u$	2	$\omega+\bar{\omega}u$	3	$\bar{\omega}+\bar{\omega}u$	1

Proposition 2.3. *[21] Let us define a map ϕ from R^n to $\mathbb{F}_4^{2n}$ as follows:*

$$\begin{array}{cccc} \phi : & R^n & \longrightarrow & \mathbb{F}_4^{2n} \\ & (a_i + b_i u, \dots, a_n + b_n u) & \longmapsto & (b_1, a_1 + b_1, \dots, b_n, a_n + b_n), \end{array}$$

where $a_i, b_i \in \mathbb{F}_4$ ($1 \le i \le n$). The map ϕ preserves the Lee weight from R^n to $\mathbb{F}_4^{2n}$, it means that, the map ϕ is a Gray map.

For a self-dual code C over R, C is a *Type II code* if the Lee weight of every codeword is divisible by 4. If not, the code C is called a *Type I code*.

Lemma 2.4. *[13, 21] Let C be a linear code of length n over R.*

(i) *We have that $|C||C^{\perp}| = |R^n|$.*
(ii) *There is a Type II code of length n over R if and only if n is even.*

From now on, we suggest some weight enumerators for a code C over R: A complete weight enumerator and a symmetrized weight enumerator of C. Let C be a linear code of length n over R, and $v = (v_1, \dots, v_n)$ be a codeword in C. Let $n_a(v)$ be the number of coordinates v_i such that $v_i = a$, where $a \in R$ and $1 \leq i \leq n$. We define the *complete weight enumerator* cwe_C of C as

$$cwe_C(x_1, \dots, x_N) := \sum_{v \in C} \prod_{j=1}^{N} x_j^{n_{a_j}(v)},$$

where x_j is an indeterminate and $a_j \in R$ for $1 \leq j \leq N$.

Let U be a fixed subgroup of the unit group of R. We note that the group U acts on R by the multiplication. We denote $a \approx b$ if $a = ub$ for some $u \in U$; it is an equivalence relation in R. Moreover, $S = \{s_1, \dots, s_{|S|}\}$ is a set of representatives of the distinct orbits of U. The symmetrized weight enumerator swe_C of C in R^n is

$$swe_C(x_1, \dots, x_{|S|}) = \sum_{c \in C} \prod_{j=1}^{|S|} x_j^{\sum_{r \approx s_j} n_r(c)}.$$

3. A Jacobi form from a linear code over R

We figure out a Jacobi form for a number field via a linear code over R. Before we do this, we first give a theta function as follows.

Let K be a number field, and Λ be a *lattice* in K^n, i.e., Λ is a free $\mathcal{O}_K$-module of rank n. Now, for each Y in Λ, we define a theta function $\Theta_{\Lambda,Y} : \mathcal{H} \to \mathbb{C}$ as

$$\begin{aligned}\Theta_{\Lambda,Y}(\vec{\tau}, \vec{z}) := \sum_{x \in \Lambda} e\Bigg[&\sum_{j=1}^{r_1} \left(\frac{1}{2} {}^t x^{(j)} x^{(j)} \tau_j + {}^t x^{(j)} Y^{(j)} z_j \right) + \sum_{j=r_1+1}^{r_1+r_2} \left\| \frac{1}{2} {}^t x^{(j)} \tau_j x^{(j)} + {}^t x^{(j)} z_j Y^{(j)} \right\|_{\mathbb{C}} \\ &+ \sum_{j=r_1+1}^{r_1+r_2} \left\| \frac{1}{2} {}^t x^{(j)} \tau_j x^{(j)} + {}^t x^{(j)} z_j Y^{(j)} \right\|_{\overline{\mathbb{C}}} \Bigg].\end{aligned}$$

In this section, let $K = \mathbb{Q}(\sqrt{5}, i)$, and then $\mathcal{O}_K = \mathbb{Z}[(1 + \sqrt{5})/2, i]$; so $r_1 = 0$ and $r_2 = 2$. Here, we can check that $R \cong \mathbb{F}_2[u, v]/\langle u^2 + u + 1, v^2 \rangle \cong \mathcal{O}_K/2\mathcal{O}_K$; the first equivalence is introduced in Section 2, and the second equivalence is from the ring isomorphism $\mathbb{F}_2[u, v]/\langle u^2 + u + 1, v^2 \rangle \to \mathcal{O}_K/2\mathcal{O}_K$,where $u + \langle u^2 + u + 1, v^2 \rangle \mapsto (1 + \sqrt{5})/2 + 2\mathcal{O}_K$, and $v + \langle u^2 + u + 1, v^2 \rangle \mapsto 1 + i + 2\mathcal{O}_K$.

Let $h : \mathcal{O}_K \to R$ be the reduction map by modulo 2, and $\tilde{h} : \mathcal{O}_K^n \to R^n$ be defined as $(x_1, \dots, x_n) \to (h(x_1), \dots, h(x_n))$. For a linear code C over R, the lattice $\Lambda(C)$ is $\frac{1}{\sqrt{2}} \tilde{h}^{-1}(C)$.

In the following theorem, we obtain the relation between the theta function and complete weight enumerator for a linear code under the previous settings.

Theorem 3.1. *Let C be a linear code of length n over R. Then we get*

$$\Theta_{\Lambda(C),\sqrt{2}(1,\dots,1)}(\vec{\tau},\vec{z}) = cwe_C\left(\omega_{1,\mu}(\vec{\tau},\vec{z}) \mid \mu \in \mathcal{O}_K/2\mathcal{O}_K\right),$$

where

$$\begin{aligned}
\omega_{1,\mu}(\vec{\tau},\vec{z}) &= \sum_{\substack{r\in\delta_K^{-1},\\ r\equiv\mu \pmod 2}} e\left[\sum_{j=1}^{r_1}\left(\frac{r^{(j)2}\tau_j}{4} + r^{(j)}z_j\right) + \sum_{j=r_1+1}^{r_1+r_2}\left\|\tfrac{1}{4}r^{(j)}\tau_j r^{(j)} + r^{(j)}z_j\right\|_{\mathbb{C}}\right.\\
&\quad \left. + \sum_{j=r_1+1}^{r_1+r_2}\left\|\tfrac{1}{4}r^{(j)}\tau_j r^{(j)} + r^{(j)}z_j\right\|_{\overline{\mathbb{C}}}\right],\\
&= \sum_{\substack{r\in\delta_K^{-1},\\ r\equiv\mu \pmod 2}} e\left[\left\|\tfrac{1}{4}r^{(1)}\tau_1 r^{(1)} + r^{(1)}z_1\right\|_{\mathbb{C}} + \left\|\tfrac{1}{4}r^{(2)}\tau_2 r^{(2)} + r^{(2)}z_2\right\|_{\mathbb{C}}\right.\\
&\quad \left. + \left\|\tfrac{1}{4}r^{(1)}\tau_1 r^{(1)} + r^{(1)}z_1\right\|_{\overline{\mathbb{C}}} + \left\|\tfrac{1}{4}r^{(2)}\tau_2 r^{(2)} + r^{(2)}z_2\right\|_{\overline{\mathbb{C}}}\right].
\end{aligned}$$

Proof. Let v be a codeword $(v_1,\dots,v_n)$ in C, and $\widetilde{v}$ be a vector $(\widetilde{v_1},\dots,\widetilde{v_n}) \in \widetilde{h}^{-1}(v)$. We easily check $\widetilde{h}$ is a ring homomorphism, thus $\widetilde{h}^{-1}(v) = \widetilde{h}^{-1}(\mathbf{0}) + \widetilde{v}$. Then we obtain that

$$\begin{aligned}
&\sum_{x\in\widetilde{h}^{-1}(v)} e\left[\left\|\tfrac{1}{4}(x_1^{(1)}\tau_1 x_1^{(1)} + \cdots + x_n^{(1)}\tau_1 x_n^{(1)}) + (x_1^{(1)} + \cdots + x_n^{(1)})z_1\right\|_{\mathbb{C}}\right.\\
&\quad + \left\|\tfrac{1}{4}(x_1^{(2)}\tau_2 x_1^{(2)} + \cdots + x_n^{(2)}\tau_2 x_n^{(2)}) + (x_1^{(2)} + \cdots + x_n^{(2)})z_2\right\|_{\mathbb{C}}\\
&\quad + \left\|\tfrac{1}{4}(x_1^{(1)}\tau_1 x_1^{(1)} + \cdots + x_n^{(1)}\tau_1 x_n^{(1)}) + (x_1^{(1)} + \cdots + x_n^{(1)})z_1\right\|_{\overline{\mathbb{C}}}\\
&\quad \left. + \left\|\tfrac{1}{4}(x_1^{(2)}\tau_2 x_1^{(2)} + \cdots + x_n^{(2)}\tau_2 x_n^{(2)}) + (x_1^{(2)} + \cdots + x_n^{(2)})z_2\right\|_{\overline{\mathbb{C}}}\right],\\
&= \sum_{x_1\in 2\mathcal{O}_K+\widetilde{v_1}} e\left[\left\|\tfrac{1}{4}x_1^{(1)}\tau_1 x_1^{(1)} + x_1^{(1)}z_1\right\|_{\mathbb{C}} + \left\|\tfrac{1}{4}x_1^{(2)}\tau_2 x_1^{(2)} + x_1^{(2)}z_2\right\|_{\mathbb{C}}\right.\\
&\qquad \left. + \left\|\tfrac{1}{4}x_1^{(1)}\tau_1 x_1^{(1)} + x_1^{(1)}z_1\right\|_{\overline{\mathbb{C}}} + \left\|\tfrac{1}{4}x_1^{(2)}\tau_2 x_1^{(2)} + x_1^{(2)}z_2\right\|_{\overline{\mathbb{C}}}\right]\\
&\quad \cdots \sum_{x_n\in 2\mathcal{O}_K+\widetilde{v_n}} e\left[\left\|\tfrac{1}{4}x_n^{(1)}\tau_1 x_n^{(1)} + x_n^{(1)}z_1\right\|_{\mathbb{C}} + \left\|\tfrac{1}{4}x_n^{(2)}\tau_2 x_n^{(2)} + x_n^{(2)}z_2\right\|_{\mathbb{C}}\right.\\
&\qquad \left. + \left\|\tfrac{1}{4}x_n^{(1)}\tau_1 x_n^{(1)} + x_n^{(1)}z_1\right\|_{\overline{\mathbb{C}}} + \left\|\tfrac{1}{4}x_n^{(2)}\tau_2 x_n^{(2)} + x_n^{(2)}z_2\right\|_{\overline{\mathbb{C}}}\right],\\
&= \prod_{\mu\in R}\omega_{1,\mu}(\vec{\tau},\vec{z})^{n_\mu(v)};
\end{aligned} \tag{3.1}$$

we only consider τ_1 and τ_2 because $r_1 = 0$ and $r_2 = 2$ as we mentioned before. And the second equation is from that

$$\begin{aligned}
\|(u_1+v_1\kappa) + (u_2+v_2\kappa)\|_{\mathbb{C}} &= (u_1+v_1 i) + (u_2+v_2 i) = \|u_1+v_1\kappa\|_{\mathbb{C}} + \|u_2+v_2\kappa\|_{\mathbb{C}},\\
\|(u_1+v_1\kappa) + (u_2+v_2\kappa)\|_{\overline{\mathbb{C}}} &= (\overline{u_1}+\overline{v_1}i) + (\overline{u_2}+\overline{v_2}i) = \|u_1+v_1\kappa\|_{\overline{\mathbb{C}}} + \|u_2+v_2\kappa\|_{\overline{\mathbb{C}}}
\end{aligned}$$

for all $u_1+v_1\kappa, u_2+v_2\kappa \in Q$. Therefore, we have that

$$\begin{aligned}
&\Theta_{\Lambda(C),\sqrt{2}(1,\dots,1)}(\vec{\tau},\vec{z})\\
&= \sum_{x\in\Lambda(C)} e\left[\left\|\frac{1}{2}{}^t x^{(1)}\tau_1 x^{(1)} + {}^t x^{(1)}z_1 Y^{(1)}\right\|_{\mathbb{C}} + \left\|\frac{1}{2}{}^t x^{(2)}\tau_2 x^{(2)} + {}^t x^{(2)}z_2 Y^{(2)}\right\|_{\mathbb{C}}\right.\\
&\quad \left. + \left\|\frac{1}{2}{}^t x^{(1)}\tau_1 x^{(1)} + {}^t x^{(1)}z_1 Y^{(1)}\right\|_{\overline{\mathbb{C}}} + \left\|\frac{1}{2}{}^t x^{(2)}\tau_2 x^{(2)} + {}^t x^{(2)}z_2 Y^{(2)}\right\|_{\overline{\mathbb{C}}}\right],\\
&= \sum_{x\in\frac{1}{\sqrt{2}}\tilde{h}^{-1}(C)} e\left[\left\|\frac{1}{2}{}^t x^{(1)}\tau_1 x^{(1)} + {}^t x^{(1)}z_1 Y^{(1)}\right\|_{\mathbb{C}} + \left\|\frac{1}{2}{}^t x^{(2)}\tau_2 x^{(2)} + {}^t x^{(2)}z_2 Y^{(2)}\right\|_{\mathbb{C}}\right.
\end{aligned}$$

$$+ \left\|\frac{1}{2}{}^t x^{(1)}\tau_1 x^{(1)} + {}^t x^{(1)} z_1 Y^{(1)}\right\|_{\overline{\mathbb{C}}} + \left\|\frac{1}{2}{}^t x^{(2)}\tau_2 x^{(2)} + {}^t x^{(2)} z_2 Y^{(2)}\right\|_{\overline{\mathbb{C}}}\Bigg],$$

$$= \sum_{v\in C}\sum_{x\in\frac{1}{\sqrt{2}}\tilde{h}^{-1}(v)} e\Bigg[\left\|\frac{1}{2}{}^t x^{(1)}\tau_1 x^{(1)} + {}^t x^{(1)} z_1 Y^{(1)}\right\|_{\mathbb{C}} + \left\|\frac{1}{2}{}^t x^{(2)}\tau_2 x^{(2)} + {}^t x^{(2)} z_2 Y^{(2)}\right\|_{\mathbb{C}}$$

$$+ \left\|\frac{1}{2}{}^t x^{(1)}\tau_1 x^{(1)} + {}^t x^{(1)} z_1 Y^{(1)}\right\|_{\overline{\mathbb{C}}} + \left\|\frac{1}{2}{}^t x^{(2)}\tau_2 x^{(2)} + {}^t x^{(2)} z_2 Y^{(2)}\right\|_{\overline{\mathbb{C}}}\Bigg],$$

$$= \sum_{v\in C}\prod_{\mu\in R}\omega_{1,\mu}(\vec{\tau},\vec{z})^{n_\mu(v)},$$

$$= cwe_C(\omega_{1,\mu}(\vec{\tau},\vec{z}) \mid \mu \in O_K/2O_K);$$

the fourth equation is from (3.1), where $n_\mu(v) := |\{j : v_j = \mu\}|$ for each $\mu \in O_K/2O_K$. The result is proved. □

We fix an O_K-basis $\{v_1, \ldots, v_n\}$ for a lattice Λ, and set L to be an $n \times n$-matrix $(v_1, \ldots, v_n)$. The matrix $Q := {}^tLL$ is a symmetric matrix, and $Q^{(j)} := {}^tL_jL_j$, where $L_j = (v_1^{(j)}, \ldots, v_n^{(j)})$ for $1 \le j \le n$. Morevoer, $Q^{(j)}$ is positive definite for $1 \le j \le r_1$.

In [24], for $b \in O_K^n$, the function $\theta_{Q,b}(\vec{\tau},\vec{z})$ is defined as

$$\theta_{Q,b}(\vec{\tau},\vec{z}) = \sum_{x\in O_K^n} e\Bigg[\Bigg(\sum_{j=1}^{r_1}\left(Q^{(j)}[x^{(j)}]\tau_j + 2{}^t x^{(j)} Q^{(j)} b^{(j)} z_j\right) + \sum_{j=r_1+1}^{r_1+r_2}\left\|{}^t x^{(j)t} L_j\tau_j L_j x^{(j)} + 2{}^t x^{(j)t} L_j z_j L_j b^{(j)}\right\|_{\mathbb{C}}$$

$$+ \sum_{j=r_1+1}^{r_1+r_2}\left\|{}^t x^{(j)t} L_j\tau_j L_j x^{(j)} + 2{}^t x^{(j)t} L_j z_j L_j b^{(j)}\right\|_{\overline{\mathbb{C}}}\Bigg)\Bigg].$$

The next theorem says that we can obtain a Jacobi form by using a linear code of length n over R.

Theorem 3.2. *We use the same notations as above. Let C be a linear code of length n over R, and b an element of $\mathbb{Z}[\alpha, i]^n$ with $\alpha = (1 + \sqrt{5})/2$. Then we get a Jacobi form of weight $\frac{n}{2}$, index $4n$ and index vector $(2, \ldots, 2)$ as follows:*

$$\sum_{\substack{p_1^2+q_1^2+\cdots+p_n^2+q_n^2=4n \\ p_\ell\neq 2,\ q_\ell=0\ for\ all\ 1\le\ell\le n}} \theta_{Q,b}(\vec{\tau},\vec{z}) + cwe_C\left(\omega_{1,\mu}(4\vec{\tau},4\vec{z}) \mid \mu \in O_K/2O_K\right),$$

where $Lb = (p_1 + q_1\alpha, \ldots, p_n + q_n\alpha) \in \mathbb{Z}[\alpha]^n$ (in particular, $\omega_{1,\mu}(\vec{\tau},\vec{z})$ is introduced in Theorem 3.1).

Proof. First, we claim that Lb is in $\mathbb{Z}[\alpha]^n$. We set an n-tuple $Lb = (t_1, \ldots, t_n)$ satisfying ${}^t(Lb)Lb = t_1^2 + \cdots + t_n^2 = 4n$, and $\overline{{}^t(Lb)}Lb = |t_1|^2 + \cdots + |t_n|^2 = 4n$; the ℓ-th component t_ℓ can be written as $p_\ell + q_\ell\alpha + r_\ell i + s_\ell\alpha i$, where $p_\ell, q_\ell, r_\ell, s_\ell \in \mathbb{Z}$ $(1 \le \ell \le n)$. It means that

$$(p_1 + q_1\alpha + r_1 i + s_1\alpha i)^2 + \cdots + (p_n + q_n\alpha + r_n i + s_n\alpha i)^2 = 4n, \tag{3.2}$$

$$(p_1 + q_1\alpha)^2 + (r_1 + s_1\alpha)^2 + \cdots + (p_n + q_n\alpha)^2 + (r_n + s_n\alpha)^2 = 4n. \tag{3.3}$$

Let σ be the complex embedding of K such that $\sigma : \sqrt{5} \mapsto -\sqrt{5}$ and $i \mapsto i$. Applying σ to (3.3), we have $(p_1 + q_1\sigma(\alpha))^2 + (r_1 + s_1\sigma(\alpha))^2 + \cdots + (p_n + q_n\sigma(\alpha))^2 + (r_n + s_n\sigma(\alpha))^2 = 4n$. It follows that

$$(\alpha p_1 - q_1)^2 + (\alpha r_1 - s_1)^2 + \cdots + (\alpha p_n - q_n)^2 + (\alpha r_n - s_n)^2 = 4n\alpha^2 = 4n + 4n\alpha. \tag{3.4}$$

We get that combining (3.3) and (3.4), we see that

$$p_1^2+q_1^2-r_1^2-s_1^2+\cdots+p_n^2+q_n^2-r_n^2-s_n^2=4n=p_1^2+q_1^2+r_1^2+s_1^2+\cdots+p_n^2+q_n^2+r_n^2+s_n^2;$$

the first equality is obtained by expanding (3.2), and the second equality is from combining (3.3) and (3.4). Thus $r_\ell=s_\ell=0$ since $r_\ell, s_\ell\in\mathbb{Z}$ for $\ell=1,\ldots,n$. Hence we proved the first claim.

By using [24, p. 41], we can say that

$$\sum_{\substack{b\in\mathbb{Z}[\alpha,i]^n,\\ Lb=(p_1+q_1\alpha,\ldots,p_n+q_n\alpha)\in\mathbb{Z}[\alpha]^n,\\ p_1^2+q_1^2+\cdots+p_n^2+q_n^2=4n}}\theta_{Q,b}(\vec{\tau},\vec{z})$$

is a Jacobi form of weight $\frac{n}{2}$, index $4n$ and index vector $(2,\ldots,2)$.

Next, when $Lb=(2,\ldots,2)$, $\theta_{Q,L^{-1}(2,\ldots,2)}(\vec{\tau},\vec{z})$ is obtained from the complete weight enumerator of the code C; in this case, the condition $p_1^2+q_1^2+\cdots+p_n^2+q_n^2=4n$ is satisfied. In detail, we have that

$$\Theta_{\Lambda(C),\sqrt{2}(1,\ldots,1)}(4\vec{\tau},4\vec{z})=\Theta_{\tilde{h}^{-1}(C),(2,\ldots,2)}(2\vec{\tau},2\vec{z})=\theta_{Q,L^{-1}(2,\ldots,2)}(\vec{\tau},\vec{z}),$$

where $b=L^{-1}(2,\ldots,2)$; the first equation can be checked easily, and the second equation follows by the below reasoning

$$\begin{aligned}
&\Theta_{\tilde{h}^{-1}(C),(2,\ldots,2)}(2\vec{\tau},2\vec{z})\\
&=\sum_{x\in\tilde{h}^{-1}(C)}e\left[\left(\sum_{j=1}^{r_1}{}^tx^{(j)}x^{(j)}\tau_j+2{}^tx^{(j)}Y^{(j)}z_j\ +\sum_{j=r_1+1}^{r_1+r_2}\left\|{}^tx^{(j)}\tau_jx^{(j)}+2{}^tx^{(j)}z_jY^{(j)}\right\|_{\mathbb{C}}\right.\right.\\
&\quad\left.\left.+\sum_{j=r_1+1}^{r_1+r_2}\left\|{}^tx^{(j)}\tau_jx^{(j)}+2{}^tx^{(j)}z_jY^{(j)}\right\|_{\overline{\mathbb{C}}}\right)\right],\\
&=\sum_{x\in\mathcal{O}_K^n}e\left[\left(\sum_{j=1}^{r_1}{}^tx^{(j)t}L_jL_jx^{(j)}\tau_j+2{}^tx^{(j)t}L_jL_jb^{(j)}z_j\ +\sum_{j=r_1+1}^{r_1+r_2}\left\|{}^tx^{(j)t}L_j\tau_jL_jx^{(j)}+2{}^tx^{(j)t}L_jz_jL_jb^{(j)}\right\|_{\mathbb{C}}\right.\right.\\
&\quad\left.\left.+\sum_{j=r_1+1}^{r_1+r_2}\left\|{}^tx^{(j)t}L_j\tau_jL_jx^{(j)}+2{}^tx^{(j)t}L_jz_jL_jb^{(j)}\right\|_{\overline{\mathbb{C}}}\right)\right],\\
&=\sum_{x\in\mathcal{O}_K^n}e\left[\left(\sum_{j=1}^{r_1}Q^{(j)}[x^{(j)}]\tau_j+2{}^tx^{(j)}Q^{(j)}b^{(j)}z_j\ +\sum_{j=r_1+1}^{r_1+r_2}\left\|{}^tx^{(j)t}L_j\tau_jL_jx^{(j)}+2{}^tx^{(j)t}L_jz_jL_jb^{(j)}\right\|_{\mathbb{C}}\right.\right.\\
&\quad\left.\left.+\sum_{j=r_1+1}^{r_1+r_2}\left\|{}^tx^{(j)t}L_j\tau_jL_jx^{(j)}+2{}^tx^{(j)t}L_jz_jL_jb^{(j)}\right\|_{\overline{\mathbb{C}}}\right)\right],\\
&=\theta_{Q,L^{-1}(2,\ldots,2)}(\vec{\tau},\vec{z})
\end{aligned}$$

(here $Y=(2,\ldots,2)$). This implies the second claim, thus this is the result here. □

We close this section with an examlple.

Example 3.3. *Let C be the linear code over R of length 2 generated by the following 1×2 matrix*

$$\begin{pmatrix} \alpha & 1+\alpha+i \end{pmatrix}$$

(here $\alpha = (1+\sqrt{5})/2$). By using Table 1, we can check that the code C is a Type II code over R. Meanwhile, $\tilde{h}^{-1}(C)$ is generated by $(\alpha, 1+\alpha+i), (2,0)$ and $(0,2)$ over O_K. And then $\{(\alpha, 1+\alpha+i), (0,2)\}$ is a basis for $\tilde{h}^{-1}(C)$; since $(\alpha, 1+\alpha+i)$ and $(0,2)$ are linearly independent over O_K, and $(2,0) = (-2+2\alpha)(\alpha, 1+\alpha+i) + (-\alpha+i-\alpha i)(0,2)$. Thus the matrix L is obtained as

$$L = \begin{pmatrix} \alpha & 0 \\ 1+\alpha+i & 2 \end{pmatrix},$$

and the matrix $Q = {}^tLL$. We note that

$$L^{-1} = \frac{1}{2\alpha}\begin{pmatrix} 2 & 0 \\ -1-\alpha-i & \alpha \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2\alpha-2 & 0 \\ -\alpha+i-\alpha i & 1 \end{pmatrix}.$$

As a result, by Theorem 3.2, we have a Jacobi form of weight 1, index 8 and index vector (2,2) as follows:

$$\begin{aligned}
\sum_{\substack{b\in\mathbb{Z}[\alpha,i]^2,\\ Lb=(p_1+q_1\alpha,p_2+q_2\alpha)\in\mathbb{Z}[\alpha]^2,\\ p_1^2+q_1^2+p_2^2+q_2^2=8}} \theta_{Q,b}(\vec{\tau},\vec{z}) &= \sum_{\substack{p_1,q_1,p_2,q_2\in\mathbb{Z},\\ p_1^2+q_1^2+p_2^2+q_2^2=8}} \theta_{Q,L^{-1}\left(\begin{smallmatrix} p_1+q_1\alpha \\ p_2+q_2\alpha\end{smallmatrix}\right)}(\vec{\tau},\vec{z}) \\
&= \theta_{Q,\frac{1}{2}\left(\begin{smallmatrix} 2\alpha-2 & 0 \\ -\alpha+i-\alpha i & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} \pm2\pm2\alpha \\ 0\end{smallmatrix}\right)}(\vec{\tau},\vec{z}) + \theta_{Q,\frac{1}{2}\left(\begin{smallmatrix} 2\alpha-2 & 0 \\ -\alpha+i-\alpha i & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} \pm2 \\ \pm2\end{smallmatrix}\right)}(\vec{\tau},\vec{z}) \\
&\quad + \theta_{Q,\frac{1}{2}\left(\begin{smallmatrix} 2\alpha-2 & 0 \\ -\alpha+i-\alpha i & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} \pm2 \\ \pm2\alpha\end{smallmatrix}\right)}(\vec{\tau},\vec{z}) + \theta_{Q,\frac{1}{2}\left(\begin{smallmatrix} 2\alpha-2 & 0 \\ -\alpha+i-\alpha i & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} \pm2\alpha \\ \pm2\end{smallmatrix}\right)}(\vec{\tau},\vec{z}) \\
&\quad + \theta_{Q,\frac{1}{2}\left(\begin{smallmatrix} 2\alpha-2 & 0 \\ -\alpha+i-\alpha i & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} \pm2\alpha \\ \pm2\alpha\end{smallmatrix}\right)}(\vec{\tau},\vec{z}) + \theta_{Q,\frac{1}{2}\left(\begin{smallmatrix} 2\alpha-2 & 0 \\ -\alpha+i-\alpha i & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 0 \\ \pm2\pm2\alpha\end{smallmatrix}\right)}(\vec{\tau},\vec{z}), \\
&= cwe_C\left(\omega_{1,\mu}(4\vec{\tau},4\vec{z}) \mid \mu\in O_K/2O_K\right) + \theta_{Q,b_1}(\vec{\tau},\vec{z}) + \theta_{Q,b_2}(\vec{\tau},\vec{z}) \\
&\quad + \theta_{Q,b_3}(\vec{\tau},\vec{z}) + \theta_{Q,b_4}(\vec{\tau},\vec{z}) + \theta_{Q,b_5}(\vec{\tau},\vec{z}) + \theta_{Q,b_6}(\vec{\tau},\vec{z}) + \theta_{Q,b_7}(\vec{\tau},\vec{z}),
\end{aligned}$$

where $b_1 = \left(\begin{smallmatrix} 2\alpha-2 & 0 \\ -\alpha+i-\alpha i & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} \pm1\pm\alpha \\ 0\end{smallmatrix}\right)$, $b_2 = \left(\begin{smallmatrix} 2\alpha-2 & 0 \\ -\alpha+i-\alpha i & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} -1 \\ -1\end{smallmatrix}\right)$, $b_3 = \pm\left(\begin{smallmatrix} 2\alpha-2 & 0 \\ -\alpha+i-\alpha i & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ -1\end{smallmatrix}\right)$, $b_4 = \left(\begin{smallmatrix} 2\alpha-2 & 0 \\ -\alpha+i-\alpha i & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} \pm1 \\ \pm\alpha\end{smallmatrix}\right)$, $b_5 = \left(\begin{smallmatrix} 2\alpha-2 & 0 \\ -\alpha+i-\alpha i & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} \pm\alpha \\ \pm1\end{smallmatrix}\right)$, $b_6 = \left(\begin{smallmatrix} 2\alpha-2 & 0 \\ -\alpha+i-\alpha i & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} \pm\alpha \\ \pm\alpha\end{smallmatrix}\right)$, $b_7 = \left(\begin{smallmatrix} 2\alpha-2 & 0 \\ -\alpha+i-\alpha i & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 0 \\ \pm1\pm\alpha\end{smallmatrix}\right)$

$$\left(\text{in detail}\ , \theta_{Q,\frac{1}{2}\left(\begin{smallmatrix} 2\alpha-2 & 0 \\ -\alpha+i-\alpha i & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 2 \\ 2\end{smallmatrix}\right)}(\vec{\tau},\vec{z}) \text{ term is equal to } cwe_C\left(\omega_{1,\mu}(4\vec{\tau},4\vec{z}) \mid \mu\in O_K/2O_K\right)\right).$$

4. Invariants via a self-dual code over R

In this section, we study invariants by using self-dual codes over R. First, we give definition for some weight enumerators in higher genus $g \geq 1$.

Definition 4.1. *Let C be a code of length n over R.*

(i) *The complete weight enumerator in genus g is*

$$cwe_{C,g}(x_a \text{ with } a\in R^g) := \sum_{c_1,\ldots,c_g\in C}\prod_{a\in R^g} x_a^{n_a(c_1,\ldots,c_g)},$$

where $n_a(c_1,\ldots,c_g)$ is the number of i such that $a = {}^t(c_{1i},\ldots,c_{gi})$.

(ii) *The symmetrized weight enumerator in genus g is*

$$swe_{C,g}(x_{[a]} \text{ with } [a] \in S^g) := \sum_{c_1,\dots c_g \in C} \prod_{[a] \in S^g} x_{[a]}^{n_{[a]}(c_1,\dots c_g)},$$

where S is the set introduced in Section 2, and $n_{[a]}(c_1, \dots, c_g)$ is the number of i satisfying $[a] = [{}^t(c_{1i}, \dots, c_{gi})]$.

The following lemma says the *MacWilliams identity* for linear codes over R. For this, we give that an $n \times n$ matrix $M = (m_{ij})$ over $\mathbb{C}$ acts on $\mathbb{C}[x_1, \dots, x_n]$ as follows:

$$M \cdot h(x_1, \dots, x_n) = h\left(\sum_{1 \le i \le n} m_{1i}x_i, \dots, \sum_{1 \le i \le n} m_{ni}x_i\right),$$

where $h(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$.

As we mentioned before, the ring R is a finite commutative local Frobenius ring of order 16. We note that the generating character χ of $R \cong \mathbb{F}_2[u, v]/\langle v^2 + v + 1, u^2\rangle$ is

$$\chi(a + bu + cv + duv) := (-1)^{a+b+c+d},$$

where $a, b, c, d \in \mathbb{F}_2$.

Lemma 4.2 is about the MacWilliams identity for a linear code over R with weight enumerators in genus g. We recall that S is a set of representatives of the distinct orbits of a fixed unit subgroup U in R.

Lemma 4.2. *Let C be a linear code of length n over R, and χ be the generating character of R. Let a_i (resp. b_j) be an element of R (resp. S) with any ordering for $1 \le i \le N$ (resp. $1 \le j \le |S|$).*

(i) *Let T_1 be a $N \times N$-matrix such that (i, j)-th component of T_1 is $\chi(a_i a_j)$ with $1 \le i, j \le N$. For complete weight enumerator, we have*

$$cwe_{C^\perp,g}(x_a) = \frac{1}{|C|^g}\left(\bigotimes^g T_1\right) \cdot cwe_{C,g}(x_a).$$

(ii) *Let T_2 be a $|S| \times |S|$-matrix such that (i, j)-th component of T_2 is $\sum_{a' \in \tau} T_{a,a'}$, where $[a] = b_i$ and $\tau = \{a' \in R : a' \approx b_j\}$. For symmetrized weight enumerator, we get*

$$swe_{C^\perp,g}(x_{[a]}) = \frac{1}{|C|^g}\left(\bigotimes^g T_2\right) \cdot swe_{C,g}(x_{[a]}).$$

Proof. We give an equivalence relation as $a \approx b$ if $a = bu$, where $u \in U$, and U is a unit subgroup in R. By the equivalence relation, the generating character for the ring R, and [28, Theorem 8.4], we obtain the result.

□

For an arbitrary unit subgroup for the ring R, we can suggest a symmetrized weight enumerator for a code over R. In the following remark, we show a symmterized weight enumerator and MacWilliams identity for a certain unit group of R.

Remark 4.3. *Let U be a subgroup of the unit group of R. There are eight subgroups of the unit group of R. For example, let $U = \langle\omega\rangle$. Then we get the set $S = \{0, 1, u, 1+u, 1+\omega u, u+\omega\}$ which is the set of representatives of the distinct orbits of U. We note that the equivalence classes are* $[0] = \{0\}$, $[1] = \{1, \omega, 1+\omega\}$, $[u] = \{u, \omega u, u+\omega u\}$, $[1+u] = \{1+u, \omega+\omega u, 1+\omega+u+\omega u\}$, $[1+\omega u] = \{1+\omega u, \omega+u+\omega u, 1+\omega+u\}$ *and* $[u+\omega] = \{u+\omega, 1+\omega+\omega u, 1+u+\omega u\}$. *The ordering in S is given as* $0, 1, u, 1+u, 1+\omega u, u+\omega$. *In this case, we obtain the following matrix T_2 as follows:*

$$T_2 = \begin{pmatrix} 1 & 3 & 3 & 3 & 3 & 3 \\ 1 & -1 & -1 & 3 & -1 & -1 \\ 1 & -1 & 3 & -1 & -1 & -1 \\ 1 & 3 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 3 \\ 1 & -1 & -1 & -1 & 3 & -1 \end{pmatrix}.$$

For the genus $g = 1$, the MacWilliams identity for a linear code C of length n over R is

$$swe_{C^\perp}(x_{[0]}, x_{[1]}, x_{[u]}, x_{[1+u]}, x_{[1+\omega u]}, x_{[u+\omega]}) = \frac{1}{|C|} swe_C(y_1, y_2, y_3, y_4, y_5, y_6),$$

where

$$\begin{aligned} y_1 = & \; x_{[0]} + 3x_{[1]} + 3x_{[u]} + 3x_{[1+u]} + 3x_{[1+\omega u]} + 3x_{[u+\omega]}, \\ y_2 = & \; x_{[0]} - x_{[1]} - x_{[u]} + 3x_{[1+u]} - x_{[1+\omega u]} - x_{[u+\omega]}, \\ y_3 = & \; x_{[0]} - x_{[1]} + 3x_{[u]} - x_{[1+u]} - x_{[1+\omega u]} - x_{[u+\omega]}, \\ y_4 = & \; x_{[0]} + 3x_{[1]} - x_{[u]} - x_{[1+u]} - x_{[1+\omega u]} - x_{[u+\omega]}, \\ y_5 = & \; x_{[0]} - x_{[1]} - x_{[u]} - x_{[1+u]} - x_{[1+\omega u]} + 3x_{[u+\omega]}, \\ y_6 = & \; x_{[0]} - x_{[1]} - x_{[u]} - x_{[1+u]} + 3x_{[1+\omega u]} - x_{[u+\omega]}, \end{aligned}$$

$x_{[i]}$ *and* y_j *are indeterminates.*

Finally, we will show invariants by the complete weight enumerator and symmetrized weight enumerator for a self-dual code over R. First, we define a subgroup G_g of $GL(N^g, \mathbb{C})$ as

$$G_g := \langle M_g, M_J, -I_{N^g} : J \text{ is any integer symmetric matrix}\rangle,$$

where $M_g = \left(\frac{-1}{\sqrt{N}}\right)^g \bigotimes^g T_1$ and $M_J = diag((-1)^{{}^t aJa}$ with $a \in R^g)$. Similarly, let us a subgroup $\hat{G}_g$ of $GL(|S|^g, \mathbb{C})$ as

$$\hat{G}_g := \langle \hat{M}_g, \hat{M}_J, -I_{|S|^g} : J \text{ is any integer symmetric matrix}\rangle,$$

where $\hat{M}_g = \left(\frac{-1}{\sqrt{N}}\right)^g \bigotimes^g T_2$ and $\hat{M}_J = diag((-1)^{{}^t aJa}$ with $a \in S^g)$.

Especially, for a self-dual code of even length over R, we obtain the result of invariant.

Theorem 4.4. *Let C be a self-dual code of even length n over R. The complete weight enumerator $cwe_{C,g}$ of C in genus g is invariant under the action of the group G_g. Similarly, the symmetrized weight enumerator $swe_{C,g}$ of C in genus g is invariant under the action of the group $\hat{G}_g$.*

Proof. We prove the statement for the complete weight enumerator for the code C. We can easily check that $-I_{n^g}$ and M_g invariant for $cwe_C(x_i)$; first, $-I_{N^g} \cdot cwe_C(x_i) = cwe_C(x_i)$ since $cwe_C(x_i) =$

$\sum_{c\in C} x_i^{n_{x_i}(c)} = \sum_{c\in C}((-1)x_i)^{n_{x_i}(c)}$ by $2 \mid n$. Second, $M_g \cdot cwe_C(x_i) = cwe_C(x_i)$ is from the MacWilliams identity Lemma 4.2. Now, we claim that M_J is in G_g. We note that

$$\begin{aligned} M_J \cdot cwe_C(x_a) &= \textstyle\sum_{c_1,\dots,c_g\in C} \prod_{a\in R^g}((-1)^{{}^t aJa} x_a)^{n_a(c_1,\dots,c_g)}, \\ &= \textstyle\sum_{c_1,\dots,c_g\in C} \prod_{a\in R^g}(-1)^{{}^t aJa\cdot n_a(c_1,\dots,c_g)} x_a^{n_a(c_1,\dots,c_g)}. \end{aligned}$$

In detail,

$$\begin{aligned} \textstyle\sum_{a\in R^g} {}^t aJa \cdot n_a(c_1,\dots,c_g) &= \textstyle\sum_{1\le i\le n} J({}^t(c_{1i},\dots,c_{gi})), \\ &= \textstyle\sum_{1\le i\le n} \left(\sum_{1\le k\le g} J_{kk}(c_{ki})^2 + 2\sum_{1\le l<m\le g} J_{lm}c_{li}c_{mi}\right), \\ &= \textstyle\sum_{1\le k\le g} J_{kk} \sum_{1\le i\le n}(c_{ki})^2 + 2\sum_{1\le l<m\le g} J_{lm}\sum_{1\le i\le n} c_{li}c_{mi}; \end{aligned}$$

the right hand side value is divisible by 2 since the code C is self-dual over R.

Thus, we have

$$M_J \cdot cwe_C(x_a) = \sum_{c_1,\dots,c_g\in C} \prod_{a\in R^g} x_a^{n_a(c_1,\dots,c_g)} = cwe_C(x_a),$$

by the above equations. We proved the statement. For symmetrized weight enumerator for C, the proof is similar with the previous case. □

If a linear code C is Type II of length n over R, then the length n is automatically even. Thus, we obtain the following corollary.

Corollary 4.5. *For a Type II code C of length n over R, $cwe_{C,g}$ and $swe_{C,g}$ are invariant under the action of the group G_g and $\hat{G}_g$, respectively.*

Proof. By Lemma 2.4 (ii), for a Type II code of length n over R, n is even. So by using Theorem 4.4, the result follows. □

5. Conclusions

We suggested a Jacobi form from a linear code C over $R := \mathbb{F}_4 + u\mathbb{F}_4$, where $u^2 = 0$. This Jacobi form is not over totally real field, and it is related to complete weight enumerator of the code C. We introduced MacWilliams identities for both, complete weight enumerator and symmetrized weight enumerator in an arbitrary genus $g \ge 1$ of a linear code over R. Moreover, we presented invariants via a self-dual code of even length over R. In the future work, we can consider another finite ring for linear codes and their various weight enumerators. From these results, we can also establish a new Jacobi form.

Acknowledgments

Boran Kim is a corresponding author and supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (NRF-2021R1C1C2012517 and NRF-2019R1I1A1A01060467). Chang Heon Kim is supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (NRF-2018R1D1A1B07045618 and 2016R1A5A1008055). Soonhak Kwon is supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (NRF-2016R1A5A1008055). Yeong-Wook Kwon is supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (NRF-2018R1D1A1B07045618 and 2020R1A4A1016649).

Conflict of interest

The authors declare no conflict of interest.

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